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A limit theorem for partial weighted sums of regression residuals

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Abstract. Let a linear regression be given. We consider the polygonal line process of weighted least squares residuals. We prove a limit theorem in Hölder spaces for this process. This extends the results of Jandhyala and MacNeill, Bischoff, Račkauskas.

Keywords: linear regression, regression residuals, partial sums process, limit theorem, Hölder space.

1. Introduction and results

Let us introduce some notations.

Let $f\colon [0,1] \to \mathbb{R}^d$ be a given function $f(t) = (f_1(t), \dots, f_d(t))^\tau$, $t \in [0,1]$, where transpose is denoted by the sign $^\tau$.

Let us define the matrix $X_n = (f(1/n), f(2/n), \dots, f(n/n))^\tau$.

Throughout we assume

$$\begin{cases} \text{matrices } A_n = \frac{1}{n} X_n^\tau X_n \text{ and} \\ A = \int_0^1 f(t) f^\tau(t)\,\mathrm{d}t \text{ are non degenerate.} \end{cases} \tag{1}$$

Our regression model is

$$y_j = f^\tau(j/n)\beta + \varepsilon_j, \quad 1 \leqslant j \leqslant n, \tag{2}$$

where $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. with mean zero and finite variance $\sigma^2 > 0$. Regression coefficients are estimated by the classical least square estimator $\widehat{\beta} = \left(X_n^\tau X_n\right)^{-1} X_n^\tau y$, where $y = (y_1, \dots, y_n)^\tau$. The residuals $\widehat{\varepsilon} = (\widehat{\varepsilon}_1, \dots, \widehat{\varepsilon}_n)^\tau$ are defined by

$$\widehat{\varepsilon}_k = y_k - \widehat{y}_k = y_k - f^\tau(j/n)\widehat{\beta}, \quad k = 1, \dots, n.$$

Let $r\colon [0,1] \to \mathbb{R}$. Define $\widehat{S}(0) = 0$ and $\widehat{S}(k) = \sum_{i \leqslant k} r(i/n)\widehat{\varepsilon}_i$, $\quad k = 0, \dots, n$.

Let us consider the polygonal line process

$$\widehat{V}_n(t) = \widehat{S}(nt) + r\big(([nt]+1)/n\big)\widehat{\varepsilon}_{[nt]+1}(nt - [nt]), \quad t \in [0,1]. \tag{3}$$

For continuously differentiable f, $r \equiv 1$, MacNeill [5] established weak convergence of $n^{-1/2}\sigma^{-1}\widehat{V}_n$ in the space $C[0,1]$. As a corollary from Bischoff [3] it follows that this convergence is true for functions f that are continuous and of bounded variation. Jandhyala and MacNeill [6] extended this result for continuously differentiable r.

The limiting process of $n^{-1/2}\sigma^{-1}\widehat{V}_n$ is

$$\widehat{V}(t)=\int_0^t r(u)\,\mathrm{d}W(u)-\int_0^t\int_0^1 r(u)f^{\tau}(u)A^{-1}f(v)\,\mathrm{d}W(v)\,\mathrm{d}u, \tag{4}$$

where $W(t)$ – Wiener process. The covariance kernel of this process can be expressed as

$$K_f(s,t)=\int_0^{\min(t,s)} r^2(u)\,\mathrm{d}u-\int_0^t\int_0^s r(u)r(v)f^{\tau}(u)A^{-1}f(v)\,\mathrm{d}u\,\mathrm{d}v.$$

We associate to α the separable Hölder space $H_\alpha^0[0,1]$ equipped with the norm $\|x\|_\alpha$. An obvious requirement for Hölder spaces we consider is that Wiener process W exists in $H_\alpha^0[0,1]$. This is true when $0<\alpha<1/2$. For more on Hölder spaces see for example Račkauskas and Suquet [2].

Next for the model (2) we introduce the admissible class of functions $\boldsymbol{f}$. The p variation on $[a,b]\subset[0,1]$ of function $\boldsymbol{g}$: $[0,1]\to\mathbb{R}^d$ is denoted by $v_{[a,b],p}(\boldsymbol{g})$. We denote the variation on $[0,1]$ by omitting the interval notation, i.e. $v_p(\boldsymbol{g})$. The space of bounded functions $\boldsymbol{g}$: $[0,1]\to\mathbb{R}^d$ of finite p variation is denoted by $V_p^d[0,1]$. This space is equipped with the norm $\|\boldsymbol{g}\|_{[p]}$.

Račkauskas [1] extended MacNeill [5] result in $H_\alpha^0[0,1]$ for $r\equiv 1$, f continuous and of p bounded variation. Necessary and sufficient condition was

$$\lim_{t\to\infty} tP\big(|\varepsilon_1|\geqslant t^{1/2-\alpha}\big)=0.$$

Our main result reads as follows.

THEOREM 1.1. *Let $0<\alpha<1/2$ and $1\leqslant p<1/(1-\alpha)$. Assume for the model* (2) *that the functions f and r are continuous and have finite p variation. Then*

$$\frac{1}{\sqrt{n}\sigma}\widehat{V}_n \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{V} \ \text{ in } H_\alpha^0[0,1], \tag{5}$$

if

$$\lim_{t\to\infty} tP\big(|\varepsilon_1|\geqslant t^{1/2-\alpha}\big)=0. \tag{6}$$

Before we proceed with the proof some remarks are in order here.

Remark 1.1. The unknown variance σ^2 in (5) can be replaced by its estimator

$$\widehat{\sigma}^2=\frac{1}{n}\sum_{k=1}^{n}\widehat{\varepsilon}_k^2. \tag{7}$$

Remark 1.2. Condition (6) yields $E|\varepsilon_1|^q$ for each $q<1/(1/2-\alpha)$.

Remark 1.3. On the other hand $E|\varepsilon_1|^{1/(1/2-\alpha)}$ yields (6).

Remark 1.4. Due to continuity of the embedding $H_\alpha^0[0,1] \hookrightarrow C[0,1]$ we obtain the following result.

COROLLARY 1.1. *Assume that the functions f and r are continuous and have finite p variation for some $1 \leqslant p < 2$. Then the convergence*

$$\frac{1}{\sqrt{n\sigma}} \widehat{V}_n \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{V} \text{ in space } C[0,1], \tag{8}$$

holds provided that $E\,|\varepsilon_1|^q$ for some $q > 2p/(2-p)$.

2. Proof of the main theorem

Here are some useful lemmas before we proceed with the proof of our main theorem.

LEMMA 2.1. *Let $p, q \geqslant 1$ and $1/p + 1/q > 1$. Let s, t such that $[s,t] \subset [0,1]$. If $g \in V_p^d[s,t]$ and $x \in V_q^1[s,t]$ then the integral $\int_s^t g(u)dx(u)$ is well defined in the Rieman-Stiltjes sense and there exists a constant $K_d > 0$ depending on d only such that*

$$\left| \int_s^t g(u)dx(u) \right| \leqslant K_d \, \|g\|_{[p]} \cdot v_{[s,t],q}^{1/q}(x),$$

Proof. The sign $[s,t]$ by the norm and variation means that they ar calculated in this interval. This lemma is a simple corollary from L.C. Young [4].

LEMMA 2.2. *Embedding $H_\alpha^0[0,1] \hookrightarrow V_{1/\alpha}^1[0,1]$ is continuous.*

LEMMA 2.3. *Embedding $V_p^d[0,1] \hookrightarrow V_{p'}^d[0,1]$ is continuous when $p' > p$.*

LEMMA 2.4. *If $p' > p$ and $g \in V_p^d[0,1]$ then*

$$\|g\|_{[p']} \leqslant \|g\|_\infty + 2^{(p'-p)/p'} v_p^{1/p'}(g)\, \|g\|_\infty^{(p'-p)/p'}.$$

Proof. Since $p' > p$ then by Lemma 2.3 we get $g \in V_{p'}^d[0,1]$. Therefore

$$\begin{aligned} v_{p'}(g) &= \sup_{(t_k)\subset[0,1]} \sum_{k=1}^m |g(t_k) - g(t_{k-1})|^p \, |g(t_k) - g(t_{k-1})|^{p'-p} \\ &\leqslant \left(2\,\|g\|_\infty\right)^{p'-p} v_p(g). \end{aligned}$$

Hence

$$\|g\|_{[p']} = \|g\|_\infty + v_{p'}^{1/p'}(g) \leqslant \|g\|_\infty + 2^{(p'-p)/p'} v_p^{1/p'}(g)\, \|g\|_\infty^{(p'-p)/p'}.$$

LEMMA 2.5. *If* $x \in H_\alpha^0[0,1]$ *then* $v_{[s,t],1/\alpha}(x) \leqslant (\omega_\alpha(x, t-s))^{1/\alpha}|t-s|$.

Proof.

$$\begin{aligned} v_{[s,t],1/\alpha}(x) &= \sup_{(t_k)\subset[s,t]} \sum_{k=1}^{m} \left(\frac{|x(t_k)-x(t_{k-1})|}{|t_k-t_{k-1}|^\alpha}\right)^{1/\alpha} |t_k-t_{k-1}| \\ &\leqslant \left(\sup_{0<t'-s'<t-s} \frac{|x(t')-x(s')|}{|t'-s'|^\alpha}\right)^{1/\alpha} \sup_{(t_k)\subset[s,t]} \sum_{k=1}^{m} |t_k-t_{k-1}| \\ &= \left(\omega_\alpha(x,t-s)\right)^{1/\alpha}|t-s|. \end{aligned}$$

LEMMA 2.6. *If* $x \in H_\alpha^0[0,1]$ *then* $v_{[s,t],1/\alpha}(x) \leqslant \|x\|_\alpha^{1/\alpha}|t-s|$.

Proof. It follows from the definition of the norm and Lemma 2.5.

Now we are ready to begin with the proof of the main theorem.

Proof (of Theorem 1.1). Set $r_n(t) = r(([nt]+1)/n)$.
Because

$$v_p(r_n) \leqslant \sup_{k,l=1,\dots,n} \sum_{k=1}^{m} |r(k/n)-r(l/n)|^p \leqslant v_p(r) < \infty$$

we have $r_n \in V_p^1[0,1]$ and

$$\|r_n\|_{[p]} = \|r_n\|_\infty + v_p^{1/p}(r_n) \leqslant \|r\|_\infty + v_p^{1/p}(r) = \|r\|_{[p]}. \tag{9}$$

Now we define operators M_n

$$M_n(x)(t) = \int_0^t r_n(u)dx(u) \tag{10}$$

and M

$$M(x)(t) = \int_0^t r(u)dx(u) \tag{11}$$

on $H_\alpha^0[0,1]$

It follows from Lemmas 2.2 and 2.1 that both integrals $\int_0^t r_n(u)dx(u)$ and $\int_0^t r(u)dx(u)$ are well defined in the Rieman–Stiltjes sense.

Lets check that both operators M_n and M map $H_\alpha^0[0,1]$ into itself.

$$\frac{|M_n(x)(t)-M_n(x)(s)|}{|t-s|^\alpha} = \frac{\left|\int_s^t r_n(u)dx(u)\right|}{|t-s|^\alpha}.$$

By using Lemmas 2.1 and 2.5

$$\sup_{0<t-s<\delta} \frac{\left|\int_s^t r_n(u)dx(u)\right|}{|t-s|^\alpha} \leqslant \sup_{0<t-s<\delta} \frac{K_1 \|r_n\|_{[p]} v^\alpha_{[s,t],1/\alpha}(x)}{|t-s|^\alpha}$$

$$\leqslant \sup_{0<t-s<\delta} \frac{K_1 \|r_n\|_{[p]} \left((\omega_\alpha(x,t-s))^{1/\alpha}|t-s|\right)^\alpha}{|t-s|^\alpha} \leqslant K_1 \|r_n\|_{[p]} \omega_\alpha(x,\delta).$$

Since $\omega_\alpha(M_n(x),\delta) \to 0$ when $\delta \to 0$, it follows $M_n(x) \in H^0_\alpha[0,1]$ when $x \in H^0_\alpha[0,1]$. The proof for $M(x) \in H^0_\alpha[0,1]$ when $x \in H^0_\alpha[0,1]$ is analogous.

It can be easily checked that for the line process (3), for all $t \in [0,1]$

$$\widehat{V}_n(t) = M_n V_n(t), \tag{12}$$

where $V_n(t) = \sum_{k\leqslant nt} \widehat{\varepsilon}_k + \widehat{\varepsilon}_{[nt]+1}(nt-[nt])$.

The limiting process $\widehat{V}$ defined by (4) can be rewritten

$$\widehat{V}(t) = MV(t), \quad t \in [0,1]., \tag{13}$$

where

$$V(t) = W(t) - \int_0^t \int_0^1 f^\tau(u) A^{-1} f(v)\,\mathrm{d}W(v)\,\mathrm{d}u. \tag{14}$$

Račkauskas has proved in [1] that condition (6) is necessary and sufficient for

$$\frac{1}{\sqrt{n}\sigma} V_n \xrightarrow[N\to\infty]{\mathcal{D}} V \text{ in the space } H^0_\alpha[0,1].$$

Due to the representations (12) and (13) it suffices to check that $M_n x_n \to Mx$ in the space $H^0_\alpha[0,1]$ for any sequence $(x_n) \subset H^0_\alpha[0,1]$ and any $x \in H^0_\alpha[0,1]$ such that $x_n \to x$. This easily follows from uniform convergence $M_n \to M$ which we are going to check.

Because r is continuous we have

$$\|r_n - r\|_\infty \to 0, \text{ when } n \to \infty. \tag{15}$$

Let $p' > p$ be such that $1/p' + \alpha > 1$ (note that $1/p + \alpha > 1$) and let $q = 1/\alpha$. By Lemmas 2.2, 2.3, 2.1 and 2.6 for all $s,t \in [0,1]$ we have

$$\begin{aligned} \left|M_n x(t) - Mx(t) - M_n x(s) + Mx(s)\right| &= \left|\int_s^t \left(r_n(u) - r(u)\right) dx(u)\right| \\ &\leqslant K_1 \|r_n - r\|_{[p']} v^{1/q}_{[s,t],q}(x) \\ &\leqslant K_1 \|r_n - r\|_{[p']} \|x\|_\alpha |t-s|^\alpha. \end{aligned}$$

It follows

$$\|M_n - M\| \leqslant \sup_{\substack{\|x\|_\alpha \leqslant 1 \\ 0<t-s<1}} K_1 \|r_n - r\|_{[p']} \|x\|_\alpha \leqslant K_1 \|r_n - r\|_{[p']}.$$

By Lemma 2.4 for the function $r_n - r$ and condition (15) we get

$$\lim_{n\to\infty} \|M_n - M\| = 0 \tag{16}$$

which completes the proof.

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REZIUMĖ

Vytenis Pažemys. Regresijos liekanų dalinių svertinių sumų ribinė teorema

Straipsnyje nagrinėjama tiesinė regresija ir tokios regresijos liekanų dalinių svertinių sumų ribinis elgesys. Įrodoma teorema apie tokių procesų konvergavimą Hiolderio erdvėse. Ši teorema apibendrina Jandhyala ir MacNeill, Bischoff, Račkausko rezultatus.