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On zeta-functions of cusp forms

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This note is a continuation of [2]. Our aim is to prove a discrete limit theorem on the complex plane $\mathbb{C}$ for zeta-functions of certain cusp forms with step of the progression h satisfying conditions discussed in [2].

More precisely, let $F(z)$ be a holomorphic cusp form of weight κ which is a normalized eigenform. Suppose that the function $F(z)$ has the following Fourier series expansion

$$F(z)=\sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi imz}, \quad c(1)=1.$$

We consider the zeta-function $\varphi(s,F)$, $s=\sigma+it$, attached to $F(z)$:

$$\varphi(s,F)=\sum_{m=1}^{\infty}\frac{c(m)}{m^s}, \quad \sigma>\frac{\kappa+1}{2}.$$

It is well known that the function $\varphi(s,F)$ is an entire function.

Suppose that $h>0$ is such that $\exp\{\frac{2\pi k}{h}\}$ is rational for some integer $k\neq 0$. Denote by k_0 the smallest of such k, and let $\exp\{\frac{2\pi k_0}{h}\}=\frac{m_0}{n_0}$, $m_0,n_0\in\mathbb{N}$, $(m_0,n_0)=1$.

Let, as usual,

$$\Omega=\prod_p \gamma_p,$$

where γ_p is the unit circle γ on the complex plane for each prime p. Ω is a compact topological group. Denote $\Omega_h=\{\omega\in\Omega\colon\ \omega(m_0)=\omega(n_0)\}$. Then Ω_h is a closed subgroup of Ω, and therefore is a compact topological group. Denote by $\omega(p)$ the projection of $\omega\in\Omega$ to the coordinate space γ_p, and let, for $m\in\mathbb{N}$,

$$\omega(m)=\prod_{p^\alpha\|m}\omega^\alpha(p).$$

Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S, and denote by m_{hH} the probability Haar measure on $(\Omega_h,\mathcal{B}(\Omega_h))$. Then in [2] it was proved that, for $\sigma>\kappa/2$,

$$\varphi(\sigma,\omega_h,F)=\sum_{m=1}^{\infty}\frac{c(m)\omega_h(m)}{m^\sigma}$$

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is a complex-valued random variable defined on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$.

THEOREM 1. *Suppose that the number h satisfies the above conditions, and $\sigma > \frac{\kappa}{2}$. Then the probability measure*

$$P_N(A) = \frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon \varphi(\sigma + imh, F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the random variable $\varphi(\sigma, \omega_h, F)$ as $N \to \infty$.

First the condition on the number h is used in a limit theorem for Dirichlet polynomial. Let

$$p_n(t) = \sum_{m=1}^{\infty} \frac{c(m)}{m^{\sigma+it}}, \quad n > \max(m_0, n_0).$$

Denote by $p_1, \dots, p_r$ distinct prime numbers which divide $n!$. Let

$$\Omega_r = \prod_{j=1}^{r} \gamma_{p_j},$$

where $\gamma_{p_j} = \gamma$ for $j = 1, \dots, r$, and

$$\Omega_{hr} = \big\{\omega \in \Omega_r\colon \omega(m_0) = \omega(n_0)\big\}.$$

Consider on $(\Omega_r, \mathcal{B}(\Omega_r))$ a probability measure

$$Q_{hN}(A) = \frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon (p_1^{imh}, \dots, p_r^{imh}) \in A\big).$$

LEMMA 1. *The probability measure Q_{hN} converges to the Haar measure m_{hr} on $(\Omega_{hr}, \mathcal{B}(\Omega_{hr}))$.*

Proof. Consider the Fourier transform

$$g_N(k_1, \dots, k_r) = \int_{\Omega_r} x_1^{k_1} \dots x_r^{k_r} \, \mathrm{d}Q_{hN},$$

$(k_1, \dots, k_r) \in \mathbb{Z}^r$, $(x_1, \dots, x_r) \in \Omega_r$, of the measure Q_{hN}. Without loss of generality we can suppose that the prime numbers $p_1, \dots, p_l$ occur in the factorization of m_0 and n_0, and let α_i be the exponent of p_i in $\frac{m_0}{n_0}$, $i = 1, \dots, l$. Then we have that

$$g_N(k_1, \dots, k_r) = \frac{1}{N+1}\sum_{m=0}^{\infty}\prod_{j=1}^{r} p_j^{imhk_j} = \frac{1}{N+1}\sum_{m=0}^{N}\exp\Big\{imh\sum_{j=1}^{r} k_j \log p_j\Big\}$$

$$= \begin{cases} 1, & \text{if } k_1 = k\alpha_1, \dots, k_l = k\alpha_l, k_{l+1} = \dots = k_r = 0, \\ \frac{1}{N+1}\frac{1-\exp\{i(N+1)h\sum_{j=1}^{r} k_j \log p_j\}}{1-\exp\{ih\sum_{j=1}^{r} k_j \log p_j\}} & \text{otherwise.} \end{cases} \qquad (1)$$

Since the logarithms of prime numbers are linearly independent over the field of rational numbers,

$$\sum_{j=1}^{r} k_j \log p_j \neq 0 \quad \text{for } (k_1, \dots, k_r) \neq (0, \dots, 0).$$

By the definition of k_0

$$\exp\left\{\frac{2\pi k_0}{h}\right\} = \frac{m_0}{n_0} = p_1^{\alpha_1} \dots p_l^{\alpha_l},$$

therefore the rational numbers

$$p_1^{k_1} \dots p_r^{k_r} \quad \text{and} \quad \exp\left\{\frac{2\pi k}{h}\right\}$$

coincide if and only if $(k_1, \dots, k_r) = (m\alpha_1, \dots, m\alpha_l, 0, \dots, 0)$. Hence

$$\sum_{j=1}^{r} k_j \log p_j \neq \frac{2\pi k}{h} \quad \text{for any } k \in \mathbb{Z},$$

for $(k_1, \dots, k_l, k_{l+1}, \dots, k_r) \neq (m\alpha_1, \dots, m\alpha_l, 0, \dots, 0)$. Consequently, (1) shows that

$$\lim_{N\to\infty} g_N(k_1, \dots, k_r) = \begin{cases} 1, & \text{if } (k_1, \dots, k_l, k_{l+1}, \dots, k_r) = (k\alpha_1, \dots, k\alpha_l, 0, \dots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Hence the lemma follows.

Let $g \in \Omega_h$ and

$$p_n(t, g) = \sum_{m=1}^{n} \frac{c(m)g(m)}{m^{\sigma+it}}.$$

LEMMA 2. *The probability measures*

$$\frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ p_n(mh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ p_n(mh, g) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

both converge weakly to the same measure as $N \to \infty$.

Proof. We use Lemma 1 and the same method as in [1].

Now we define, for $\sigma_1 > \frac{1}{2}$,

$$\varphi_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

and

$$\varphi_n(s, \omega_h, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_h(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

It is not difficult to see that the later two series converge absolutely for $\sigma > \kappa/2$.

LEMMA 3. *Suppose that $\sigma > \frac{\kappa}{2}$. Then there exists a probability measure P_n on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that both the measures*

$$\frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh;\ F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh, \omega_h;\ F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converge weakly to P_n as $N \to \infty$.

Proof. The proof of the lemma uses Lemma 3 and is similar to that of Lemma 3.1 of [1].

Now we give a result of the ergodic theory. Let $a_h = \{p^{-ih}\colon p \text{ is prime}\}$. We define a transformation f_h on Ω_h by $f_h(\omega_h) = a_h\omega_h$, $\omega_h \in \Omega_h$. Then f_h is a measurable measure preserving transformation on $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$.

LEMMA 4. *The transformation f_h is ergodic.*

Proof. The proof with minor changes coincides with that of Lemma 5.1 of [1].
Using Lemma 4 and the classical Birkhoff theorem, we obtain that, for $\sigma > \frac{\kappa}{2}$,

$$\sum_{m=0}^{N} \big|\varphi(\sigma + imh, \omega_h;\ F\big|^2 = \mathrm{O}(N), \quad N \to \infty,$$

for almost all $\omega_h \in \Omega_h$. The estimate

$$\sum_{m=0}^{N} \big|\varphi(\sigma + imh;\ F)\big|^2 = \mathrm{O}(N), \quad N \to \infty,$$

for $\sigma > \frac{\kappa}{2}$ follows from the estimate [3]

$$\int_0^T \big|\varphi(\sigma + it;\ F)\big|^2 = \mathrm{O}(T), \quad T \to \infty.$$

The last two mean-value estimates allow us to obtain the following statement.

LEMMA 5. *Let $\sigma > \frac{\kappa}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \big|\varphi(\sigma + imh;\ F) - \varphi_n(\sigma + imh;\ F)\big| = 0$$

and

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left|\varphi(\sigma + imh, \omega_h; F) - \varphi_n(\sigma + imh, \omega_h; F)\right| = 0$$

for almost all $\omega_h \in \Omega_h$.

Proof of Theorem 1. First, using Lemmas 3 and 5, by traditional way we show that there exists a probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the probability measures P_N and

$$\frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh, \omega_h; F) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

both converge weakly to P as $N \to \infty$. It remains to check that P is the distribution of the random variable $\varphi(\sigma, \omega_h; F)$.

Let $A \in \mathcal{B}(\mathbb{C})$ be a continuity set of the measure P. Then by the above remark, for $\omega_h \in \Omega_h$,

$$\lim_{N\to\infty} \frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh, \omega_h; F) \in A\big) = P(A). \tag{2}$$

Now we fix the set A and define the random variable η on $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$ by the formula

$$\eta(\omega_h) = \begin{cases} 1, & \text{if } \varphi(\sigma, \omega_h; F) \in A, \\ 0, & \text{if } \varphi(\sigma, \omega_h; F) \notin A. \end{cases}$$

Then

$$E\eta = \int_{\Omega_h} \eta \, \mathrm{d}m_{hH} = m_{hH}\big(\omega_h \in \Omega_h\colon\ \varphi(\sigma, \omega_h; F)\big) \tag{3}$$

is the distribution of the random variable $\varphi(\sigma, \omega_h; F)$. However, by Lemma 4 and the Birkhoff theorem

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \eta\big(f_h^m(\omega_h)\big) = E\eta \tag{4}$$

for almost all $\omega_h \in \Omega_h$. On the other hand, the definition of η and f_h shows that

$$\frac{1}{N+1} \sum_{m=0}^{N} \eta\big(f_h^m(\omega_h)\big) = \frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh, \omega_h; F) \in A\big).$$

This together with (3) and (4) yields

$$\lim_{N\to\infty} \frac{1}{N+1}\#\big(0 \leqslant m \leqslant N\colon\ \varphi(\sigma + imh, \omega_h; F) \in A\big) = P_\varphi(A)$$

for almost all $\omega_h \in \Omega_h$, where P_φ is the distribution of the random element $\varphi(\sigma, \omega_h; F)$. Hence and from (2) we have that

$$P(A) = P_\varphi(A)$$

for all continuity sets A of the measure P. From this, clearly, the theorem follows.

References

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2. A. Laurinčikas, The definition of one complex-valued random variable (this volume).
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REZIUMĖ

A. Laurinčikas. Apie parabolinių formų dzeta funkcijas

Įrodoma diskreti ribinė teorema kompleksinėje plokštumoje parabolinių formų dzeta funkcijoms, kai progresijos žingsnis h turi savybę: $\exp\{\frac{2\pi k}{h}\}$ yra racionalus kuriems nors $k \neq 0$.