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Joint Weighted Limit Theorems for General Dirichlet Series

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Abstract. In the paper,two joint weighted limit theorems in the sense of weak convergence of probability measures on the complex plane for general Dirichlet series are obtained.The first of them gives only the existence of the limit measure,while in the second theorem,under some additional hypothesis on the weight function,the explicit form of the limit measure is presented. Namely,the limit measure coincides with the distribution of some random element related to considered Dirichlet series.

Keywords: General Dirichlet series, joint limit theorem, probability measure, weak convergence, weighted limit theorem.

AMS Subject Classification: 11M41.

1 Introduction

Let $s = \sigma + it$ denote a complex variable, and $\{a_m : m \in \mathbb{N}\}$ and $\{\lambda_m : m \in \mathbb{N}\}$ be a sequence of complex numbers and an increasing sequence of real numbers, respectively, $\lim_{m\to\infty} \lambda_m = +\infty$. The series of the type

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}$$

is called a general Dirichlet series. It is well known that the region of convergence as well as of absolute convergence of Dirichlet series is a half-plane.

The first probabilistic results for Dirichlet series were obtained by Bohr and Jessen [2, 3]. They obtained prototypes of modern limit theorems in the sense of weak convergence of probability measures for the Riemann zeta-function $\zeta(s)$ which, for $\sigma > 1$, is defined by an ordinary Dirichlet series ($\lambda_m = \log m$)

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with coefficients $a_m \equiv 1$. Modern limit theorems for $\zeta(s)$ and other zeta and L-functions can be found in [11, 12, 18, 21, 22]. Limit theorems of such a type for general Dirichlet series were proved in [4, 5, 8, 13, 14, 15, 16, 17, 20].

Weighted limit theorems for general Dirichlet series were began to study in [6], where the case of weak convergence of probability measures on the complex plane was investigated. In [9], a joint generalization with a fixed system of exponents of theorems from [6] was given. Finally, in [7] weighted limit theorems in the space of meromorphic functions for general Dirichlet series were obtained. The aim of this paper is to prove joint weighted limit theorems on the complex plane for general Dirichlet series with a non-fixed system of exponents.

For $r \in \mathbb{N} \setminus \{1\}$, let $\{a_{mj}\colon m \in \mathbb{N}\}$ and $\{\lambda_{mj}\colon m \in \mathbb{N}\}$ be a sequence of complex numbers and an increasing sequence of real numbers, respectively, $\lim\limits_{m\to\infty} \lambda_{mj} = +\infty$, and, for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} \mathrm{e}^{-\lambda_{mj} s}, \quad j = 1, \dots, r.$$

Additionally, we assume that the function $f_j(s)$, $j = 1, \dots, r$, can be meromorphically continued to the region $\sigma > \sigma_{1j}$, $\sigma_{1j} < \sigma_{aj}$, all poles in this region are included in a compact set, and that, for $\sigma > \sigma_{1j}$, σ is not the real part of a pole of $f_j(s)$, the estimates

$$f_j(\sigma + it) = \mathrm{O}\left(|t|^{a_j}\right), \quad a_j = a_j(\sigma) > 0, \quad |t| \geq t_0 > 0, \tag{1.1}$$

and

$$\int_{-T}^{T} |f_j(\sigma + it)|^2 \, \mathrm{d}t = \mathrm{O}(T), \quad T \to \infty, \tag{1.2}$$

are satisfied.

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, and

$$U = U(T, w) = \int_{T_0}^{T} w(t) \, \mathrm{d}t.$$

We suppose that $\lim_{T\to\infty} U(T, w) = +\infty$, and that, for $\sigma > \sigma_{1j}$, σ is not the real part of a pole of $f_j(s)$, and all $v \in \mathbb{R}$, the estimate

$$\int_{T_0+v}^{T+v} w(t - v)|f_j(\sigma + it)|^2 \, \mathrm{d}t = \mathrm{O}(U(1 + |v|)), \quad j = 1, \dots, r, \tag{1.3}$$

holds. For example, if $w(t) = t^{-1}$, then the estimate (1.2) implies (1.3).

Denote by $\mathcal{B}(S)$ the class of Borel sets of a metric space S, and define the probability measure

$$P_{T,\underline{\sigma},w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t\colon \underline{f}(\underline{\sigma}+it) \in A\}} \, \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}^r),$$

where I_A denotes the indicator function of the set A, $\mathbb{C}$ is the complex plane, $\mathbb{C}^r = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r}$, $\underline{\sigma} = (\sigma_1, \dots, \sigma_r)$, and

$$\underline{f}(\underline{\sigma} + it) = (f_1(\sigma_1 + it), \dots, f_r(\sigma_r + it)).$$

Theorem 1. *For $j = 1, \ldots, r$, suppose that $\sigma_j > \sigma_{1j}$, and the functions $f_j(s)$ and $w(t)$ satisfy* (1.1) *and* (1.3). *Then on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ there exists a probability measure $P_{\underline{\sigma},w}$ such that the measure $P_{T,\underline{\sigma},w}$ converges weakly to $P_{\underline{\sigma},w}$ as $T \to \infty$.*

It is important to identify the limit measure $P_{\underline{\sigma},w}$ in Theorem 1. For this, we need additional hypotheses on the functions $w(t)$ and $f_j(s)$, $j = 1, \ldots, r$, as well as some notation and definitions. First of all, we suppose that

$$\lambda_{mj} \geq c(\log m)^{\theta_j} \tag{1.4}$$

with some positive constants c_j and θ_j, $j = 1, \ldots, r$.

Denote by γ the unit circle $\{s \in \mathbb{C}: |s| = 1\}$ on the complex plane, and define the infinite-dimensional torus $\Omega = \prod_{m=1}^{\infty} \gamma_m$, where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure $\hat{m}_H$ exists, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), \hat{m}_H)$.

Now let $\Omega^r = \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Denote by $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ the elements of Ω^r, with $\omega_j \in \Omega_j$, $j = 1, \ldots, r$. Then Ω^r again is a compact topological group, and on $(\Omega^r, \mathcal{B}(\Omega)^r)$, the probability Haar measure m_H can be defined. Thus, we obtain a new probability space $(\Omega^r, \mathcal{B}(\Omega)^r, m_H)$. Note, that the Haar measure m_H is the product of Haar's measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \ldots, r$.

The estimates (1.2) and (1.4) imply [15] that, for $\sigma > \sigma_{1j}$,

$$f_j(\sigma, \omega_j) = \sum_{m=1}^{\infty} a_{mj} \omega_j(m) \mathrm{e}^{-\lambda_{mj}\sigma},$$

where $\omega_j(m)$ denotes the projection of $\omega_j \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}$, is a complex-valued random element on the probability space $(\Omega_j, \mathcal{B}(\Omega_j), m_{jH})$, $j = 1, \ldots, r$. On the probability space $(\Omega^r, \mathcal{B}(\Omega)^r, m_H)$, define the $\mathbb{C}^r$-valued random element $\underline{F}(\underline{\sigma}, \underline{\omega})$ by the formula

$$\underline{F}(\underline{\sigma}, \underline{\omega}) = (f_1(\sigma_1, \omega_1), \ldots, f_r(\sigma_r, \omega_r)),$$

and denote by P_X the distribution of the random element X. In particular, the distribution $P_{\underline{F}}$ of the random element $\underline{F}(\underline{\sigma}, \underline{\omega})$ is defined by

$$P_{\underline{F}}(A) = m_H\left(\underline{\omega} \in \Omega^r\colon \underline{F}(\underline{\sigma}, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

For identification of the limit measure in Theorem 1, we need one more hypothesis on the weight function $w(t)$. Denote by $\mathbb{E}X$ the expectation of the random element X. Let $X(\tau, \omega)$ be an arbitrary ergodic process, $\mathbb{E}|X(\tau, \omega)| < \infty$, with sample path integrable almost surely in the Riemann sense over every finite interval. We suppose that

$$\frac{1}{U} \int_{T_0}^{T} w(t) X(t + v, \omega)\, \mathrm{d}t = \mathbb{E}X(0, \omega) + r_T\,(1 + |v|)^{\alpha} \tag{1.5}$$

almost surely for all $v \in \mathbb{R}$ with some $\alpha > 0$, where $r_T \to 0$ as $T \to \infty$. If in (1.5) $w(t) \equiv 1$ and $v = 0$, then (1.5) becomes the well-known Birkhoff–Khintchine theorem. Define $\mu(T) = \inf_{t\in[T_0,T]} w(t)$. If

$$w(T)\mu^{-1}(T) = \mathrm{O}(1),$$

then the weight function $w(t)$ satisfies (1.5) with $\alpha = 1$ [19]. Let

$$\Lambda = \{\lambda_{mj}\colon m \in \mathbb{N},\ j = 1, \dots, r\}.$$

Theorem 2. *Suppose that the set Λ is linearly independent over the field of rational numbers, and inequality* (1.4) *holds. For $j = 1, \dots, r$, let $\sigma_j > \sigma_{1j}$, and let the functions $f_j(s)$ and $w(t)$ satisfy* (1.1)–(1.3) *and* (1.5). *Then the probability measure $P_{T,\underline{\sigma},w}$ converges weakly to the measure $P_{\underline{F}}$ as $T \to \infty$.*

2 Limit Theorems on Ω^r

In this section, we consider the weak convergence of the probability measure

$$Q_{T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon ((\mathrm{e}^{-i\lambda_{m1}t}\colon m\in\mathbb{N}),\dots,(\mathrm{e}^{-i\lambda_{mr}t}\colon m\in\mathbb{N}))\in A\}}\,\mathrm{d}t, \quad A \in \mathcal{B}(\Omega^r),$$

as $T \to \infty$.

Theorem 3. *On $(\Omega^r, \mathcal{B}(\Omega^r))$, there exists a probability measure Q such that the measure Q_T converges weakly to Q as $T \to \infty$.*

Proof. The dual group of Ω^r is isomorphic to $D = \bigoplus_{j=1}^{r}\bigoplus_{m=1}^{\infty} \mathbb{Z}_{mj}$, where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $m \in \mathbb{N}$ and $j = 1, \dots, r$. An element $\underline{k} = (k_{mj}) \in D$, where only finite number of integers k_{mj} are distinct from zero, acts on Ω by the formula

$$\underline{\omega} \to \underline{\omega}^{\underline{k}} = \prod_{j=1}^{r}\prod_{m=1}^{\infty} \omega_j^{k_{mj}}(m).$$

Therefore, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is of the form

$$g_T(\underline{k}) = \int_{\Omega^r} \prod_{j=1}^{r}\prod_{m=1}^{\infty} \omega_j^{k_{mj}}(m)\mathrm{d}Q_T = \frac{1}{U}\int_{T_0}^{T} w(t) \prod_{j=1}^{r}\prod_{m=1}^{\infty} \mathrm{e}^{-i\lambda_{mj}k_{mj}t}\,\mathrm{d}t,$$

where, as above, only a finite number of integers k_{mj} are distinct from zero. Thus, we have that

$$g_T(\underline{k}) = \frac{1}{U}\int_{T_0}^{T} w(t)\exp\Big\{-it\sum_{j=1}^{r}\sum_{m=1}^{\infty}\lambda_{mj}k_{mj}\Big\}\mathrm{d}t.$$

If

$$(\Lambda, \underline{k}) \overset{def}{=} \sum_{j=1}^{r}\sum_{m=1}^{\infty}\lambda_{mj}k_{mj} = 0,$$

then, clearly, $g_T(\underline{k}) = 1$. If $(\Lambda, \underline{k}) \neq 0$, then the integration by parts shows that

$$g_T(\underline{k}) = \mathrm{O}\left((U(\Lambda, \underline{k}))^{-1}\right).$$

Thus, we have that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } (\Lambda, \underline{k}) = 0, \\ 0 & \text{if } (\Lambda, \underline{k}) \neq 0. \end{cases}$$

Now, by Theorem 1.4.2 of [10], we find that the measure Q_T converges weakly to the measure Q defined by the Fourier transform

$$\begin{cases} 1 & \text{if } (\Lambda, \underline{k}) = 0, \\ 0 & \text{if } (\Lambda, \underline{k}) \neq 0, \end{cases}$$

as $T \to \infty$. □

Theorem 4. *Suppose that the set Λ is linearly independent over the field of rational numbers. Then the measure Q_T converges weakly to Haar measure m_H as $T \to \infty$.*

Proof. Since the set Λ is linearly independent over the field of rational numbers, $(\Lambda, \underline{k}) = 0$ if and only if $\underline{k} = \underline{0}$. Therefore, repeating the proof of Theorem 3, we obtain that the measure Q_T converges weakly to the measure with Fourier transform

$$\begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Since the latter Fourier transform is of the Haar measure m_H, the theorem is proved. □

3 Absolutely Convergent Series

Let $\sigma_{2j} > \sigma_{aj} - \sigma_{1j}$, and, for $m, n \in \mathbb{N}$,

$$v_{mj}(n) = \exp\left\{-\mathrm{e}^{(\lambda_{mj} - \lambda_{nj})\sigma_{2j}}\right\}, \quad j = 1, \ldots, r.$$

For $\sigma > \sigma_{1j}$ and $\underline{\hat{\omega}} \in \Omega^r$, define

$$f_{nj}(s) = \sum_{m=1}^{\infty} a_{mj} v_{mj}(n) \mathrm{e}^{-\lambda_{mj} s},$$

$$f_{nj}(s, \hat{\omega}_j) = \sum_{m=1}^{\infty} a_{mj} v_{mj}(n) \hat{\omega}_j(m) \mathrm{e}^{-\lambda_{mj} s}, \quad j = 1, \ldots, r.$$

In [15], it was proved that the series for $f_{nj}(s)$, and thus for $f_{nj}(s, \hat{\omega}_j)$, is absolutely convergent for $\sigma > \sigma_{1j}$, $j = 1, \ldots, r$. Let

$$\underline{f}_n(\underline{\sigma} + it) = (f_{n1}(\sigma_1 + it), \ldots, f_{nr}(\sigma_r + it))$$

and

$$\underline{f}_n(\underline{\sigma}+it,\hat{\underline{\omega}}) = (f_{n1}(\sigma_1+it,\hat{\underline{\omega}}_1),\dots,f_{nr}(\sigma_r+it,\hat{\underline{\omega}}_r)).$$

In this section, we consider the weak convergence of the probability measures

$$P_{T,n,\underline{\sigma},w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon\ \underline{f}_n(\underline{\sigma}+it)\in A\}}\,\mathrm{d}t, \quad A\in\mathcal{B}(\mathbb{C}^r),$$
$$\hat{P}_{T,n,\underline{\sigma},w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon\ \underline{f}_n(\underline{\sigma}+it,\hat{\underline{\omega}})\in A\}}\,\mathrm{d}t, \quad A\in\mathcal{B}(\mathbb{C}^r).$$

Theorem 5. *For $j=1,\dots,r$, let $\sigma_j>\sigma_{1j}$. Then on $(\mathbb{C}^r,\mathcal{B}(\mathbb{C}^r))$, there exists a probability measure $P_{n,\underline{\sigma},w}$ such that the measure $P_{T,n,\underline{\sigma},w}$ converges weakly to $P_{n,\underline{\sigma},w}$ as $T\to\infty$.*

Proof. Define the function $h_{n,\underline{\sigma}}:\Omega^r\to\mathbb{C}^r$ by the formula

$$h_{n,\underline{\sigma}}(\underline{\omega}) = \Bigl(\sum_{m=1}^{\infty} a_{m1}v_{m1}(n)\omega_1(m)\mathrm{e}^{-\lambda_{m1}\sigma_1},\dots,\sum_{m=1}^{\infty} a_{mr}v_{mr}(n)\omega_r(m)\mathrm{e}^{-\lambda_{mr}\sigma_r}\Bigr).$$

Since the series

$$\sum_{m=1}^{\infty} a_{mj}v_{mj}(n)\mathrm{e}^{-\lambda_{mj}\sigma_j}$$

converges absolutely, the series

$$\sum_{m=1}^{\infty} a_{mj}v_{mj}(n)\omega_j(m)\mathrm{e}^{-\lambda_{mj}\sigma_j}$$

converges uniformly in ω_j, $j=1,\dots,r$. Therefore, the function $h_{n,\underline{\sigma}}$ is continuous. Moreover,

$$h_{n,\underline{\sigma}}\bigl((\mathrm{e}^{-i\lambda_{m1}t}\colon m\in\mathbb{N}),\dots,(\mathrm{e}^{-i\lambda_{mr}t}\colon m\in\mathbb{N})\bigr) = \underline{f}_n(\underline{\sigma}+it).$$

Therefore, $P_{T,n,\underline{\sigma},w} = Q_{T,w}h^{-1}_{n,\underline{\sigma}}$, where $Q_{T,w}$ is the measure in Theorem 3. Hence, the continuity of $h_{n,\underline{\sigma}}$, Theorem 3 and Theorem 5.1 of [1] show that the measure $P_{T,n,\underline{\sigma},w}$, as $T\to\infty$, converges to $Qh^{-1}_{n,\underline{\sigma}}$, where Q is the limit measure in Theorem 3. □

Theorem 6. *For $j=1,\dots,r$, let $\sigma_j>\sigma_{1j}$, and suppose that the set Λ is linearly independent over the field of rational numbers. Then on $(\mathbb{C}^r,\mathcal{B}(\mathbb{C}^r))$, there exists a probability measure $P_{n,\underline{\sigma},w}$ such that the measures $P_{T,n,\underline{\sigma},w}$ and $\hat{P}_{T,n,\underline{\sigma},w}$ both converge weakly to $P_{n,\underline{\sigma},w}$ as $T\to\infty$.*

Proof. Since the set Λ is linearly independent over the field of rational numbers, using Theorem 4 and repeating the proof of Theorem 5, we find that the measure $P_{T,n,\underline{\sigma},w}$ converges weakly to $m_H h^{-1}_{n,\underline{\sigma}}$ as $T\to\infty$.

Now consider the measure $\hat{P}_{T,n,\underline{\sigma},w}$. Define $\hat{h}_{n,\underline{\sigma}} : \Omega^r \to \mathbb{C}^r$ by the formula

$$\hat{h}_{n,\underline{\sigma}}(\underline{\omega}) = \Bigg(\sum_{m=1}^{\infty} a_{m1} v_{m1}(n) \hat{\omega}_1(m) \omega_1(m) \mathrm{e}^{-\lambda_{m1}\sigma_1}, \ldots \\ \sum_{m=1}^{\infty} a_{mr} v_{mr}(n) \hat{\omega}_r(m) \omega_r(m) \mathrm{e}^{-\lambda_{mr}\sigma_r} \Bigg).$$

Then, similarly to the case of $P_{T,n,\underline{\sigma},w}$, we obtain that the measure $\hat{P}_{T,n,\underline{\sigma},w}$ converges weakly to $m_H \hat{h}^{-1}_{n,\underline{\sigma}}$ as $T \to \infty$. It remains to show that $m_H h^{-1}_{n,\underline{\sigma}} = m_H \hat{h}^{-1}_{n,\underline{\sigma}}$. For this, we take $h : \Omega^r \to \Omega^r$ defined by $h(\underline{\omega}) = \underline{\omega}\ \underline{\hat{\omega}}$. Then $\hat{h}_{n,\underline{\sigma}}(\underline{\omega}) = h_{n,\underline{\sigma}}(h(\underline{\omega}))$, and the invariance of the Haar measure m_H shows that

$$m_H \hat{h}^{-1}_{n,\underline{\sigma}} = m_H (h_{n,\underline{\sigma}}(h))^{-1} = (m_H h^{-1}) h^{-1}_{n,\underline{\sigma}} = m_H h^{-1}_{n,\underline{\sigma}}. \qquad \square$$

4 Approximation in the Mean

Let, for $\underline{\omega} \in \Omega^r$,

$$\underline{f}(\underline{\sigma} + it, \underline{\omega}) = (f_1(\sigma_1 + it, \omega_1), \ldots, f_r(\sigma_r + it, \omega_r)).$$

In this section, we will approximate $\underline{f}(\underline{\sigma} + it)$ and $\underline{f}(\underline{\sigma} + it, \underline{\omega})$ in the mean by $\underline{f}_n(\underline{\sigma} + it)$ and $\underline{f}_n(\underline{\sigma} + it, \underline{\omega})$, respectively. Let, for $\underline{z}_1 = (z_{11}, \ldots, z_{1r}) \in \mathbb{C}^r$ and $\underline{z}_2 = (z_{21}, \ldots, z_{2r}) \in \mathbb{C}^r$,

$$\varrho_r(\underline{z}_1, \underline{z}_2) = \Big(\sum_{j=1}^{r} |z_{1j} - z_{2j}|^2 \Big)^{\frac{1}{2}}$$

be the metric in $\mathbb{C}^r$ inducing its topology.

Theorem 7. *For $j = 1, \ldots, r$, let $\sigma_j > \sigma_{1j}$. Then, under hypotheses of Theorem Theorem 1,*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t) \varrho_r(\underline{f}(\underline{\sigma} + it), \underline{f}_n(\underline{\sigma} + it))\, \mathrm{d}t = 0.$$

Proof. Under hypotheses of the theorem, in [6], Theorem 9, it was proved that

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t) |f_j(\sigma_j + it) - f_{nj}(\sigma_j + it)|\, \mathrm{d}t = 0, \quad j = 1, \ldots, r. \tag{4.1}$$

In view of the inequality $(|a| + |b|)^{\frac{1}{2}} \le |a|^{\frac{1}{2}} + |b|^{\frac{1}{2}}$, we have that

$$\varrho_r(\underline{z}_1, \underline{z}_2) \le \sum_{j=1}^{r} |z_{1j} - z_{2j}|.$$

Therefore, the theorem is a result of (4.1). $\square$

Theorem 8. *For $j = 1, \dots, r$, let $\sigma_j > \sigma_{1j}$. Then, under hypotheses of Theorem 2,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) \varrho_r\big(\underline{f}(\underline{\sigma}+it, \underline{\omega}), \underline{f}_n(\underline{\sigma}+it, \omega)\,\mathrm{d}t = 0$$

for almost all $\underline{\omega} \in \Omega^r$.

Proof. Since the set Λ is linearly independent over the field of rational numbers, obviously, each set $\{\lambda_{mj}: m \in \mathbb{N}\}$, $j = 1, \dots, r$, is as well. Therefore, by Theorem 13 from [6], we have that, for $j = 1, \dots, r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) |f_j(\sigma_j+it, \omega_j) - f_{nj}(\sigma_j+it, \omega_j)|\,\mathrm{d}t = 0$$

for almost all $\omega_j \in \Omega_j$. Since the Haar measure m_H is the product of the measures m_{jH}, $j = 1, \dots, r$, hence the theorem follows in the same way as Theorem 7. □

5 Proof of Theorem 1

By Theorem 5, the probability measure $P_{T,n,\underline{\sigma},w}$ converges weakly to $P_{n,\underline{\sigma},w}$ as $T \to \infty$.

Lemma 1. *The family of probability measures $\{P_{n,\underline{\sigma},w}: n \in \mathbb{N}\}$ is tight.*

Proof. For $M > 0$, we have

$$\begin{aligned}
\frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t:\ \varrho_r(\underline{f}_{nj}(\underline{\sigma}+it,\underline{0})>M\}}\,\mathrm{d}t \quad &\le \frac{1}{MU} \int_{T_0}^{T} w(t) \varrho_r\big(\underline{f}_n(\underline{\sigma}+it, \underline{0})\,\mathrm{d}t \\
\le \frac{1}{MU} \int_{T_0}^{T} w(t) \varrho_r\big(\underline{f}(\underline{\sigma}+it), \underline{f}_n(\sigma+it)\big)\,\mathrm{d}t &+ \frac{1}{MU} \int_{T_0}^{T} w(t) \varrho_r\big(\underline{f}(\underline{\sigma}+it), \underline{0}\big)\,\mathrm{d}t.
\end{aligned}$$

Therefore, Theorem 7 and (1.3) with $v = 0$ show that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t:\ \varrho_r(\underline{f}_n(\underline{\sigma}+it,0)>M\}}\,\mathrm{d}t \\
&\quad \le \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{MU} \int_{T_0}^{T} w(t) \varrho_r\big(\underline{f}(\underline{\sigma}+it), f_n(\sigma+it)\big)\,\mathrm{d}t \\
&\qquad + \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{MU} \sum_{j=1}^{r} \int_{T_0}^{T} w(t) |f_j(\sigma_j+it)|\,\mathrm{d}t \\
&\quad \ll \frac{1}{M} + \limsup_{T\to\infty} \frac{1}{MU} \sum_{j=1}^{r} \Big(\int_{T_0}^{T} w(t)\,\mathrm{d}t \int_{T_0}^{T} w(t) |f_j(\sigma_j+it))|^2\,\mathrm{d}t \Big)^{\frac{1}{2}} \\
&\quad \le R/M, \qquad (5.1)
\end{aligned}$$

where $R < \infty$. Now let ϵ be an arbitrary positive number, and $M = R\epsilon^{-1}$. Then (5.1) and properties of weak convergence of probability measures imply

$$\begin{aligned}
&P_{n,\underline{\sigma},w}\left(\{\underline{s} \in \mathbb{C}^r\colon\, \varrho_r(\underline{s},\underline{0}) > M\}\right)\\
&\quad\le \liminf_{T\to\infty} P_{T,n,\underline{\sigma},w}\left(\{\underline{s} \in \mathbb{C}^r\colon\, \varrho_r(\underline{s},\underline{0}) > M\}\right)\\
&\quad= \liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon\, \varrho_r(\underline{f}_n(\underline{\sigma}+it,\underline{0})>M\}}\,\mathrm{d}t\\
&\quad\le \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon\, \varrho_r(\underline{f}_n(\underline{\sigma}+it,\underline{0})>M\}}\,\mathrm{d}t \le \varepsilon. \qquad (5.2)
\end{aligned}$$

Let $K_\varepsilon = \{\underline{s} \in \mathbb{C}^r\colon\, \varrho_r(\underline{s},\underline{0}) \le M\}$. Then the set K_ε is compact in $\mathbb{C}^r$, and, by (5.2),

$$P_{n,\underline{\sigma},w}(K_\varepsilon) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{P_{n,\underline{\sigma},w}\colon\, n \in \mathbb{N}\}$ is tight. $\square$

Proof of Theorem 1. On a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$, define a random variable $\theta = \theta_T$ by

$$\mathbb{P}(\theta \in A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_A\,\mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}).$$

Let $\underline{X}_{T,n,w}(\underline{\sigma}) = \underline{f}_n(\underline{\sigma}+i\theta_T)$. Then the assertion of Theorem 5 can be rewritten in the form

$$\underline{X}_{T,n,w}(\underline{\sigma}) \underset{T\to\infty}{\xrightarrow{\mathcal{D}}} \underline{X}_{n,w}(\underline{\sigma}), \qquad (5.3)$$

where $\underline{X}_{n,w}(\underline{\sigma})$ is a $\mathbb{C}^r$-valued random element with the distribution $P_{n,\underline{\sigma},w}$, and $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Since the family $\{P_{n,\underline{\sigma},w}\colon\, n \in \mathbb{N}\}$ is tight, by the Prokhorov theorem, see [1], it is relatively compact. Thus, there exists a sequence $\{P_{n_k,\underline{\sigma},w}\} \subset \{P_{n,\underline{\sigma},w}\}$ such that $P_{n_k,\underline{\sigma},w}$ converges weakly to a certain probability measure $P_{\underline{\sigma},w}$ on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $k \to \infty$. In other words,

$$\underline{X}_{n_k,w}(\underline{\sigma}) \underset{k\to\infty}{\xrightarrow{\mathcal{D}}} P_{\underline{\sigma},w}. \qquad (5.4)$$

Define $\underline{X}_{T,w}(\underline{\sigma}) = \underline{f}(\underline{\sigma} + i\theta_T)$. Then, by Theorem 7, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\varrho_r(\underline{X}_{T,w}(\underline{\sigma}), \underline{X}_{T,n,w}(\underline{\sigma})) \ge \varepsilon\right)\\
&\quad= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon\, \varrho_r(\underline{f}(\underline{\sigma}+it),\underline{f}_n(\underline{\sigma}+it))\ge\varepsilon\}}\,\mathrm{d}t\\
&\quad\le \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U\varepsilon}\int_{T_0}^{T} w(t)\varrho_r(\underline{f}(\underline{\sigma}+it), \underline{f}_n(\underline{\sigma}+it))\,\mathrm{d}t = 0.
\end{aligned}$$

This, relations (5.3) and (5.4) together with Theorem 4.2 of [1] yield

$$\underline{X}_{T,w}(\underline{\sigma}) \underset{T\to\infty}{\xrightarrow{\mathcal{D}}} P_{\underline{\sigma},w}, \qquad (5.5)$$

hence, $P_{T,\underline{\sigma},w}$ converges weakly to $P_{\underline{\sigma},w}$ as $T \to \infty$. $\square$

6 Proof of Theorem 1.2

Define one more probability measure

$$\hat{P}_{T,\underline{\sigma},w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t\colon\, \underline{f}(\underline{\sigma}+it,\omega)\in A\}}\,\mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Theorem 9. *Under hypotheses of Theorem 2 on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$, there exists a probability measure $P_{\underline{\sigma},w}$ such that the measures $P_{T,\underline{\sigma},w}$ and $\hat{P}_{T,\underline{\sigma},w}$ both converge weakly to $P_{\underline{\sigma},w}$ as $T \to \infty$.*

Proof. By Theorem 6, the probability measures $P_{T,n,\underline{\sigma},w}$ and $\hat{P}_{T,n,\underline{\sigma},w}$ both converge weakly to the same measure $P_{n,\underline{\sigma},w}$ as $T \to \infty$. As in the proof of Theorem 1, we have that relations (5.3)–(5.5) hold, and the measure $P_{T,\underline{\sigma},w}$ converges weakly to $P_{\underline{\sigma},w}$. Moreover, the relation (5.5) shows that the measure $P_{\underline{\sigma},w}$ is independent on the sequence $\{P_{n_k,\underline{\sigma},w}\}$. Therefore, we have that

$$\underline{X}_{n,w}(\underline{\sigma}) \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\sigma},w}. \tag{6.1}$$

Now define

$$\underline{Y}_{T,n,w}(\underline{\sigma}) = \underline{f}_n(\underline{\sigma} + i\theta_T, \underline{\omega}), \quad \underline{Y}_{T,w}(\underline{\sigma}) = \underline{f}(\underline{\sigma} + i\theta_T, \underline{\omega}).$$

Then repeating the arguments of the proof of Theorem 1, applying Theorem 8 and using (6.1), we find that

$$\underline{Y}_{T,w}(\underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\sigma},w}.$$

This means that the measure $\hat{P}_{T,\underline{\sigma},w}$ also converges to the measure $P_{\underline{\sigma},w}$ as $T \to \infty$. □

Define, for $t \in \mathbb{R}$,

$$\underline{a}_t = \left\{\left((\mathrm{e}^{-i\lambda_{m1}t}\colon m \in \mathbb{N}), \ldots, (\mathrm{e}^{-i\lambda_{mr}t}\colon m \in \mathbb{N})\right)\right\}.$$

Then $\{\underline{a}_t\colon t \in \mathbb{R}\}$ is a one-parameter group. Define the one-parameter family $\{\underline{\varphi}_t\colon t \in \mathbb{R}\}$ of transformations on Ω^r by $\underline{\varphi}_t(\underline{\omega}) = \underline{a}_t\underline{\omega}$, $\underline{\omega} \in \Omega^r$. Then $\{\underline{\varphi}_t\colon t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω^r. A set $A \in \mathcal{B}(\Omega^r)$ is invariant with respect to the group $\{\underline{\varphi}_t\colon t \in \mathbb{R}\}$ if, for every $t \in \mathbb{R}$, the sets A and $A_t = \underline{\varphi}_t(A)$ differ one from another only by a set of zero m_H-measure. The one-parameter group $\{\underline{\varphi}_t\colon t \in \mathbb{R}\}$ is called ergodic if its σ-field of invariant sets consists only from sets having m_H-measure 0 or 1.

Lemma 2. *Suppose that the set Λ is linearly independent over the field of rational numbers. Then the one-parameter group $\{\underline{\varphi}_t\colon t \in \mathbb{R}\}$ is ergodic.*

Proof. Let χ be a character of Ω^r. As we have seen in the proof of Theorem Theorem 3,

$$\chi(\underline{\omega}) = \prod_{j=1}^{r}\prod_{m=1}^{\infty} \omega_j^{k_{mj}}(m),$$

where only a finite number of integers k_{mj} are distinct from zero.

Suppose that χ is a non-trivial character. Then we have

$$\chi(\underline{a}_t) = \prod_{j=1}^{r} \prod_{m=1}^{\infty} \mathrm{e}^{-i\lambda_{mj} k_{mj} t},$$

where only a finite number of integers k_{mj} are distinct from zero. Since the set Λ is linearly independent over the field of rational numbers,

$$\prod_{j=1}^{r} \prod_{m=1}^{\infty} \mathrm{e}^{\lambda_{mj} k_{mj}} \neq 1$$

for $\underline{k} \neq \underline{0}$. Consequently, there exists a $t_0 \in \mathbb{R}$ such that $\chi(\underline{a}_{t_0}) \neq 1$.

Now we take an invariant set $A \in \mathcal{B}(\Omega)$ with respect to $\{\underline{\varphi}_t \colon t \in \mathbb{R}\}$. Then, for almost all $\underline{\omega} \in \Omega^r$ with respect to the measure m_H,

$$I_A(\underline{a}_t\ \underline{\omega}) = I_A(\underline{\omega}).$$

Therefore, the Fourier transform $\hat{I}_A$ of I_A is

$$\begin{aligned}\hat{I}_A(\chi) &= \int\limits_{\Omega^r} \chi(\underline{\omega}) I_A(\underline{\omega}) m_H(\mathrm{d}\underline{\omega}) = \int\limits_{\Omega^r} \chi(\underline{\omega}) I_A(\underline{a}_{t_0}\ \underline{\omega}) m_H(\mathrm{d}\underline{\omega}) \\ &= \chi(\underline{a}_{t_0}) \int\limits_{\Omega} \chi(\underline{\omega}) I_A(\underline{\omega}) m_H(\mathrm{d}\underline{\omega}) = \chi(\underline{a}_{t_0}) \hat{I}_A(\chi).\end{aligned}$$

Since $\chi(\underline{a}_{t_0}) \neq 1$, hence we have that $\hat{I}_A(\chi) = 0$ for all non-trivial characters χ of Ω^r.

Now let χ_0 be the trivial character of Ω^r, that is, $\chi_0(\underline{\omega}) = 1$ for all $\underline{\omega} \in \Omega^r$. We put $\hat{I}_A(\chi_0) = u$. Then, using the orthogonality of characters

$$\int\limits_{\Omega^r} \chi(\underline{\omega}) m_H(\mathrm{d}\underline{\omega}) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain that, for any character χ of Ω^r,

$$\hat{I}_A(\chi) = u \int\limits_{\Omega^r} \chi(\underline{\omega}) m_H(\mathrm{d}\underline{\omega}) = u\hat{1}(\chi) = \hat{u}(\chi).$$

This shows that $I_A(\underline{\omega}) = u$ for almost all $\underline{\omega} \in \Omega^r$. Since $u = 0$ or $u = 1$, either $m_H(A) = 0$ or $m_H(A) = 1$, i. e., the group $\{\underline{\varphi}_t \colon t \in \mathbb{R}\}$ is ergodic. □

Proof of Theorem 2. In view of Theorem 9, it suffices to show that the limit measure $P_{\underline{\sigma},w}$ coincides with P_F.

Let $A \in \mathcal{B}(\mathbb{C}^r)$ be an arbitrary fixed continuity set of the measure $P_{\underline{\sigma},w}$. Then, by Theorem 9 and Theorem 2.1 of [1],

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t\colon \underline{f}(\underline{\sigma}+it)\in A\}}\,\mathrm{d}t = P_{\underline{\sigma},w}(A). \tag{6.2}$$

On the probability space $(\Omega^r, \mathcal{B}(\Omega^r))$, define the random variable θ by the formula

$$\theta(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{F}(\underline{\sigma}, \underline{\omega}) \in A, \\ 0 & \text{if } \underline{F}(\underline{\sigma}, \underline{\omega}) \notin A. \end{cases}$$

Clearly,

$$\mathbb{E}\theta = \int\limits_{\Omega^r} \theta(\underline{\omega}) \mathrm{d}m_H = m_H \left(\underline{\omega} \in \Omega^r \colon \underline{F}(\underline{\sigma}, \underline{\omega}) \in A\right) = P_{\underline{F}}(A). \tag{6.3}$$

Lemma 2 implies the ergodicity of the random process $\theta(\underline{\varphi}_t(\underline{\omega}))$. Therefore, (1.5) with $v = 0$ shows that

$$\lim_{T \to \infty} \frac{1}{U} \int\limits_{T_0}^{T} w(t)\theta(\underline{\varphi}_t(\underline{\omega}))\, \mathrm{d}t = \mathbb{E}\theta. \tag{6.4}$$

Moreover, the definitions of θ and $\underline{\varphi}_t(\underline{\omega})$ yield

$$\begin{aligned} \frac{1}{U} \int_{T_0}^{T} w(t)\theta(\underline{\varphi}_t(\underline{\omega}))\, \mathrm{d}t &= \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t \colon \underline{F}(\underline{\sigma}, \underline{\varphi}_t(\underline{\omega})) \in A\}}\, \mathrm{d}t \\ &= \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t \colon \underline{f}(\underline{\sigma} + it, \underline{\omega}) \in A\}}\, \mathrm{d}t. \end{aligned}$$

This together with (6.3) and (6.4) shows that

$$\lim_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t \colon \underline{f}(\underline{\sigma} + it, \underline{\omega}) \in A\}}\, \mathrm{d}t = P_{\underline{F}}(A).$$

Therefore, in view of (6.2), for all continuity sets A of $P_{\underline{\sigma}, w}$, the equality

$$P_{\underline{\sigma}, w}(A) = P_{\underline{F}}(A)$$

holds. Hence, it is true for all $A \in \mathcal{B}(\mathbb{C}^r)$. This completes the proof. □

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