Liet. matem. rink., **44**, *spec. nr.*, 2004, 830–833

Stochastic differential equations with bad coefficients: a short note on the weak approximations

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Motivation

Let us consider a one-dimensional stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \quad t \geqslant 0, \tag{1}$$

where $W = \{W_t,\ t \geqslant 0\}$ is a one-dimensional Brownian motion.

It is well known that if the coefficients b and σ are smooth enough, its Euler approximation X^h $(h > 0)$

$$\begin{aligned} X_0^h &= x_0, \\ X_t^h &= X_{kh}^h + b(X_{kh}^h)(t - kh) + \sigma(X_{kh}^h)(W_t - W_{kh}), \quad t \in [kh, (k+1)h], \end{aligned} \tag{2}$$

is a first order approximation of the (1) in the weak sense for some (sufficiently smooth) class of functions $f\colon \mathbf{R} \to \mathbf{R}$, i.e.,

$$Ef(X_T) - Ef(X_T^h) = \mathrm{O}(h), \quad h \to 0.$$

The problem is that it is not known what happens to the rate of convergence of the Euler scheme when coefficients of the (1) are not smooth enough ("bad"). In [1] it is proved that in the case where the diffusion coefficient σ is constant and b satisfies the Lipshitz condition, i.e., there exists $L > 0$ such that

$$\bigl|b(x) - b(y)\bigr| \leqslant L|x - y|, \quad x, y \in \mathbf{R},$$

for every $f \in C_p^3$, the order of the Euler approximation X^h in the weak sense is still one. This result dictated the idea of modifying an original stochastic differential equation (1) in such a way that we would be able to use the stated result and claim that Euler scheme for this new stochastic differential equation works with the rate of convergence equal to one.

Idea

Let us consider the following modified stochastic differential equation (instead of (1)):

$$Y_t = g(X_t). \tag{3}$$

Then, using Ito formulae

$$\begin{aligned} \mathrm{d}Y_t &= g'(X_t)\,\mathrm{d}X_t + \frac{1}{2}g''(X_t)\,\mathrm{d}\langle X\rangle_t \\ &= g'(X_t)\big(b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\big) + \frac{1}{2}g''(X_t)\sigma^2(X_t)\,\mathrm{d}t \\ &= Ag(X_t)\,\mathrm{d}t + \sigma(X_t)g'(X_t)\,\mathrm{d}W_t \\ &= Ag\big(g^{-1}(Y_t)\big)\,\mathrm{d}t + \sigma g'(X_t)\,\mathrm{d}W_t = \bar{b}(Y_t)\,\mathrm{d}t + \bar{\sigma}(Y_t)\,\mathrm{d}W_t, \end{aligned} \tag{4}$$

where $\bar{b}(t) = Ag(g^{-1}(t))$ and $\bar{\sigma}(t) = \sigma g'(g^{-1}(Y_t))$. If $\bar{\sigma}$ was constant (say equal to one) and $\bar{b}$ satisfied the Lipschitz condition, we would be able to apply the result from [1] to the Euler approximation for the equation (3) Y^h, $h > 0$. For $\bar{\sigma}$ to be equal to one:

$$g'(t) = \frac{1}{\sigma(t)}.$$

Therefore

$$g(t) = \int_0^t \frac{\mathrm{d}x}{\sigma(x)}.$$

Further calculations, including the assumption that $\bar{b}$ must satisfy the Lipschitz condition, gives us the following requirements for the coefficient σ from (1):

$$0 < \delta \leqslant \sigma(t), \quad \sigma'(t) \text{ must satisfy Lipschitz condition.}$$

The drawback of the idea is that the model we get though has the desired rate of convergence equal to one, but is not exactly an Euler schema.

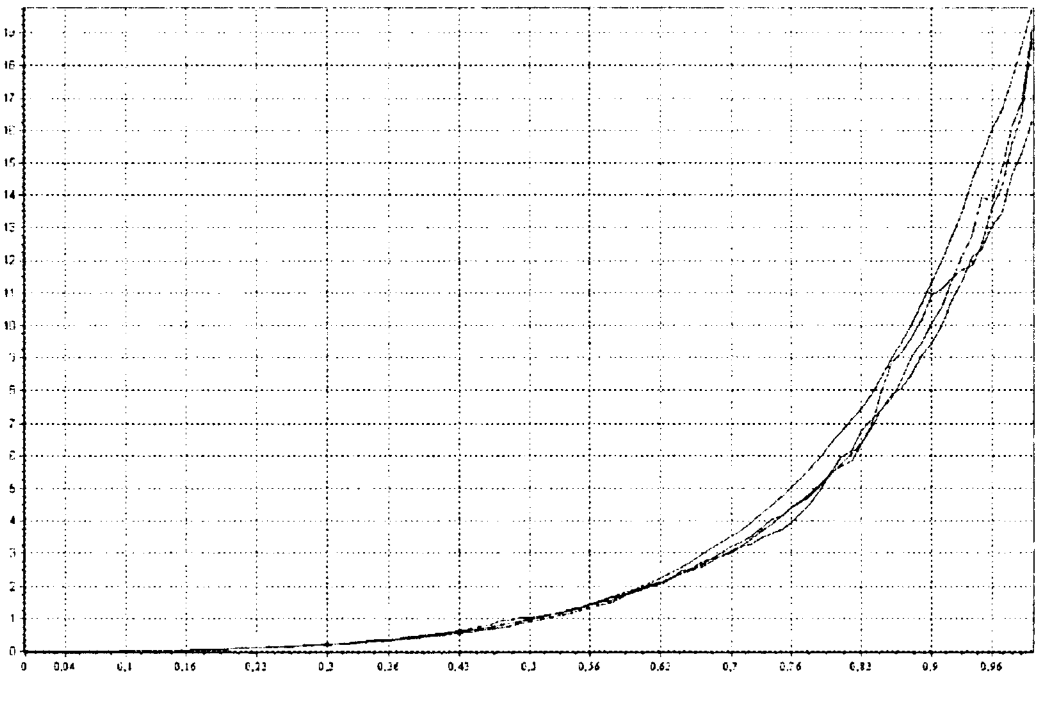

Fig. 1. f-Euler approximation. Approximation step $h = 0.01$, number of simulated trajectories $\gamma = 100$. Solid line: Milshtein approximation (number of simulated trajectories $\gamma = 1000000$) representing an exact solution.

The model

Let τ_n be a partition of $[0..T]$

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$$

and denote, as usual, $\Delta t_i := t_i - t_{i-1}$ and $\Delta W_{t_i} := W_{t_i} - W_{t_{i-1}}$. Realizing the idea above, we get the following modified Euler schema (let us call it a f-Euler approximation):

$$\begin{aligned}
Y_0^h &= \int_0^{x_0} \frac{\mathrm{d}x}{\sigma(x)},\\
Y_{t_1}^h &= Y_0^h + \bar{b}(Y_0^h)\Delta t_1 + \Delta W_{t_1},\\
&\ldots\\
Y_{t_i}^h &= Y_{t_{i-1}}^h + \bar{b}(Y_{t_{i-1}}^h)\Delta t_i + \Delta W_{t_i},\\
&\ldots\\
Y_{t_n}^h &= Y_{t_{n-1}}^h + \bar{b}(Y_{t_{n-1}}^h)\Delta t_n + \Delta W_{t_n},\\
\hline
X_{t_1}^h &= g^{-1}(Y_{t_1}^h),\\
&\ldots\\
X_{t_i}^h &= g^{-1}(Y_{t_i}^h),\\
&\ldots\\
X_{t_n}^h &= g^{-1}(Y_{t_n}^h).
\end{aligned}$$

Values of X_t^h on the interval (t_{i-1}, t_i) are obtained by simple linear interpolation of the points $(t_{i-1}, X_{t_{i-1}}^h)$ and $(t_i, X_{t_i}^h)$.

The value of $\mathbf{E}f(X_{t_i})$ is evaluated by averaging the values $f(X_{t_i}^{h,j})$, where $X_{t_i}^{h,j}$, $j = 1, 2, \ldots, \gamma$ are γ simulated trajectories of the approximation of the solution to stochastic differential equation (1).

An example

Let us consider the following process:

$$X_t = \int_0^t |X_s - 0.5|\,\mathrm{d}s + \int_0^t \sqrt{1 + X_s^2}\,\mathrm{d}W_s, \quad t \in [0, T].$$

Also, let $f(x) = x^3$. Then, using a computer and programming skills we will illustrate the proposed model of f-Euler scheme with the diagrams 1 and 2. Fig. 2 represents a comparison of Euler and modified Euler scheme. It seems, visually at least, that f-Euler scheme behaves slightly better than Euler scheme.

Because the explicit solution to this equation is not known, we use the Milstein approximation scheme with number of trajectories γ equal to 1000000 for the exact solution.

References

1. V. Mackevicius, On the convergence rate of Euler scheme for SDE with Lipschitz drift and constant diffusion, *Acta Applicandae Mathematicae*, **78**, 301–310 (2003).

REZIUMĖ

A. Lenkšas. Stochastinės diferencialinės lygtys su blogais koeficientais: silpnosios aproksimacijos

Yra žinoma, kad esant pakankamai geriems stochastinės diferencialinės lygties koeficientams b ir σ bei pakankamai glodžioms funkcijoms f, silpnoji Eulerio aproksimacija yra pirmosios eilės. Nelabai aišku ar eilė lieka lygi vienetui, jei koeficientai yra „blogi" (nėra pakankamai glodūs). Pasinaudojant [1] straipsnyje gautu rezultatu, Eulerio schema modifikuojama taip, kad ir tokiems koeficientams esant, gautoji schema būtų pirmosios eilės silpnoji stochastinės diferencialinės lygties aproksimacija.

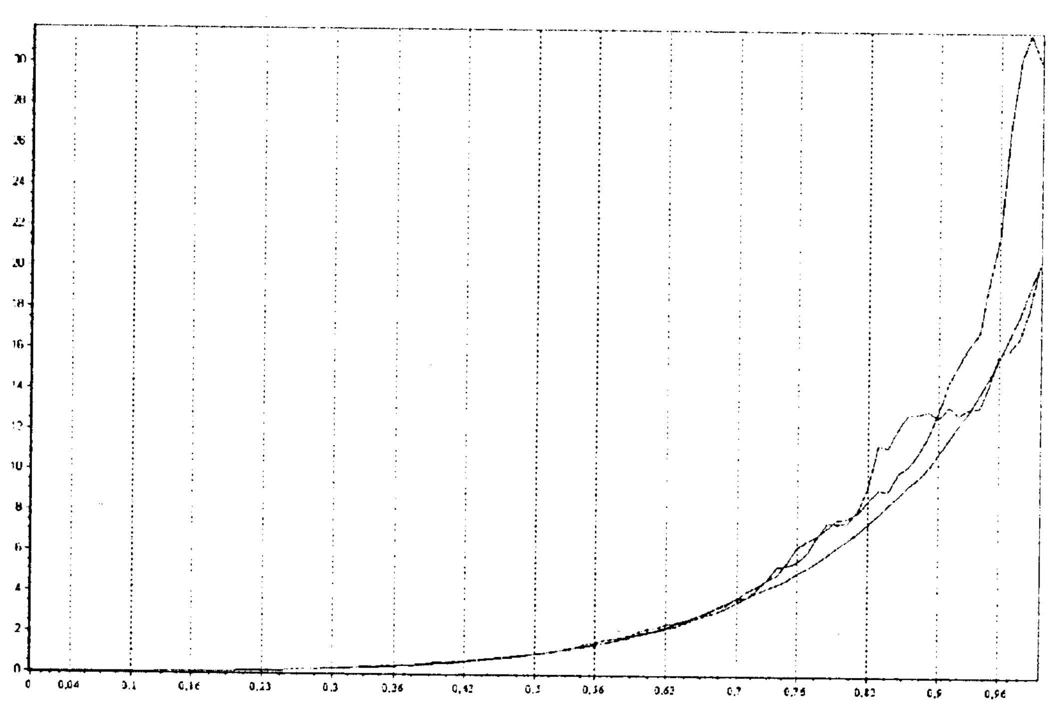

Fig. 2. A comparison of Euler and f-Euler approximations. Approximation step $h = 0.01$, number of simulated trajectories $\gamma = 100$. Solid line: Milshtein approximation (number of simulated trajectories $\gamma = 1000000$) representing an exact solution.