

## On the mean value of coefficients of certain cusp forms

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Let  $F(z)$  be a normalized eigenform of weight  $\kappa$  with the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}, \quad c(1) = 1.$$

The zeta-function  $\varphi(s, F)$ ,  $s = \sigma + it$ , attached to  $F(z)$ , for  $\sigma > \frac{\kappa+1}{2}$ , is defined by

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and it is analytically continuable to an entire function.

In [1] the mean value

$$\sum_{m \leq x} h(m)$$

was considered. Here  $h(m) = g_w^2(m)m^{1-\kappa}$ , where  $g_w(m)$  is defined by

$$\varphi^w(s, w) = \sum_{m=1}^{\infty} \frac{g_w(m)}{m^s}, \quad \sigma > \frac{\kappa+1}{2}.$$

Note that  $g_w(m)$  is a multiplicative function,

$$g_w(p^k) = \sum_{l=0}^k d_w(p^l)\alpha^l(p)d_w(p^{k-l})\beta^{k-l}(p), \quad (1)$$

where

$$d_w(p^k) = \frac{w(w+1)\dots(w+k-1)}{k!}, \quad k = 1, 2, \dots,$$

and

$$c(p) = \alpha(p) + \beta(p).$$

The aim of this note is to give an asymptotic formula for the mean value of  $h(m)$  with explicitly given remainder term.

Let

$$m(h, x) = \prod_{p \leq x} \left( 1 + \sum_{\alpha=1}^{\infty} \frac{h(p^\alpha)}{p^\alpha} \right),$$

$\Gamma(s)$  denote the Euler gamma-function, and let  $\gamma$  be the Euler constant.

Let  $c_p = c(p)p^{\frac{1-\kappa}{2}}$ . Then by (1)  $g_w(p) = w(\alpha(p) + \beta(p)) = wc(p)$ . Hence  $h(p) = w^2 c_p^2$ . In [2] it was proved that, for  $x \rightarrow \infty$ ,

$$\sum_{p \leq x} c_p^2 \log p = x(1 + o(1)).$$

Consequently, we have that there exists a function  $r_1(x)$ ,  $r_1(x) \rightarrow 0$  as  $x \rightarrow \infty$ , such that

$$\sum_{p \leq x} h(p) \log p = w^2 x + w^2 x r_1(x). \tag{2}$$

Let  $r_2(x) = \sup_{z \geq x} |r_1(z)|$ . Then in view of (2)

$$\sup_{z \geq x} \left| \frac{1}{z} \sum_{p \leq z} h(p) \log p - w^2 \right| = w^2 r_2(x).$$

Define

$$r_3(x) = \max(w^2 r_2(x), (\log x)^{-1}),$$

and put

$$r(x) = \max(w^{-2} r_3(x), (\log x)^{-1}).$$

Then, clearly,  $r(x) \rightarrow 0$  as  $x \rightarrow \infty$ , and  $r(x) \geq r_2(x)$ .

Suppose that  $y = x^{\frac{1}{\sqrt{\log \log x}}}$ . Let

$$S_1(x, y) = \sum_{m_1 \leq x} h(m_1),$$

and  $\epsilon(x) = r(y)$ .

**THEOREM.** Let  $\frac{c}{\sqrt{\log \log x}} \leq |w| \leq \frac{1}{2}$  and  $\text{Re } w^2 > 0$ . Then uniformly in  $w$  and  $x$

$$\begin{aligned} \sum_{m \leq x} h(m) &= \frac{e^{-\gamma w^2 x}}{\Gamma(w^2) \log x} m(h, x) \\ &\times (\sqrt{\log \log x})^{O(|w|^2 \epsilon(x))} \left( 1 + O\left( |w|^2 \exp\left\{ -\frac{\log x}{\sqrt{\log \log x}} \right\} \right) \right) \end{aligned}$$

$$\begin{aligned}
& + O\left(\frac{|w|^2 x m(|h|, x)}{\log x} (\sqrt{\log \log x})^{-1+O(|w|^2 \varepsilon(x))}\right) \\
& + O\left(\frac{|w|^2 \varepsilon(x) x}{\log x} m(|h|, x) (\sqrt{\log \log x})^{|w|^2}\right).
\end{aligned}$$

The theorem is based on the following assertion.

LEMMA 1. Let  $\frac{c}{\sqrt{\log \log x}} \leq |w| \leq \frac{1}{2}$  and  $\operatorname{Re} w^2 > 0$ . Then uniformly in  $w$  and  $x$

$$\begin{aligned}
S_1(x, y) &= 1 + \frac{x \rho'_w(\sqrt{\log \log x})}{\log y} - \frac{w^2 y}{\log y} \\
&+ O\left(|w|^2 x \frac{r(y)}{\log y} (\sqrt{\log \log x})^{|w|^2-1} \frac{m(|h|, x)}{m(|h|, y)}\right. \\
&\quad \left.+ \frac{x r(y) \log y}{y} (\sqrt{\log \log x})^{|w|^2-1}\right).
\end{aligned}$$

Here  $\rho_w(u)$ , for  $u \geq 0$ , is a continuous solution of the difference – differential equation

$$u \rho'_w(u) = w^2 \rho_w(u - 1),$$

and  $\rho_w(u) = 0$  for  $u < 0$ .

*Proof.* Let

$$Z(x, y) = \sum_{m_1 \leq x} h(m_1) \log m_1.$$

Then, obviously,

$$Z(x, y) = S_1(x, y) \log x - \int_1^x S_1(u, y) \frac{du}{u}. \quad (3)$$

Moreover, by (2) and Lemma 5 of [1]

$$\begin{aligned}
Z(x, y) &= \sum_{\substack{p^\alpha \leq x \\ p > y}} h(p^\alpha) \log p^\alpha \sum_{\substack{m_1 \leq \frac{x}{p^\alpha} \\ (m_1, p) = 1}} h(m_1) \\
&= \sum_{m_1 \leq \frac{x}{y}} h(m_1) \sum_{\substack{p^\alpha \leq \frac{x}{m_1} \\ p > y}} h(p^\alpha) \log p^\alpha \\
&= \sum_{m_1 \leq \frac{x}{y^2}} h(m_1) \sum_{\substack{p^{\alpha+\beta} \leq \frac{x}{m_1} \\ p > y, p \nmid m_1}} h(p^\alpha) h(p^\beta) \log p^\alpha
\end{aligned}$$

$$\begin{aligned}
 &= \sum_{m_1 \leq \frac{x}{y^2}} h(m_1) \left( w^2 \left( \frac{x}{m_1} - 1 \right) + O \left( |w|^2 \left( r(y) \frac{x}{m_1} + \sqrt{\frac{x}{m_1}} \right) \right) \right) \\
 &\quad + O \left( |w|^4 \frac{x}{y} \sum_{m_1 \leq \frac{x}{y^2}} \frac{|h(m_1)|}{m_1} \right). \tag{4}
 \end{aligned}$$

This and (3) show that, for  $x = y^t$ ,

$$\begin{aligned}
 t S_1(y^t, y) - \int_0^t S_1(y^u, y) du &= O \left( |w|^2 r(y) \frac{y^t}{\log y} \sum_{m_1 \leq y^t} \frac{|h(m_1)|}{m_1} \right) \\
 &= O \left( |w|^2 r(y) \frac{y^t}{\log y} \frac{m(|h|, y^t)}{m(|h|, y)} \right).
 \end{aligned}$$

Therefore

$$\frac{1}{t} (S_1(y^t, y) - 1) - \frac{1}{t^2} \int_1^t (S_1(y^u, y) - 1) du = O \left( |w|^2 r(y) \frac{y^t}{t^2 \log y} \frac{m(|h|, y^t)}{m(|h|, y)} \right),$$

and

$$\frac{1}{t} \int_1^t (S_1(y^u, y) - 1) du = O \left( |w|^2 r(y) \frac{y^t}{t^2 \log^2 y} \frac{m(|h|, y^t)}{m(|h|, y)} \right). \tag{5}$$

Since

$$t = \frac{\log x}{\log y} = \frac{w^2 r_3(y) y^t \log y^t}{w^2 r_3(y) y^t \log y} = O \left( \frac{|w|^2 r(y) y^t}{\log y} \right),$$

(3)–(5) imply

$$t S_1(y^t, y) = \frac{w^2 y^t}{\log y} \sum_{m_1 \leq y^{t-1}} \frac{h(m_1)}{m_1} - \frac{w^2 y}{\log y} S_1(y^{t-1}, y) + O \left( |w|^2 \frac{r(y) y^t m(|h|, y^t)}{m(|h|, y)} \right),$$

and

$$t y^{-t} S_1(y^t, y) - w^2 \int_0^{t-1} y^{-u} S_1(y^u, y) du = O \left( |w|^2 \frac{r(y) m(|h|, y^t)}{\log y m(|h|, y)} \right). \tag{6}$$

Now let

$$y^{-t} S_1(y^t, y) = y^{-t} + \frac{\rho'_w(t)}{\log y} - w^2 \delta(t) \frac{y^{1-t}}{\log y} F(t, y) \frac{r(y)}{\log y}, \tag{7}$$

where

$$\delta(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

Using (6), similarly as in [1], we find that

$$F(t, y) = O\left(|w|^2 t^{|w|^2-1} \frac{m(|h|, y^t)}{m(|h|, y)} + t^{|w|^2-1} \frac{\log^2 y}{y}\right).$$

This and (7), since  $t = \frac{\log x}{\log y} = \sqrt{\log \log x}$ , yield the lemma.

*Proof of the Theorem.* Summing by parts, from (2) we deduce

$$\sum_{y < p \leq x} \frac{h(p)}{p} = w^2 \log \sqrt{\log \log x} + O(|w|^2 r(y) \log \sqrt{\log \log x}).$$

Therefore

$$\begin{aligned} \frac{m(h, x)}{m(h, y)} &= \prod_{y < p \leq x} \left(1 + \sum_{\alpha=1}^{\infty} \frac{h(p^\alpha)}{p^\alpha}\right) = \exp\left\{\sum_{y < p \leq x} \left(\frac{h(p)}{p} + O\left(\frac{|w|^2}{p^2}\right)\right)\right\} \\ &= \exp\left\{\sum_{y < p \leq x} \frac{h(p)}{p}\right\} \left(1 + O\left(\frac{|w|^2}{y}\right)\right) \\ &= (\sqrt{\log \log x})^{w^2 + O(|w|^2 r(y))} \left(1 + O\left(\frac{|w|^2}{y}\right)\right), \end{aligned} \quad (8)$$

and

$$\frac{m(|h|, x)}{m(|h|, y)} \geq \left|\frac{m(h, x)}{h, y}\right| = (\sqrt{\log \log x})^{\operatorname{Re} w^2 + O(|w|^2 r(y))} \left(1 + O\left(\frac{|w|^2}{y}\right)\right). \quad (9)$$

Let positive integers  $m_2$  be free of prime divisors greater than  $y$ , and

$$S_2(x, y) = \sum_{m_2 > x} \frac{h(m_2)}{m_2}, \quad S_3(x, y) = \sum_{m_2 \leq x} h(m_2).$$

Then in [1] it was proved that

$$S_2(x, y) = O\left(m(|h|, y) \exp\{-t \log t + O(|w|^2 t)\}\right). \quad (10)$$

$$S_3(x, y) = O\left(\frac{|w|^2 m(|h|, y)}{\log x} \exp\{-t \log t + O(|w|^2 t)\}\right). \quad (11)$$

We have

$$\sum_{m \leq x} h(m) = \sum_{m_2 \leq \frac{x}{y}} h(m_2) \left(S_1\left(\frac{x}{m_2}, y\right) - 1\right) + \sum_{m_2 \leq x} h(m_2). \quad (12)$$

Using Lemma 1, we find that the first sum in the right-hand side of (12) is

$$\begin{aligned} & \frac{x}{\log x} \sum_{m_2 \leq \frac{x}{y}} \frac{h(m_2)}{m_2} \rho'_w \left( \frac{\log \frac{x}{m_2}}{\log y} \right) - w^2 \frac{y}{\log y} \sum_{m_2 \leq \frac{x}{y}} h(m_2) \\ & + O \left( \left( |w|^2 x \frac{r(y)}{\log y} t^{|w|^2-1} \frac{m(|h|, x)}{m(|h|, y)} + \frac{x r(y) \log y}{y} t^{|w|^2-1} \right) \sum_{m_2 \leq \frac{x}{y}} \frac{|h(m_2)|}{m_2} \right). \end{aligned}$$

Thus, the equality (12) can be written in the form

$$\begin{aligned} \sum_{m \leq x} h(m) &= \frac{x m(h, y)}{\log y} \rho'_w(t) - \frac{w^2 x}{\log y} \sum_{m_2 > \frac{x}{y}} \frac{h(m_2)}{m_2} \\ &- \frac{x}{\log y} \int_1^t \left( \sum_{m_2 > \frac{x}{yu}} \frac{h(m_2)}{m_2} \right) \rho''_w(u) du - \frac{w^2 y}{\log y} \sum_{m_2 < \frac{x}{y}} h(m_2) + \sum_{m_2 \leq x} h(m_2) \\ &+ O \left( \frac{|w|^2 x r(y)}{\log y} t^{|w|^2-1} m(|h|, x) + \frac{x r(y) \log y}{y} t^{|w|^2-1} m(|h|, x) \right). \end{aligned} \quad (13)$$

By (10), (11) and (8), (9)

$$\begin{aligned} \frac{w^2 x}{\log y} \sum_{m_2 > \frac{x}{y}} \frac{h(m_2)}{m_2} &= O \left( \frac{|w|^2 x m(|h|, x)}{\log x} \exp\{-ct\} \right), \\ \sum_{m_2 \leq x} h(m_2) &= O \left( \frac{|w|^2 x m(|h|, x)}{\log x} \exp\{-c_1 t\} \right), \\ \frac{w^2 y}{\log y} \sum_{m_2 \leq \frac{x}{y}} h(m_2) &= O \left( \frac{|w|^2 x m(|h|, x)}{\log x} \exp\{-c_2 t\} \right). \end{aligned}$$

The term with integral is

$$O \left( \frac{|w|^2 x m(|h|, x)}{\log x} t^{-1} \right),$$

while in view of Lemma 6 from [1] and (8)

$$\begin{aligned} \frac{x m(h, y)}{\log y} \rho'_w(t) &= \frac{e^{-\gamma w^2} x m(h, x)}{\Gamma(w^2) \log x} (\sqrt{\log \log x})^{O(|w|^2 r(y))} \left( 1 + O \left( \frac{|w|^2}{y} \right) \right) \\ &+ O \left( \frac{|w|^2 x m(|h|, x)}{\log x} \right) (\sqrt{\log \log x})^{-1+O(|w|^2 r(y))}. \end{aligned}$$

All these estimates together with (13) and

$$\frac{x}{\log y} \int_1^t \left( \sum_{m_2 > \frac{x}{y^u}} \right) \rho_w''(u) du = O\left(\frac{|w|^2 x m(|h|, x)}{\log x}\right) (\sqrt{\log \log x})^{-1}$$

prove the theorem.

### References

1. A. Laurinčikas, J. Steuding, On zeta-functions associated to certain cusp form, I, *CEJM*, **1**, 1–18 (2004).
2. R.A. Rankin, An  $\Omega$ -result for the coefficients of cusp forms, *Math. Ann.*, **239**, 239–250 (1973).

### REZIUMĖ

*R. Ivanauskaitė. Apie kai kurių parabolinių formų dzeta funkcijų koeficientų vidurkius*

Gauta parabolinių formų dzeta funkcijų koeficientų vidurkio asimptotika.