

Universality of Dirichlet L -functions with shifted characters

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Abstract. We study Dirichlet L -functions with shifted characters $L(s, \chi, H) = \sum_{n=1}^{\infty} \frac{\chi(n+H)}{n^s}$, where H is an integer. Here we obtain a joint universality theorem for such functions.

Keywords: Dirichlet L -functions, universality theorem.

1. Introduction

For Dirichlet L -functions we have the joint universality theorem (Lemma 2, see below) which says that a finite collection of Dirichlet L -functions with nonequivalent characters can approximate a finite collection of any analytic *nonvanishing* functions. In [1] we considered Dirichlet L -functions with shifted characters

$$L(s, \chi, H) = \sum_{n=1}^{\infty} \frac{\chi(n+H)}{n^s},$$

where χ is a Dirichlet character and H is an integer. Note that this function can be expressed by Hurwitz zeta-functions, also by Dirichlet L -functions:

$$\begin{aligned} L(s, \chi, H) &= \frac{1}{q^s} \sum_{k=1}^{q-1} \chi(k+H) \zeta\left(s, \frac{k}{q}\right) + \frac{1}{q^s} \chi(H) \zeta(s) \\ &= \frac{1}{q-1} \sum_{n=1}^{q-1} \eta_n(\chi, H) L(s, \chi_n) + \frac{1}{q^s} \chi(H) \zeta(s), \end{aligned} \quad (1)$$

where $\chi_n, n = 1, \dots, q-1$, are all Dirichlet characters mod q and

$$\eta_n(\chi, H) = \sum_{k=0}^{q-1} \chi(k+H) \bar{\chi}_n(k) \quad (n = 1, \dots, q-1). \quad (2)$$

From this we see that $L(s, \chi, H)$ can be continued meromorphically to the whole complex plane with at most one simple pole at $s = 1$. For this function, in general, the Riemann hypothesis is not true. Let $q > 3$ be prime and q not divide H . Then in [1] we with Steuding showed that for any σ_1 and σ_2 with $1/2 < \sigma_1 < \sigma_2 < 1$, there

exists a constant $c = c(H, \sigma_1, \sigma_2) > 0$ such that for sufficiently large T the function $L(s, \chi, H)$ has more than cT zeros in the rectangle $\sigma_1 < \sigma < \sigma_2, |t| < T$.

Here we prove a kind of joint universality theorem for $L(s, \chi, H)$.

THEOREM 1. *Let $0 < r < \frac{1}{4}$ and $H_j \in \mathbb{Z}, j = 1, \dots, m$. Let $3 < q_1 < \dots < q_m$ be primes and $\chi_j \pmod{q_j}, j = 1, \dots, m$ be Dirichlet characters. Let $f_1(s), \dots, f_m(s)$ be any continuous functions on the disc $|s| \leq r$, analytic in the interior of this disc, and if q_j divides H_j , let $f_j(s)$ also be nonvanishing. Then, for every $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{2T} \text{meas} \left\{ \tau \in [-T, T], \max_{1 \leq j \leq m} \max_{|s| \leq r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_j, H_j\right) - f_j(s) \right| < \epsilon \right\} > 0.$$

Here and further $\text{meas}\{A\}$ denotes the Lebesgue measure of the set A . We say that two characters are equivalent if they are generated by the same primitive character. Note, that characters χ_1, \dots, χ_n from Theorem 1 are pairwise not equivalent. The proof of Theorem 1 makes use of the following Voronin theorem on the joint universality of the Dirichlet L -functions:

LEMMA 2 (Voronin [2]). *Let $0 < r < \frac{1}{4}$. Let χ_1, \dots, χ_m be pairwise non equivalent Dirichlet characters. Let $f_1(s), \dots, f_m(s)$ be any nonvanishing continuous functions on the disc $|s| \leq r$, and analytic in the interior of this disc. Then, for every $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{2T} \text{meas} \left\{ \tau \in [-T, T], \max_{1 \leq j \leq m} \max_{|s| \leq r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_j\right) - f_j(s) \right| < \epsilon \right\} > 0.$$

Also we need some information on coefficients $\eta_n(H)$. In [1] the following property was proved.

LEMMA 3. *Let $q > 3$ be a prime and q not divide H . Then there exist at least two different nonprincipal characters, say χ_i and χ_j , such that the numbers $\eta_i(H)$ and $\eta_j(H)$ are nonzero.*

Proof of Theorem 1. If χ_1 denotes the principal character $\pmod{q_j}$, then we have $\zeta(s) = \left(1 - \frac{1}{q_j^s}\right)^{-1} L(s, \chi_1)$. Thus, in view of (1), for j such that q_j not divides H_j , we have

$$\begin{aligned} L(s, \chi_j, H_j) &= \frac{1}{q_j - 1} \sum_{n=2}^{q_j-1} \eta_n(\chi_j, H_j) L(s, \chi_n) \\ &\quad + \left(\frac{\eta_1(\chi_j, H_j)}{q_j - 1} + \frac{\chi_j(H_j)}{q_j^s} \left(1 - \frac{1}{q_j^s}\right)^{-1} \right) L(s, \chi_1) \\ &= \frac{1}{q_j - 1} \sum_{n=2}^{q_j-1} \eta_n(\chi_j, H_j) L(s, \chi_n) \end{aligned}$$

$$+ \frac{1}{q_j - 1} \left(\eta_1(H_j) + \frac{q_j - 1}{q_j^s - 1} \chi_j(H_j) \right) L(s, \chi_1). \quad (3)$$

By Lemma 3 we can suppose that $\eta_2(\chi_j, H_j)$ and $\eta_3(\chi_j, H_j)$ are nonzero. Let

$$M := 1 + \max_{\substack{|s| \leq r \\ j=1, \dots, m}} |f_j(s)|.$$

Then $f_j(s) + M \neq 0$. For such j we define

$$f_{2,j}(s) := \frac{f_j(s) + M}{\eta_2(\chi_j, H_j)},$$

$$f_{3,j}(s) := -\frac{M}{\eta_3(\chi_j, H_j)},$$

$$f_{1,j}(s) := f_{4,j}(s) = \dots = f_{q-1,j}(s) = \delta > 0$$

and for j such that q_j divides H_j (in this case $f_j \neq 0$),

$$f_{n,j}(s) := f_j(s),$$

where the constant $\delta > 0$ will be chosen later. Note that for j such that q_j divides H_j , we have $L(s, \chi_j, H_j) = L(s, \chi_j)$. By Lemma 2,

$$\liminf_{T \rightarrow \infty} \frac{1}{2T} \text{meas} \left\{ \tau \in [-T, T], \max_{1 \leq j \leq m} \max_{1 \leq n \leq q_j - 1} \max_{|s| \leq r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_n \pmod{q_j}\right) - f_{n,j}(s) \right| < \delta \right\} > 0.$$

For $|s - 3/4| \leq r < 1/4$ and $\tau \in \mathbb{R}$ the quantity

$$\frac{q_j - 1}{q_j^{s+i\tau} - 1}$$

is bounded. By this, choosing sufficiently small δ , we obtain the theorem.

References

1. R. Garunkštis, J. Steuding, Twists of Lerch zeta-functions, *Lith. Math. J.*, **41**(2), 135–142 (2001).
2. S.M. Voronin, *Analytical Properties of the Dirichlet Generating Series of Arithmetical Objects*, Doctoral thesis, Moscow (1977) (in Russian).

REZIUMĖ

R. Garunkštis. Dirichlet L -funkcijų su paslinktais charakteriais universalumas

Mes studijuojame Dirichlet L -funkcijas su paslinktais charakteriais $L(s, \chi, H) = \sum_{n=1}^{\infty} \frac{\chi(n+H)}{n^s}$, čia H yra sveikasis skaičius. Šiame darbe įrodome daugiamatę universalumo teoremą šioms funkcijoms.