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Joint Universality of Dirichlet L-Functions and Periodic Hurwitz Zeta-Functions*

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Abstract. In the paper, we prove that every system of analytic functions can be approximated simultaneously uniformly on compact subsets of some region by a collection consisting of shifts of Dirichlet L-functions with pairwise non-equivalent characters and periodic Hurwitz zeta-functions with parameters algebraically independent over the field of rational numbers.

Keywords: Dirichlet L-function, limit theorem, periodic Hurwitz zeta-function, space of analytic functions, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

Since a remarkable Voronin's work [24] on the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, it is known that the majority of other zeta and L-functions also are universal in the sense that their shifts approximate uniformly on compact subsets of certain regions wide classes of analytic functions, for results and references, see [1, 4, 6, 7, 12, 17, 22]. Also, a more complicated approximation property of zeta and L-functions – the joint universality – is known. In this case, we deal with a simultaneous approximation of a given system of analytic functions. The first result in this direction also is due to Voronin who obtained [23] the joint universality of Dirichlet L-functions. The

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joint universality for Hurwitz zeta-functions was proved in [20] and [9]. We observe that Dirichlet L-functions have Euler product over primes while Hurwitz zeta-functions $\zeta(s,\alpha)$, $0<\alpha\le 1$, do not have Euler product, except for the cases $\zeta(s,1)=\zeta(s)$ and $\zeta\big(s,1/2\big)=\big(2^s-1\big)\zeta(s)$.

In [19] H. Mishou began to study the joint universality for zeta-functions having and having no Euler product over primes. He proved a joint universality theorem for the Riemann zeta-function and Hurwitz zeta-function with transcendental parameter α.

Let $D=\{s\in\mathbb{C}\colon \frac{1}{2}<\sigma<1\}$. Denote by $\mathcal{K}$ the class of compact subsets of D with connected complements. Moreover, let $\mathcal{H}_0(K)$ and $\mathcal{H}(K)$, $K\in\mathcal{K}$, be the classes of continuous non-vanishing and continuous on K functions, respectively, which are analytic in the interior of K. Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then the Mishou theorem is stated as follows: Suppose that α is transcendental. Let $K_1,K_2\in\mathcal{K}$, and $f_1\in\mathcal{H}_0(K_1)$, $f_2\in\mathcal{H}(K_2)$. Then, for every $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]\colon \sup_{s\in K_1}\big|\zeta(s+i\tau)-f_1(s)\big|<\varepsilon,\\ \sup_{s\in K_2}\big|\zeta(s+i\tau,\alpha)-f_2(s)\big|<\varepsilon\Big\}>0.$$

We call a property of $\zeta(s)$ and $\zeta(s,\alpha)$ in the later theorem the mixed joint universality.

In [5], the Mishou theorem was generalised for a periodic zeta-function and a periodic Hurwitz zeta-function. Let $\mathfrak{a}=\{a_m\colon m\in\mathbb{N}\}$ and $\mathfrak{b}=\{b_m\colon m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ be two periodic sequences of complex numbers. Then the periodic zeta-function $\zeta(s;\mathfrak{a})$ and the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$ are defined, for $\sigma>1$, by

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}\quad\text{and}\quad \zeta(s,\alpha;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^s},$$

respectively, and by analytic continuation elsewhere, except for a possible poles at $s=1$.

In [10] a mixed universality theorem was extended to a collection consisting of several periodic zeta-functions and periodic Hurwitz zeta-functions.

In the case of periodic Hurwitz zeta-functions, the following more general joint universality can be considered. For $j=1,\dots,r$, α_j, let $0<\alpha_j\le 1$ be a fixed parameter, $l_j\in\mathbb{N}$, $\mathfrak{a}_{jl}=\{a_{mjl}\colon m\in\mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl}\in\mathbb{N}$, and $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$ be the corresponding periodic Hurwitz zeta-function. In [8, 14, 15], the joint universality for the functions

$$\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r}) \tag{1.1}$$

was obtained. Later, the mixed joint universality for system (1.1) extended by some zeta-functions having Euler product was began to study. In [3], the Riemann zeta-function was added to the system (1.1). In the subsequent papers

[13, 16, 21], the function $\zeta(s)$ was replaced by zeta-functions of certain cusp forms. Namely, the paper [21] is devoted to a mixed joint universality theorem for the zeta-function $\zeta(s,F)$ attached to a normalized Hecke eigen cusp form F and the functions (1.1), in [16], the function $\zeta(s,F)$ was replaced by a zeta-function of a new form, and in [13], the case of a zeta-function of a cusp form F with respect to the Hecke subgroup with Dirichlet character was considered.

The aim of this paper is to extend the system (1.1) by a collection of Dirichlet L-functions. The extension of the class of jointly universal functions is motivated by wide theoretical and practical applications of universality (functional independence, zero-distribution, various value denseness problems, approximation and estimation of complicated analytic functions and their functionals).

Let χ be a Dirichlet character modulo q. We remind that the corresponding Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma > 1$, by the series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and is analytically continued to an entire function provided χ is a non-principal character. For the principal character χ_0,

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \Big(1 - \frac{1}{p^s}\Big).$$

Let $k_j = [k_{j1}, \dots, k_{jl_j}]$ be the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, and

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_jj1} & a_{k_jj2} & \dots & a_{k_jjl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

The main result of the paper is contained in the following theorem.

Theorem 1. *Suppose that* $\chi_1, \dots, \chi_d$ *are pairwise non-equivalent Dirichlet characters, the numbers* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over the field of rational numbers, and that* $\mathrm{rank}(A_j) = l_j$, $j = 1, \dots, r$. *For* $j = 1, \dots, d$, *let* $K_j \in \mathcal{K}$ *and* $f_j \in \mathcal{H}_0(K_j)$, *and, for* $j = 1, \dots, r$, $l = 1, \dots, l_j$, *let* $K_{jl} \in \mathcal{K}$ *and* $f_{jl} \in \mathcal{H}(K_{jl})$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T]\colon \sup_{1\le j\le d} \sup_{s\in K_j} \big|L(s+i\tau,\chi_j) - f_j(s)\big| < \varepsilon, \\ \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} \big|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)\big| < \varepsilon\Big\} > 0.$$

2 Multidimensional Limit Theorem

The main ingredient of the proof of Theorem 1 is a limit theorem for probability measures in the multidimensional space of analytic functions.

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and let

$$H_{d,u} = H_{d,u}(D) = H^d(D) \times H^u(D), \quad u = \sum_{j=1}^{r} l_j.$$

Let $\mathcal{B}(S)$ stand for the σ-field of Borel sets of the space S. This section is devoted to weak convergence of probability measures defined by terms of Dirichlet L-functions and periodic Hurwitz zeta-functions in the space $(H_{d,u}, \mathcal{B}(H_{d,u}))$.

Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p, \qquad \Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The tori Ω and Ω_1 are compact topological Abelian groups. Define

$$\Omega^\kappa = \Omega \times \Omega_{11} \times \cdots \times \Omega_{1r},$$

where $\Omega_{1j} = \Omega_1$, for all $j = 1, \dots, r$, and $\kappa = 1 + r$. Then Ω^κ also is a compact topological group. Therefore on $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa))$, the probability Haar measure m_H^κ can be defined. This gives the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$. Let $\Omega_1^r = \Omega_{11} \times, \cdots, \times \Omega_{1r}$. Then we have that the measure m_H^κ is the product of the probability Haar measures m_H and m_H^r on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega^r, \mathcal{B}(\Omega^r))$, respectively.

Now, on the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define a $H_{d,u}$-valued random element. We denote by ω_p the projection of $\omega \in \Omega$ to γ_p, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_{1j}$ to γ_m. Let, for brevity, $\underline{\omega} = (\omega, \omega_1, \dots, \omega_r)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\chi} = (\chi_1, \dots, \chi_d)$ and $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$. Let the $H_{d,u}$-valued random element $F(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ be given by

$$\begin{aligned} F(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(& L(s, \omega, \chi_1), \dots, L(s, \omega, \chi_d), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}) \dots, \\ & \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big), \end{aligned}$$

where

$$L(s, \omega, \chi_j) = \prod_p \Big(1 - \frac{\chi_j(p)}{p^s}\Big)^{-1}, \quad j = 1, \dots, d,$$

and

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r,\ l = 1, \dots, l_j.$$

Denote by P_F the distribution of the random element $F(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, i.e., for $A \in \mathcal{B}(H_{d,u})$,

$$P_F(A) = m_H^\kappa\big(\underline{\omega} \in \Omega^\kappa\colon F(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\big).$$

Moreover,

$$\begin{aligned} F(s, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(& L(s, \chi_1), \dots, L(s, \chi_d), \zeta(s, \alpha_1; \mathfrak{a}_{11}) \dots, \\ & \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big). \end{aligned}$$

Theorem 2. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$. Then*

$$P_T(A) \stackrel{def}{=} \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0,T] \colon F(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\bigr\}, \quad A \in \mathcal{B}(H_{d,u}),$$

converges weakly to P_F as $T \to \infty$.

Limit theorems for as a wide system of functions as in Theorem 2 are not known, however, their proofs differ from that, for example, in [4] only by details. Therefore, we present a shortened proof of Theorem 2. Denote by $\mathcal{P}$ the set of all prime numbers.

Lemma 1. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then*

$$Q_T(A) \stackrel{def}{=} \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0,T] \colon \bigl(\bigl(p^{-i\tau} \colon p \in \mathcal{P}\bigr), \bigl((m+\alpha_1)^{-i\tau} \colon m \in \mathbb{N}_0\bigr), \ldots, \bigl((m+\alpha_r)^{-i\tau} \colon m \in \mathbb{N}_0\bigr)\bigr) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(\Omega^\kappa\bigr),$$

converges weakly to the Haar measure m_H^κ as $T \to \infty$.

Proof of the lemma is given in [10, Theorem 3].

Let $\sigma_0 > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\biggl\{-\biggl(\frac{m}{n}\biggr)^{\sigma_0}\biggr\}, \qquad v_n(m,\alpha) = \exp\biggl\{-\biggl(\frac{m+\alpha}{n+\alpha}\biggr)^{\sigma_0}\biggr\}.$$

Define

$$L_n(s,\chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) v_n(m)}{m^s}, \quad j = 1, \ldots, d,$$

$$\zeta_n(s,\alpha_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl} v_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \ l = 1, \ldots, l_j,$$

and, for $\underline{\omega}_0 = (\omega_0, \omega_{10}, \ldots, \omega_{r0}) \in \Omega^\kappa$,

$$L_n(s,\chi_j,\omega_0) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega_0(m) v_n(m)}{m^s}, \quad j = 1, \ldots, d,$$

$$\zeta_n(s,\alpha_j,\omega_{0j}; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_{0j}(m) v_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \ l = 1, \ldots, l_j.$$

All later series are absolutely convergent for $\sigma > \frac{1}{2}$. Here $\omega(p)$ is extended to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^k \mid m \\ p^{k+1} \nmid m}} \omega^k(p), \quad m \in \mathbb{N}.$$

Let, for brevity,

$$F_n(s,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}}) = \big(L_n(s,\chi_1),\dots,L_n(s,\chi_d),\zeta_n(s,\alpha_1;\mathfrak{a}_{11})\dots, \\ \zeta_n(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta_n(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_r;\mathfrak{a}_{rl_r})\big),$$

and

$$F_n(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}) = \big(L_n(s,\omega,\chi_1),\dots,L_n(s,\omega,\chi_d),\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{11})\dots, \\ \zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})\big).$$

Lemma 2. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then on $(H_{n,r},\mathcal{B}(H_{n,r}))$, there exists a probability measure P_n such that*

$$P_{T,n}(A) \stackrel{def}{=} \frac{1}{T}\operatorname{meas}\big\{\tau\in[0,T]\colon F_n(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})\in A\big\},\quad A\in\mathcal{B}(H_{d,u}),$$
$$\hat{P}_{T,n}(A) \stackrel{def}{=} \frac{1}{T}\operatorname{meas}\big\{\tau\in[0,T]\colon F_n(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\omega}_0;\underline{\mathfrak{a}})\in A\big\},\quad A\in\mathcal{B}(H_{d,u}),$$

both converge weakly to P_n as $T\to\infty$.

Proof. Define $h_n : \Omega^\kappa\to H_{d,r}$ by the formula $h_n(\underline{\omega}) = F_n(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$. In view of the absolute convergence of the series for $L_n(s,\chi_j,\omega)$ and $\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})$, the function h_n is continuous, and

$$h_n\big((p^{-i\tau}\colon p\in\mathcal{P}),\big((m+\alpha_1)^{-i\tau}\colon m\in\mathbb{N}_0\big),\dots,\big((m+\alpha_r)^{-i\tau}\colon m\in\mathbb{N}_0\big)\big) \\ = F_n(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}}).$$

Thus, we have that $P_{T,n}=Q_T h_n^{-1}$. This, continuity of h_n, and Lemma 1 show that $P_{T,n}$ converges weakly to $P_n = m_H^\kappa h_n^{-1}$ as $T\to\infty$.

The invariance of the Haar measure m_H^κ allows to prove that the measure $\hat{P}_{T,n}$ also converges weakly to P_n. □

Define

$$\hat{P}_T(A) = \frac{1}{T}\operatorname{meas}\big\{\tau\in[0,T]\colon F(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\in A\big\},\quad A\in\mathcal{B}(H_{d,u}).$$

Lemma 3. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then on $(H_{d,u},\mathcal{B}(H_{d,u}))$, there exists a probability measure P such that P_T and $\hat{P}_T$ both converge weakly to P as $T\to\infty$.*

Proof. To prove the lemma it suffices to pass from $F_n(s,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})$ and $F_n(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$ to $F(s,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})$ and $F(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$, respectively. For this, define a metric on $H_{d,u}$ which induces the topology of uniform convergence on compacta. Let $\{K_v\colon v\in\mathbb{N}\}\subset D$ be a sequences on compact subsets such that $D=\bigcup_{v=1}^\infty K_v$, $K_l\subset K_{l+1}$, $l\in\mathbb{N}$, and if $K\subset D$ is a compact subset, then $K\subset K_v$, for some v. For $g_1,g_2\in H(D)$, define

$$\rho(g_1,g_2)=\sum_{v=1}^\infty 2^{-v}\frac{\sup_{s\in K_v}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_v}|g_1(s)-g_2(s)|}.$$

Then ρ is a metric on $H(D)$ inducing the topology of uniform convergence on compacta. For $\underline{g}_j = (g_{j1}, \dots, g_{j,d+u}) \in H_{d,u}$, $j = 1, 2$, we put

$$\rho_{d+u}(\underline{g}_1, \underline{g}_2) = \max_{1 \le l \le d+u} \rho(g_{1l}, g_{2l}).$$

Then ρ_{d+u} is a desired metric on $H_{d,u}$. Let ρ_d and ρ_r be analogical metrics on $H^d(D)$ and $H^u(D)$, respectively. We put

$$\underline{L}(s, \underline{\chi}) = \big(L(s, \chi_1), \dots, L(s, \chi_d)\big),$$
$$\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big)$$

and

$$\underline{L}_n(s, \underline{\chi}) = \big(L_n(s, \chi_1), \dots, L_n(s, \chi_d)\big), \quad \underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(\zeta_n(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{rl_r})\big).$$

Then, from the proof of a limit theorem for Dirichlet L-functions in [11], it follows that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_d\big(\underline{L}(s + i\tau, \underline{\chi}), \underline{L}_n(s + i\tau, \underline{\chi})\big)\, \mathrm{d}\tau = 0.$$

Similarly, in [8], it was obtained that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_u\big(\underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\big)\, \mathrm{d}\tau = 0.$$

Two last equalities together with the definition of the metric ρ_κ show that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_{d+u}\big(F(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}), F_n(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})\big)\, \mathrm{d}\tau = 0. \quad (2.1)$$

Now let θ be a random variable defined on a certain probability space ($\hat{\Omega}$, $\mathcal{A}, \mathbb{P}$) and uniformly distributed on $[0, 1]$. Define the $H_{d,u}$-valued random element $\underline{X}_{T,n} = \underline{X}_{T,n}(s) = \big(X_{T,n,1}(s), \dots, X_{T,n,d}(s), \hat{X}_{T,n,1,1}(s), \dots, \hat{X}_{T,n,1,l_1}(s), \dots, \hat{X}_{T,n,r,1}(s), \dots, \hat{X}_{T,n,r,l_r}(s)\big) = F_n(s + i\theta T, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})$, and denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then, by Lemma 2, we have that

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n, \quad (2.2)$$

where $\underline{X}_n = \big(X_{n,1}, \dots, X_{n,d}, \hat{X}_{n,1,1}, \dots, \hat{X}_{n,1,l_1}, \dots, \hat{X}_{n,r,1}, \dots, \hat{X}_{n,r,l_r}\big)$ is a $H_{d,r}$-valued random element having the distribution P_n (P_n is the limit measure in Lemma 2). We will prove that the family of probability measures $\{P_n \colon n \in \mathbb{N}\}$ is tight.

Since the series for $L_n(s, \chi_j)$ converges absolutely for $\sigma > \frac{1}{2}$, we find that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_v} \big|L_n(s + i\tau, \chi_j)\big|\, \mathrm{d}\tau \le C_v R_{jv}, \quad (2.3)$$

where

$$R_{jv} = \left(\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_v}} \right)^{\frac{1}{2}}, \quad j = 1, \dots, d, \ l \in \mathbb{N},$$

with some $C_v > 0$ and $\sigma_v > \frac{1}{2}$. Similarly, the absolute convergence for $\zeta_n(s, \alpha_j, ; \mathfrak{a}_{jl})$ leads to the estimate

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} \left| \zeta_n(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) \right| \mathrm{d}\tau \le \hat{C}_v R_{jlv}, \tag{2.4}$$

where

$$R_{jlv} = \left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m + \alpha_j)^{2\hat{\sigma}_v}} \right)^{\frac{1}{2}},$$

with some $\hat{C}_v > 0$ and $\hat{\sigma}_v > \frac{1}{2}$.

Now let $\varepsilon > 0$ be arbitrary fixed number,

$$M_{jv} = M_{jv}(\varepsilon) = C_v R_{jv} 2^{v+1} d \varepsilon^{-1}, \quad j = 1, \dots, d,$$

and

$$M_{jlv} = M_{jlv}(\varepsilon) = \hat{C}_v R_{jlv} 2^{v+1} u \varepsilon^{-1}, \quad j = 1, \dots, r.$$

Then the bounds (2.3) and (2.4) imply

$$\begin{aligned}
&\limsup_{T \to \infty} \mathbb{P}\Big(\exists j = 1, \dots, d\colon \sup_{s \in K_v} \left| X_{T,n,j}(s) \right| > M_{jv} \\
&\qquad \text{or } \exists (j,l), \ j = 1, \dots, r, \ l = 1, \dots, l_j\colon \sup_{s \in K_v} \left| \hat{X}_{T,n,j,l}(s) \right| > M_{jlv} \Big) \\
&\quad \le \sum_{j=1}^{d} \limsup_{T \to \infty} \mathbb{P}\Big(\sup_{s \in K_j} \left| X_{T,n,j}(s) \right| > M_{jv} \Big) \\
&\qquad + \sum_{j=1}^{r} \sum_{l=1}^{l_j} \limsup_{T \to \infty} \mathbb{P}\Big(\sup_{s \in K_j} \left| \hat{X}_{T,n,j,l}(s) \right| > M_{jlv} \Big) \\
&\quad \le \sum_{j=1}^{d} \frac{1}{M_{jv}} \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} \left| L_n(s + i\tau, \chi_j) \right| \mathrm{d}\tau \\
&\qquad + \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{1}{M_{jlv}} \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_v} \left| \zeta_n(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) \right| \mathrm{d}\tau \\
&\quad < \frac{\varepsilon}{2^{v+1}} + \frac{\varepsilon}{2^{v+1}} = \frac{\varepsilon}{2^v}, \quad v \in \mathbb{N}.
\end{aligned}$$

This together with (2.2), for all $n \in \mathbb{N}$, gives

$$\begin{aligned}
&\mathbb{P}\Big(\exists j = 1, \dots, d\colon \sup_{s \in K_v} \left| X_{n,j}(s) \right| > M_{jv} \text{ or } \exists (j,l), \ j = 1, \dots, r, \\
&\qquad l = 1, \dots, l_j\colon \sup_{s \in K_v} \left| \hat{X}_{n,j,l}(s) \right| > M_{jlv} \Big) \le \frac{\varepsilon}{2^v}, \quad v \in \mathbb{N}.
\end{aligned} \tag{2.5}$$

Define the set

$$K_{d,u} = K_{d,u}(\varepsilon) = \Big\{(g_1,\dots,g_d,\hat{g}_{11},\dots,\hat{g}_{1l_1},\dots,\hat{g}_{r1},\dots,\hat{g}_{r,l_r}) \in H_{d,u}: \sup_{s\in K_v}\big|g_j(s)\big| \le M_{jv},\ j=1,\dots,r, \sup_{s\in K_v}\big|\hat{g}_{jl}(s)\big| \le M_{jlv},\ j=1,\dots,r,\ l=1,\dots,l_j,\ v\in\mathbb{N}\Big\}.$$

Then $K_{d,u}$ is a compact subset in $H_{d,u}$. Moreover, in view of (2.5), for all $n \in \mathbb{N}$,

$$\mathbb{P}(\underline{X}_n \in K_{d,u}) > 1 - \varepsilon \sum_{v=1}^{\infty} \frac{1}{2^v} = 1 - \varepsilon,$$

or, by the definition of $\underline{X}_n$, we find that $P_n\big(K_{d,u}(\varepsilon)\big) > 1-\varepsilon$, for all $n \in \mathbb{N}$.

Thus, we have proved that the family of probability measures $\{P_n: n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem, see [2], it is relatively compact. Therefore, there exists a sequence $\{P_{n_k}\} \subset P_n$ such that P_{n_k} converges weakly to a certain probability measure P on $(H_{d,u}, \mathcal{B}(H_{d,u}))$ as $k \to \infty$. In other words,

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{2.6}$$

Now, using (2.1), (2.2), (2.6) and Theorem 4.2 of [2], we obtain that

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{2.7}$$

where, $\underline{X}_T = \underline{X}(s) = F(s+i\theta T, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})$. Thus, we proved that P_T converges weakly to P as $T \to \infty$.

Similarly to a relation (2.1), we find that, for almost all $\underline{\omega} \in \Omega^\kappa$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_{d+u}\big(F(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}), F_n(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})\big)\,\mathrm{d}\tau = 0. \tag{2.8}$$

Let

$$\underline{\hat{X}}_{T,n} = \underline{\hat{X}}(s) = F_n(s+i\theta T, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then, by Lemma 2

$$\underline{\hat{X}}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} P_n. \tag{2.9}$$

Moreover, (2.7) shows that

$$\underline{\hat{X}}_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{2.10}$$

Now (2.8)–(2.10) and Theorem 4.2 of [2] show that

$$\underline{\hat{X}}_T \xrightarrow[T\to\infty]{\mathcal{D}} P.$$

where

$$\underline{\hat{X}}_T = \underline{\hat{X}}_T(s) = F(s+i\theta T, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then, $\hat{P}_T$ also converges weakly to P as $T \to \infty$. The lemma is proved. □

Proof of Theorem 2. We apply standard arguments. Let A be a fixed continuity set of the measure P in Lemma 3. Then we have that

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\bigl(\tau\in[0,T]\colon F(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\omega};\mathfrak{a})\in A\bigr)=P(A). \tag{2.11}$$

We have to prove that $P=P_F$. For this, on the probability space $(\Omega^\kappa,\mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define the random variable

$$\xi(\underline{\omega})=\begin{cases}1, & \text{if } F(s,\underline{\chi},\underline{\alpha},\underline{\omega};\mathfrak{a})\in A,\\ 0, & \text{otherwise.}\end{cases}$$

Obviously, the expectation $\mathbb{E}\xi$ of $\xi(\underline{\omega})$ is of the form:

$$\mathbb{E}\xi=\int_{\Omega^\kappa}\xi(\underline{\omega})\,\mathrm{d}m_H^\kappa=m_H^\kappa\bigl(\underline{\omega}\in\Omega^\kappa\colon F(s,\underline{\chi},\underline{\alpha},\underline{\omega};\mathfrak{a})\in A\bigr). \tag{2.12}$$

In the sequel, we apply the ergodic theory. We consider the one parameter group $\{\Phi_\tau\colon\tau\in\mathbb{R}\}$ of measurable measure preserving transformations on Ω^κ given by

$$\Phi_\tau(\underline{\omega})=\bigl(\bigl(p^{-i\tau}\colon p\in\mathcal{P}\bigr),\bigl((m+\alpha_1)^{-i\tau}\colon m\in\mathbb{N}_0\bigr),\dots,\bigl((m+\alpha_r)^{-i\tau}\colon m\in\mathbb{N}_0\bigr)\bigr)\underline{\omega}.$$

In [10, Lemma 7], it is proved that the group $\{\Phi_\tau\colon\tau\in\mathbb{R}\}$ is ergodic. Hence, the random process $\xi(\Phi_\tau(\underline{\omega}))$ is ergodic as well. Therefore, the Birkhoff–Khintchine theorem implies the inequality

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\xi\bigl(\Phi_\tau(\underline{\omega})\bigr)\,\mathrm{d}\tau=\mathbb{E}\xi. \tag{2.13}$$

On the other hand, the definitions of ξ and Φ_τ yield

$$\frac{1}{T}\int_0^T\xi\bigl(\Phi_\tau(\underline{\omega})\bigr)\,\mathrm{d}\tau=\frac{1}{T}\operatorname{meas}\bigl\{\tau\in[0,T]\colon F(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\omega};\mathfrak{a})\in A\bigr\}.$$

This, (2.12) and (2.13) show that

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\bigl\{\tau\in[0,T]\colon F(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\omega};\mathfrak{a})\in A\bigr\}=P_F(A).$$

Then in view of (2.11), $P(A)=P_F(A)$. Since A was an arbitrary continuity set of P, the later equality holds for all continuity sets of P. It is well known that continuity sets constitute a determining class. Consequently, $P(A)=P_F(A)$ for all $A\in\mathcal{B}(H_{d,u})$, and the theorem is proved. □

3 Support

This section is devoted to the explicit form of the support S_{P_F} of the probability measure P_F. By the definition, S_{P_F} is a minimal closed subset of $H_{d,u}$ such that $P_F(S_{P_F})=1$. Define

$$S=\bigl\{g\in H(D)\colon g(s)\neq 0 \text{ or } g(s)\equiv 0\bigr\}.$$

Theorem 3. *Suppose that $\chi_1, \dots, \chi_d$ are pairwise non-equivalent Dirichlet characters, and parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the support of P_F is the set $S^d \times H^u(D)$.*

Proof. The set $H_{d,u}$ is separable, therefore [2],

$$\mathcal{B}(H_{d,u}) = \mathcal{B}\big(H^d(D)\big) \times \mathcal{B}\big(H^u(D)\big).$$

Therefore, it suffices to consider $P_F(A \times B)$, where $A = \mathcal{B}(H^d(D))$ and $B = \mathcal{B}(H^u(D))$. Let

$$\begin{aligned}
&\underline{L}(s, \underline{\chi}, \omega) = \big(L(s, \chi_1, \omega), \dots, L(s, \chi_d, \omega)\big),\\
&\underline{\zeta}(s, \underline{\alpha}, \underset{\sim}{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots,\\
&\qquad \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}, \omega_r)\big),
\end{aligned}$$

where $\underset{\sim}{\omega} = (\omega_1, \dots, \omega_r)$. Since the measure m_H^κ is a product of the measures m_H and m_H^r, we have that

$$\begin{aligned}
P_F(A \times B) &= m_H^\kappa\big(\underline{\omega} \in \Omega^\kappa \colon F(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A \times B\big)\\
&\quad m_H\big(\omega \in \Omega \colon \underline{L}(s, \underline{\chi}, \omega) \in A\big) \times m_H^r\big(\underset{\sim}{\omega} \in \Omega^r \colon \underline{\zeta}(s, \underline{\alpha}, \underset{\sim}{\omega}; \underline{\mathfrak{a}}) \in B\big). \qquad (3.1)
\end{aligned}$$

In [11], it was obtained that S^d is a minimal closed set such

$$m_H\big(\omega \in \Omega \colon \underline{L}(s, \underline{\chi}, \omega) \in S^d\big) = 1,$$

and in [8], it was proved that $H^u(D)$ is a minimal closed set such that

$$m_H^r\big(\underset{\sim}{\omega} \in \Omega^r \colon \underline{\zeta}(s, \underline{\alpha}, \underset{\sim}{\omega}; \underline{\mathfrak{a}}) \in H^u(D)\big) = 1.$$

These equalities together with (3.1) prove the theorem. □

4 Proof of Theorem 1

We will apply the Mergelyan theorem on approximation of analytic functions by polynomials [18] which asserts that if $K \subset \mathbb{C}$ is a compact subset with connected complement, and $g(s)$ is a function continuous on K and analytic in the interior of K, then, for every $\varepsilon > 0$, there is a polynomial $p(s)$ such that

$$\sup_{s \in K} \big|g(s) - p(s)\big| < \varepsilon.$$

Proof of Theorem 1. By the Mergelyan theorem, there exist polynomials $p_j(s)$, $j = 1, \dots, d$, and $p_{jl}(s)$, $j = 1, \dots, r$, $l = 1, \dots, l_j$, such that

$$\sup_{1 \le j \le d} \sup_{s \in K_j} \big|f_j(s) - p_j(s)\big| < \frac{\varepsilon}{4} \qquad (4.1)$$

and

$$\sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} \big|f_{jl}(s) - p_{jl}(s)\big| < \frac{\varepsilon}{2}. \qquad (4.2)$$

If ε is sufficiently small, we have that $p_j(s) \neq 0$ on K_j, $j = 1, \dots, d$. Therefore, there exists a continuous branch $\log p_j(s)$ which is analytic in the interior of K_j, $j = 1, \dots, d$. Applying the Mergelyan theorem once more, we can find polynomials $q_j(s)$, $j = 1, \dots, d$, such that

$$\sup_{1 \le j \le d} \sup_{s \in K_j} \bigl| p_j(s) - \mathrm{e}^{q_j(s)} \bigr| < \frac{\varepsilon}{4}.$$

Combining this with (4.1) gives

$$\sup_{1 \le j \le d} \sup_{s \in K_j} \bigl| f_j(s) - \mathrm{e}^{q_j(s)} \bigr| < \frac{\varepsilon}{2}. \tag{4.3}$$

By Theorem 3,

$$\bigl(\mathrm{e}^{q_1(s)}, \dots, \mathrm{e}^{q_d(s)}, p_{11}(s), \dots, p_{1l_1}(s), \dots, p_{r_1}(s), \dots, p_{rl_r}(s)\bigr) \in S_{P_F}.$$

Therefore, setting

$$G = \Bigl\{ \underline{g} \in H_{d,r} \colon \sup_{1 \le j \le d} \sup_{s \in K_j} \bigl| g_j(s) - \mathrm{e}^{q_j(s)} \bigr| < \frac{\varepsilon}{2}, \\ \sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} \bigl| g_{jl}(s) - p_{jl}(s) \bigr| < \frac{\varepsilon}{2} \Bigr\},$$

we obtain, by Theorem 2, that

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\bigl\{ \tau \in [0, T] \colon F(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) \in G \bigr\} \ge P_F(G) > 0.$$

This, definition of G and (4.2) and (4.3) complete the proof of the theorem. □

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