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A two-dimensional limit theorem for Lerch zeta-functions. II

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Abstract. We prove a two-dimensional limit theorem for Lerch zeta-functions with transcendental and rational parameters.

Keywords: Lerch zeta-function, probability measure, weak convergence.

Let $s = \sigma + it$ be a complex variable, and $0 < \lambda < 1$ and $0 < \alpha \leqslant 1$ be fixed parameters. The Lerch-zeta function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by the Dirichlet series $L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$, and, because of $0 < \lambda < 1$, is analytically continued to an entire function.

Probabilistic limit theorems for the function $L(\lambda, \alpha, s)$ with transcendental and rational parameter α were proved in [2] while the case of algebraic irrational parameter α was considered in [3, 4, 6, 7].

In [5], we proved a limit theorem on the complex plane for a pair $(L(\lambda_1, \alpha_1, s), L(\lambda_2, \alpha_2, s))$, when α_1 and α_2 are transcendental and algebraic irrational numbers, respectively. The aim of this note is to prove a limit theorem of such a kind when the number α_2 is rational. To state the theorem we need some notation and definitions. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, by $\operatorname{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let

$$\nu_T(\cdots) = \frac{1}{T} \operatorname{meas}\{t \in [0, T]\colon \cdots\},$$

where in place of dots a condition satisfied by t is to be written. Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$. Define $\Omega_1 = \prod_{m=0}^{\infty} \gamma_m$ and $\Omega_2 = \prod_p \gamma_p$, where $\gamma_m = \gamma$ and $\gamma_p = \gamma$ for all $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and primes p, respectively. Denote by $\omega_1(m)$ and $\omega_2(p)$ the projections of $\omega_1 \in \Omega_1$ to γ_m and of $\omega_2 \in \Omega_2$ to γ_p, respectively. Moreover, we extend the function $\omega_2(p)$ to the set $\mathbb{N}$ by the formula $\omega_2(m) = \prod_{p^l \| m} \omega_2^l(p), m \in \mathbb{N}$, where $p^l \| m$ means that $p^l | m$ but $p^{l+1} \nmid m$.

Let $\Omega = \Omega_1 \times \Omega_2$. Then Ω is a compact topological Abelian group, therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Suppose that $\alpha_2 = \frac{a}{q}$, $0 < a < q$, $(a, q) = 1$. Denote by $\omega = (\omega_1, \omega_2)$ the elements of Ω, and put, for brevity, $\underline{\alpha} = (\alpha_1, \alpha_2)$, $\underline{\lambda} = (\lambda_1, \lambda_2)$, $\underline{\sigma} = (\sigma_1, \sigma_2)$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $\mathbb{C}^2$-valued random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \omega)$, for $\min(\sigma_1, \sigma_2) > \frac{1}{2}$, by

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \omega) = \bigl(L(\lambda_1, \alpha_1, \sigma_1, \omega_1), L(\lambda_2, \alpha_2, \sigma_2, \omega_2)\bigr),$$

where

$$L(\lambda_1,\alpha_1,\sigma_1,\omega_1)=\sum_{m=0}^{\infty}\frac{e^{2\pi i\lambda_1 m}\omega_1(m)}{(m+\alpha_1)^{\sigma_1}}$$

and

$$L(\lambda_2,\alpha_2,\sigma_2,\omega_2)=e^{-\frac{2\pi i a}{q}}q^s\omega_2(q)\sum_{\substack{m=1\\ m\equiv a(\mathrm{mod}\, q)}}^{\infty}\frac{e^{\frac{2\pi i m}{q}}}{m^{\sigma_2}}.$$

Let $\underline{L}(\underline{\lambda},\underline{\alpha},\underline{\sigma}+it)=(L(\lambda_1,\alpha_1,\sigma_1+it),L(\lambda_2,\alpha_2,\sigma_2+it))$.

Theorem 1. *Suppose that the number α_1 is transcendental, $\alpha_2=\frac{a}{q}$, $0<a<q$, $(a,q)=1$, and $\min(\sigma_1,\sigma_2)>\frac{1}{2}$. Then the probability measure*

$$P_T(A)\stackrel{\mathrm{def}}{=}\nu_T\big(\underline{L}(\underline{\lambda},\underline{\alpha},\underline{\sigma}+it)\in A\big),\quad A\in\mathcal{B}\big(\mathbb{C}^2\big),$$

converges weakly to the distribution of the random element $\underline{L}(\underline{\lambda},\underline{\alpha},\underline{\sigma},\omega)$ as $T\to\infty$.

Let $\mathcal{P}$ denote the set of all prime numbers. Since α_1 is transcendental, the set $\{\log(m+\alpha_1)\colon m\in\mathbb{N}_0\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. The set $\{\log p\colon p\in\mathcal{P}\}$ is also linearly independent over $\mathbb{Q}$. Therefore, it is not difficult to prove that the set

$$L(\alpha_1)\stackrel{\mathrm{def}}{=}\big\{\log(m+\alpha_1)\colon m\in\mathbb{N}_0\big\}\cup\{\log p\colon p\in\mathcal{P}\}$$

is linearly independent as well. This leads to the following lemma.

Lemma 1. (*See* [8].) *Suppose that the number α_1 is transcendental. Then the probability measure*

$$Q_T(A)\stackrel{\mathrm{def}}{=}\nu_T\Big(\Big(\big((m+\alpha_1)^{-it}\colon m\in\mathbb{N}_0\big),\big(p^{-it}\colon p\in\mathcal{P}\big)\Big)\in A\Big),\quad A\in\mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H as $T\to\infty$.

Let $\sigma_1>\frac{1}{2}$ be a fixed number, and $v_n(m,\alpha_1)=\exp\{-(\frac{m+\alpha_1}{n+\alpha_1})^{\sigma_1}\}$, $v_n(m)=\exp\{-(\frac{m}{n})^{\sigma_1}\}$. Define $\underline{L}_n(\underline{\lambda},\underline{\alpha},s)=(L_n(\lambda_1,\alpha_1,s),L_n(\lambda_2,\alpha_2,s))$, where

$$L_n(\lambda_1,\alpha_1,s)=\sum_{m=0}^{\infty}\frac{e^{2\pi i\lambda_1 m}v_n(m,\alpha_1)}{(m+\alpha_1)^s}$$

and

$$L_n(\lambda_2,\alpha_2,s)=\sum_{\substack{m=1\\ m\equiv a(\mathrm{mod}\, q)}}^{\infty}\frac{e^{\frac{2\pi i m}{q}}e^{-\frac{2\pi i a}{q}}q^s v_n(m)}{m^s}.$$

By contour integration it is proved, see, for example, [2], that the series for $L_n(\lambda_1,\alpha_1,s)$ and $L_n(\lambda_2,\alpha_2,s)$ both converge absolutely for $\sigma>\frac{1}{2}$.

Let, for $\omega = (\omega_1, \omega_2) \in \Omega$,

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, s) = \big(L_n(\lambda_1, \alpha_1, \omega_1, s), L_n(\lambda_2, \alpha_2, \omega_2, s)\big),$$

where

$$L_n(\lambda_1, \alpha_1, \omega_1, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda_1 m}\omega_1(m)v_n(m, \alpha_1)}{(m+\alpha_1)^s}$$

and

$$L_n(\lambda_2, \alpha_2, \omega_2, s) = \sum_{\substack{m=1 \\ m\equiv a(\mathrm{mod}\, q)}}^{\infty} \frac{e^{\frac{2\pi i\lambda_2 m}{q}} e^{-\frac{2\pi i a}{q}} \omega_2(m)\omega_2(q)q^s v_n(m)}{m^s}.$$

Since $|\omega_1(m)| = |\omega_2(m)| = 1$, the later two series also converge absolutely for $\sigma > \frac{1}{2}$.

On $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$, define the probability measures $P_{T,n}(A) = \nu_T(\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}+it) \in A)$ and $\widehat{P}_{T,n}(A) = \nu_T(\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}+it) \in A)$.

Lemma 2. *Suppose that* $\min(\sigma_1, \sigma_2) > \frac{1}{2}$*. Then on* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$*, there exists a probability measure* P_n *such that both the measures* $P_{T,n}$ *and* $\widehat{P}_{T,n}$ *converge weakly to* P_n *as* $T \to \infty$*.*

Proof. Define the function $h_n : \Omega \to \mathbb{C}^2$ by the formula $h_n(\omega) = \underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma})$. Then the function is continuous, and

$$h_n\Big(\big((m+\alpha_1)^{-it}\colon m \in \mathbb{N}_0\big), \big(p^{-it}\colon p \in \mathcal{P}\big)\Big) = \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}+it).$$

Therefore, $P_{T,n} = Q_T h_n^{-1}$. This, the continuity of h_n, Lemma 1, and Theorem 5.1 of [1] show that $P_{T,n}$ converges weakly to $m_H h_n^{-1}$ as $T \to \infty$.

By the same arguments, using the invariance of the Haar measure m_H, we obtain that the measure $\widehat{P}_{T,n}$ also converges weakly to $m_H h_n^{-1}$ as $T \to \infty$. □

For $\underline{z}_1 = (z_{11}, z_{12})$, $\underline{z}_2 = (z_{21}, z_{22}) \in \mathbb{C}^2$, let $\rho_2(\underline{z}_1, \underline{z}_2) = (\sum_{j=1}^{2} |z_{j1} - z_{j2}|^2)^{\frac{1}{2}}$.

Lemma 3. *Suppose that* $\min(\sigma_1, \sigma_2) > \frac{1}{2}$*. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \rho_2\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}+it), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}+it)\big)\, dt = 0,$$

and, for almost all $\omega \in \Omega$*,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \rho_2\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}+it), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}+it)\big)\, dt = 0.$$

Proof. The lemma follows from corresponding one-dimensional relations, see [2], and from the definition of the metric ρ_2. □

Define one more probability measure

$$\widehat{P}_T(A) = \nu_T\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}+it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^2).$$

Lemma 4. *Suppose that* $\min(\sigma_1, \sigma_2) > \frac{1}{2}$*. Then on* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$*, there exists a probability measure* P *such that both the measures* P_T *and* $\widehat{P}_T$ *converge weakly to* P *as* $T \to \infty$*.*

Proof. We remind that P_n is the limit measure in Lemma 2. First we observe that the family of probability measures $\{P_n \colon n \in \mathbb{N}_0\}$ is tight. This is obtained by using Lemma 2 and the fact that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \bigl|L_n(\lambda_1, \alpha_1, \sigma_1 + it)\bigr|^2 \, dt = \sum_{m=0}^{\infty} \frac{v_n^2(m, \alpha_1)}{(m+\alpha_1)^{2\sigma_1}} \leqslant \sum_{m=0}^{\infty} \frac{1}{(m+\alpha_1)^{2\sigma_1}} < \infty$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \bigl|L_n(\lambda_2, \alpha_2, \sigma_2 + it)\bigr|^2 \, dt = \sum_{\substack{m=1 \\ m\equiv a (\mathrm{mod}\, q)}}^{\infty} \frac{q^{2\sigma_2} v_n^2(m)}{m^{2\sigma_2}} \leqslant \sum_{\substack{m=1 \\ m\equiv a (\mathrm{mod}\, q)}}^{\infty} \frac{q^{2\sigma_2}}{m^{2\sigma_2}} < \infty.$$

By the Prokhorov theorem, the tightness implies a relative compactness. Therefore, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ as $k \to \infty$.

Let θ be a random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ and uniformly distributed on $[0,1]$. Define $\underline{X}_{T,n}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + i\theta T)$, and denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then, by Lemma 2, we have that

$$\underline{X}_{T,n}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}), \tag{1}$$

where $\underline{X}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma})$ is the $\mathbb{C}^2$-valued random element with distribution P_n. Moreover, by the above remark, we have that

$$\underline{X}_{n_k}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{2}$$

Let $\underline{X}_T(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) = \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + i\theta T)$. Then we deduce from Lemma 3 that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\bigl(\rho_2(\underline{X}_T(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}), \underline{X}_{T,n}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma})) \geqslant \varepsilon\bigr)$$
$$\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_2\bigl(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it)\bigr) \, dt = 0.$$

This, (1), (2) and Theorem 4.2 of [1] show that $\underline{X}_T(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} P$, or, in other words, the measure P_T converges weakly to P as $T \to \infty$. The latter relation also shows that the measure P is independent on the choice of the sequence $\{P_{n_k}\}$. Thus, we have that

$$\underline{X}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{3}$$

Define $\underline{\widehat{X}}_{T,n}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma} + i\theta T)$ and $\underline{\widehat{X}}_T(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}) = \underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma} + i\theta T)$. Then, repeating the above arguments for the random elements $\underline{\widehat{X}}_{T,n}(\underline{\lambda}, \underline{\alpha},$

$\omega, \underline{\sigma})$ and $\widehat{\underline{X}}_T(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma})$ with using of Lemmas 2 and 3, and the relation (3), we obtain that the measure $\widehat{P}_T$ also converges weakly to P as $T \to \infty$. □

Proof of Theorem 1. In view of Lemma 4, it remains to show that the measure P coincides with the distribution of the random element $\underline{L}$.

Let A be a continuity set of the measure P. Then by Lemma 4 we have that

$$\lim_{T\to\infty} \nu_T\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma} + it) \in A\big) = P(A). \tag{4}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable ξ by

$$\xi(\omega) = \begin{cases} 1 & \text{if } \underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly, the expectation

$$\mathbb{E}\xi = \int_\Omega \xi(\omega)\, dm_H = m_H\big(\omega \in \Omega\colon \underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma}) \in A\big) = P_{\underline{L}}(A), \tag{5}$$

where $P_{\underline{L}}$ is the distribution of the random element $\underline{L}$.

Let, for $t \in \mathbb{R}$, $a_t = \big(((m+\alpha_1)^{-it}\colon m \in \mathbb{N}_0), (p^{-it}\colon p \in \mathcal{P})\big)$, and $\varphi_t(\omega) = \omega a_t$, $\omega \in \Omega$. Then $\{\varphi_t\colon t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω. Since the set $L(\alpha_1)$ is linearly independent over $\mathbb{Q}$, by a standard method can be proved that the group $\{\varphi_t\colon t \in \mathbb{R}\}$ is ergodic. Hence, the random process $\xi(\varphi_t(\omega))$ is ergodic as well. Therefore, the Birkhoff–Khintchine theorem shows that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \xi\big(\varphi_t(\omega)\big)\, dt = \mathbb{E}\xi. \tag{6}$$

On the other hand, by the definition of ξ and φ_t we find that

$$\frac{1}{T}\int_0^T \xi\big(\varphi_t(\omega)\big)\, dt = \nu_T\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma} + it) \in A\big).$$

This, (5) and (6) yield

$$\lim_{T\to\infty} \nu_T\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \omega, \underline{\sigma} + it \in A\big) = P_{\underline{L}}(A).$$

Therefore, by (4), $P(A) = P_{\underline{L}}(A)$ for all continuity sets A of P. Hence, $P(A) = P_{\underline{L}}(A)$ for all $A \in \mathcal{B}(\mathbb{C}^2)$. □

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REZIUMĖ

Dvimatė ribinė teorema Lercho dzeta funkcijoms. II

D.R. Genienė

Straipsnyje įrodoma dvimatė ribinė teorema Lercho dzeta funkcijoms su transcendenčiuoju ir racionaliuoju parametrais.

Raktiniai žodžiai: Lercho dzeta funkcija, tikimybinis matas, silpnasis konvergavimas.