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Spline curves on torus

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1. Introduction

The problem of curve modeling on surface is actual and has many practical applications in CAGD systems. Curves and surfaces are commonly presented in parametric form in CAGD. Therefore we consider only rational curves.

Rational C^1 spline curves on the torus surface are considered using a universal rational parametrization approach. The problem of spline interpolation is reduced to similar problem on projective line. Global splines are composed of different local biquadratic parameterizations. The theory of universal rational parametrization of toric surfaces [2] in the particular case of torus surface is presented in Section 2. This leads to the construction of rational C^1 continuous spline on torus in Section 3. Full classification of low degree splines on torus is presented. Finally in Section 4 the application of splines in blending surface construction is reviewed.

2. Universal rational parametrization of torus

Consider a torus T in projective space $\mathbb{R}P^3$ presented in homogeneous coordinates: $T : (x_1^2 + x_2^2 + x_3^2 + (a^2 - b^2)x_0^2)^2 = 4ax_0^2(x_1^2 + x_2^2)$, $a > b > 0$. Universal rational parametrization of torus [3]:

$$U_T:\ (\mathbb{C}^*)^4 \to \mathbb{R}^4$$
$$\begin{pmatrix} z_0\\ z_1\\ z_2\\ z_3\end{pmatrix} \mapsto \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_0\end{pmatrix} = \begin{pmatrix} 2|z_3|^2\mathrm{Re}(z_0z_1z_2)\mathrm{Im}(z_0z_1z_2)\\ |z_3|^2(\mathrm{Re}^2(z_0z_1z_2) - \mathrm{Im}^2(z_0z_1z_2))\\ \frac{2}{\sqrt{a^2-1}}|z_0|^2\mathrm{Re}(z_1\bar{z}_2z_3)\mathrm{Im}(z_1\bar{z}_2z_3)\\ |z_0|^2(\frac{1}{a+b}\mathrm{Re}^2(z_1\bar{z}_2z_3) + \frac{1}{a-b}\mathrm{Im}^2(z_1\bar{z}_2z_3))\end{pmatrix}, \tag{1}$$

where $x_i \in \mathbb{R}[u]$, $z_i \in \mathbb{C}[u]$ are polynomials, $gcd(x_0, x_1, x_2, x_3) = 1$.

Define a projection on torus as a composition $\Pi_T = \Pi \circ U_T$, $\Pi_T : (\mathbb{C}^*)^4 \to \mathbb{R}^4\backslash(\mathbf{0}) \to T \subset \mathbb{R}P^3$. Projection Π_T maps any curve $z(u) = (z_0(u), z_1(u), z_2(u), z_3(u)) : \mathbf{I} \to (\mathbb{C}^*)^4$ to the curve on torus $c = \Pi_T \circ z$, $c(u) = (x_0(u), x_1(u), x_2(u), x_3(u)) : \mathbf{I} \to T$. $(\mathbb{C}^*)^4$ is a multiplicative group. Define the action of subgroup $G_T = \mathbb{R}^* \times (\mathbb{C}^*)^2 \times \mathbb{R}^*$ in $(\mathbb{C}^*)^4$:

$$(r, \lambda, \mu, s) * (z_0, z_1, z_2, z_3) = (r\lambda\bar{\mu}z_0, \bar{\lambda}z_1, \mu z_2, s\lambda\mu z_3). \tag{2}$$

Since $U_T((r,\lambda,\mu,s)*(z_0,z_1,z_2,z_3)) = |r|^2|\lambda|^6|\mu|^6|s|^2U_T(z_0,z_1,z_2,z_3)$, then any orbit of this action coincides with some projecting fiber of Π_T.

Π_T may be decomposed into two mappings [3]: $\Pi_T = K \circ \Pi^4$, $\Pi^4 : (\mathbb{C}^*)^4 \to (\mathbb{R}P^1)^4$, $K : (\mathbb{R}P^1)^4 \to T \subset \mathbb{R}P^3$, $\Pi_T : (z_0,z_1,z_2,z_3) \to (\mathrm{Re}z_0 : \mathrm{Im}z_0, \mathrm{Re}z_1 : \mathrm{Im}z_1, \mathrm{Re}z_2 : \mathrm{Im}z_2, \mathrm{Re}z_3 : \mathrm{Im}z_3) \to (x_0,x_1,x_2,x_3)$, where $\Pi : \mathbb{C}^* \cong \mathbb{R}^2 \setminus 0 \to \mathbb{R}P^1$ is standard projection to the real projective line.

3. Spline curves on torus. Curve types.

Any Bézier curve on torus $c(u) : \mathbf{I} \to T$ is the projection Π_T of the curve $z(u) : \mathbf{I} \to (\mathbb{C}^*)^4$. Π^4 maps curve $z(u)$ to curve $c_\diamond(u) = (c_0(u), c_1(u), c_2(u), c_3(u))$ in $(\mathbb{R}P^1)^4$ or to four curves $c_i : \mathbf{I} \to \mathbb{R}P^1$, $c_i = (\mathrm{Re}z_i : \mathrm{Im}z_i)$, $i = 0, \cdots, 3$.

Define a quadruple degree of $c(u)$ as a vector $(d_0,d_1,d_2,d_3) = (\deg c_0, \deg c_1, \deg c_2, \deg c_3)$. Then $\deg c = 2(d_0 + d_1 + d_2 + d_3)$. Denote $\mathbf{c}_{(d_0,d_1,d_2,d_3)}$ all curves of degree (d_0, d_1, d_2, d_3). Curves on torus may be divided into types according to quadruple degree.

There are 4 types of conics on T: horizontal and vertical circles $\mathbf{c}_{1,0,0,0}$ and $\mathbf{c}_{0,0,0,1}$, and two families of Villarceau circles $\mathbf{c}_{0,1,0,0}$ and $\mathbf{c}_{0,0,1,0}$ (Fig. 1 the first and the second).

There are 6 types of quartics on T: $\mathbf{c}_{1,1,0,0}$, $\mathbf{c}_{1,0,1,0}$, $\mathbf{c}_{1,0,0,1}$, $\mathbf{c}_{0,1,1,0}$, $\mathbf{c}_{0,1,0,1}$, $\mathbf{c}_{0,0,1,1}$. Each quartic is defined by two circles from different families. The schemes of combination of some ruling circles and corresponding quartic is presented in (Fig. 1 three on the right). Each combination defines different local biquadratic parametrization of quartic.

The quartic spline on torus is the projection of curve $c_\diamond = (c_0, c_1, c_2, c_3)$, where two of the functions c_0, c_1, c_2, c_3 have degree 1, and the rest two have degree 0 (Fig. 2). Let $\deg(c_1) = \deg(c_2) = 0$. c_0 and c_3 are generated as rational C^1 continuous splines of degree 1. For details see [1]. Then the projection of $c_\diamond$ is a spline quartic on torus (Fig. 2 left).

Since rational splines are constructed on the real projective line $\mathbb{R}P^1$, they have to be monotonic [1]. Monotonicity fails if curve on torus touches any of the ruling circles.

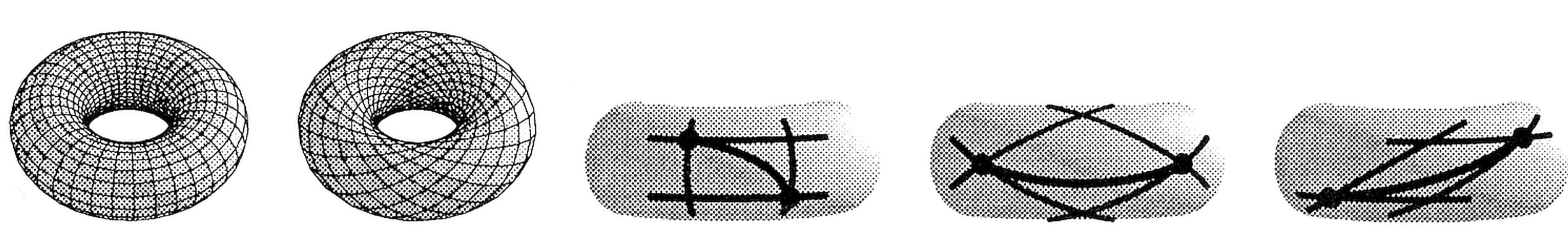

Fig. 1. Circles on torus (two on the left) and $\mathbf{c}_{1,0,0,1}$, $\mathbf{c}_{0,1,1,0}$, $\mathbf{c}_{1,0,1,0}$ type quartics on torus.

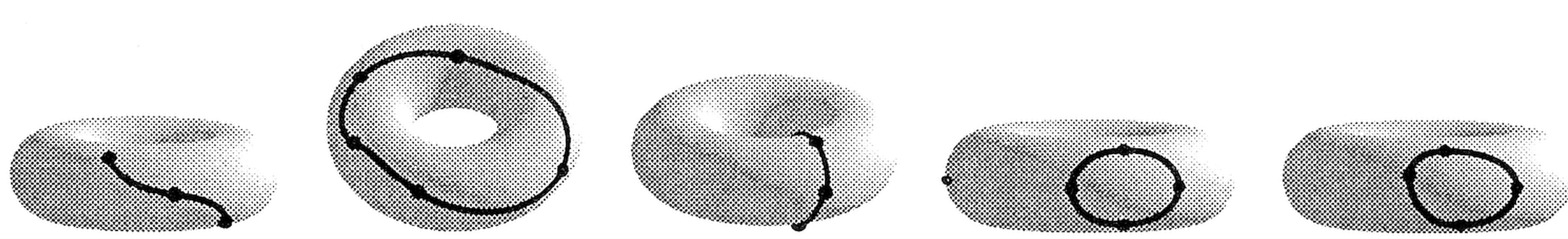

Fig. 2. Quartics on torus.

This condition reduces the ability to rule the shape of curve, e.g. when constructing closed splines. To solve this problem compose the global spline of arcs with different local biquadratic parametrization according to (2). For example see scheme (Fig. 3 left). If the spline through points on torus P_0, P_1, P_2, P_3 is constructed as $c_{(1,0,0,1)}$ curve, the monotonicity fails in P_0 and P_3. Then local parametrization of spline arcs P_0P_1 and P_2P_3 with fail end points may be changed as shown in scheme (Fig. 3 middle).

The arc P_0P_1 is drawn as $c_{(1,1,0,0)} = K(f(u), g(u), 1, 1)$ curve. At the point P_1 the parametrization is changed according to (2) as $(f, g, 1, 1) \to ((f + g)/(1 - fg), 1, 1, g) = (f_*, 1, 1, g_*)$ and the arc P_1P_2 is drawn as $c_{(1,0,0,1)} = K(f_*, 1, 1, g_*)$ curve. At the point P_2 the parametrization is changed $(f_*, 1, 1, g_*) \to (1, f_*, 1, (g_* - f_*)/(1+f_*g_*)) = (1, f_{**}, 1, g_{**})$ and the arc P_2P_3 is drawn as $c_{(0,1,0,1)} = K(1, f_{**}, 1, g_{**})$ curve. Choosing locally most convenient parametrization one may construct the spline of arbitrary shape (Fig. 2).

Lets consider the problem: Construct the closed spline on the side of torus interpolating 4 points P_0, P_2, P_4, P_6 with given derivatives v_0, v_2, v_4, v_6 (Fig. 3 right). Points may be asymmetric. (Fig. 3 right) shows the parametrization scheme with fewest of parametrization switch points. Each quarter arc between two given points is constructed as biarc in two parameterizations. The arc P_0P_2 is constructed as biarc, where middle point P_1 is deduced from condition that derivatives of arcs P_0P_1 and P_1P_2 at P_1 must be equal. The arc P_0P_1 is $c_{(0,1,0,1)}$ type curve and the arc P_1P_2 is $c_{(1,1,0,0)}$ type curve. All spline quarters are constructed in the same way (Fig. 2 the fourth and fifth).

4. Application in blending surface construction

Spline curves may be used in blending surface between two natural quadric (cylinders, cones, spheres) construction. Canal blending surface is constructed as rolling ball blend [4]. Two skew cylinders with radii $r1$ and $r2$ and perpendicular axes correspond to lines $Q_1 = (x_0, x_1, 0, -hx_0, r_1x_0)$ and $Q_2 = (x_0, 0, x_2, hx_0, r_2x_0)$ in $\mathbb{R}P^4$. All spheres tangent to both cylinders make surface S in $\mathbb{R}P^4$ [4] described by the

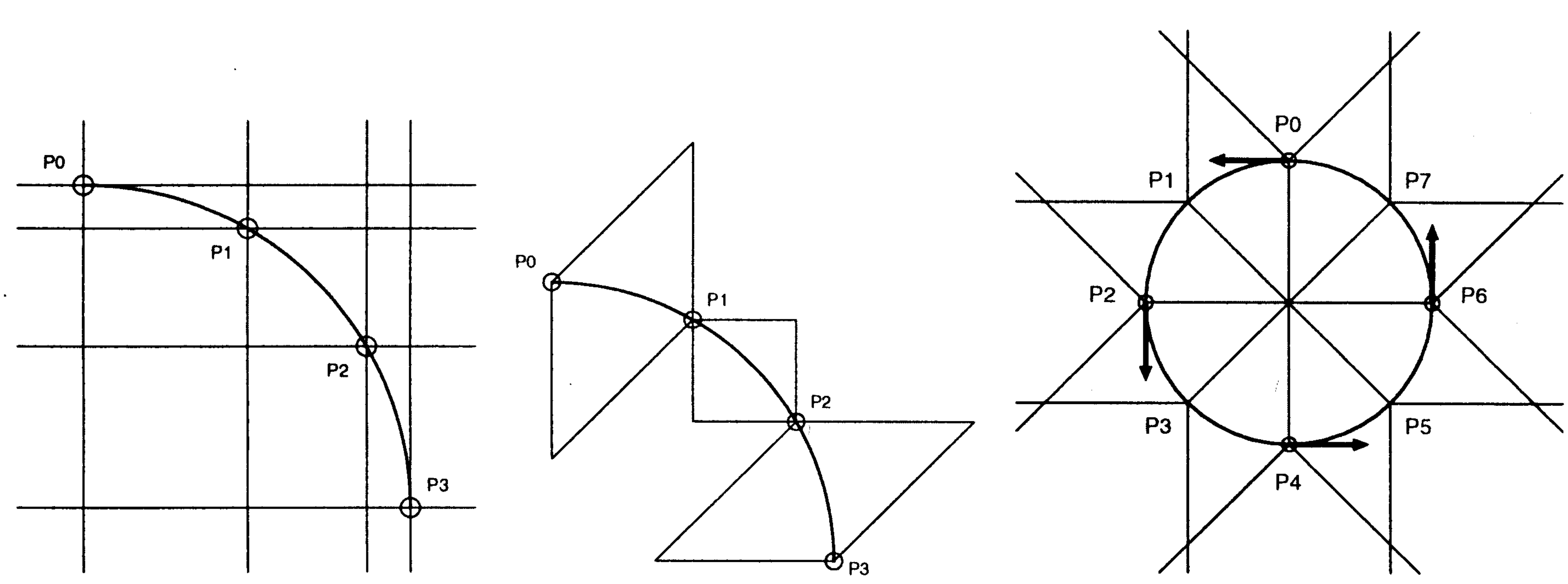

Fig. 3. The schemes of local parametrization of splines (lines represent the circles on torus, arcs represent the spline arcs).

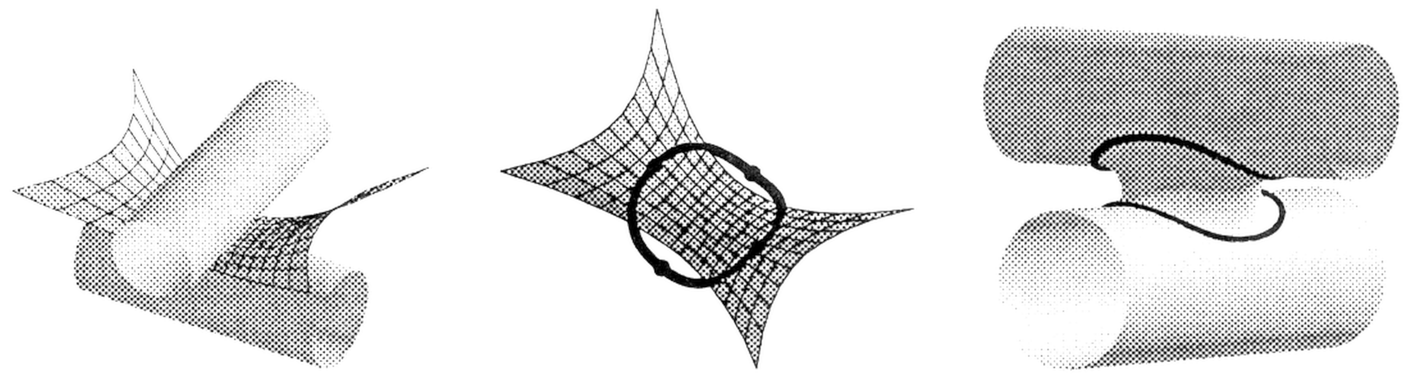

Fig. 4. The projection of centers surface to $\mathbb{R}^3$, spline on centers surface and corresponding blending canal surface.

system: $x_2^2 + (x_3 + hx_0)^2 = (r_1x_0 - x_4)^2, x_1^2 + (x_3 - hx_0)^2 = (r_2x_0 - x_4)^2$. Universal parametrization of this surface is analogous to (1). Curve on S defines blending surface in $\mathbb{R}^3$. The shape of blending surface is described by the shape of the curve. Thus the C^1 quartic splines on this surface are of great use in blending surface shape modeling (Fig. 4).

5. Conclusions

In this paper new construction of spline curves on torus is described. The proposed splines are of minimal degree 4 (degree 2 curves are circles on torus). The technique of local parametrization switching makes it possible to get open and closed curves of free form on torus. Similar methods may be used to produce splines on other toric surfaces.

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REZIUMĖ

M. Kazakevičiūtė, R. Krasauskas. Splaininės kreivės ant toro

Išnagrinėta nauja racionalių C^1 splaininių kreivių ant toro konstrukcija, remiantis universalios racionalios parametrizacijos teorija. Interpoliavimo splainais ant toro problema supaprastinta iki interpoliavimo projektyvinėje tiesėje uždavinio. Splainai sudaromi iš skirtingai lokaliai parametrizuotų dalių. Pateiktas splaininių kreivių taikymas, sudarant jungiamuosius kanalinius paviršius.