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A Joint Limit Theorem for Periodic Hurwitz Zeta-Functions with Algebraic Irrational Parameters

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Abstract. In the paper, a joint limit theorem for weakly convergent probability measures in $\mathbb{C}^r$ for periodic Hurwitz zeta-functions with algebraic irrational parameters satisfying certain independence conditions is obtained.

Keywords: Hurwitz zeta-function, joint limit theorem, probability measure, weak convergence.

AMS Subject Classification: 11M41.

1 Introduction

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$, and let α, $0 < \alpha \leq 1$, be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and is meromorphically continued to the whole complex plane by using the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha + l}{k}\right),$$

where $\zeta(s, \alpha)$ denotes the classical Hurwitz zeta-function. The point $s = 1$ is the unique possible simple pole of $\zeta(s, \alpha; \mathfrak{a})$.

In [6], the second author began to characterize the asymptotic behaviour of the function $\zeta(s, \alpha; \mathfrak{a})$ by limit theorems on the weak convergence of probability measures. We discussed the cases of transcendental, rational and algebraic irrational parameter α. The simplest of them is the case of transcendental α because of the linear independence over the field of rational numbers $\mathbb{Q}$ of the set $L(\alpha) = \{\log(m+\alpha)\colon m \in \mathbb{N}_0\}$. The case of rational α is based on the linear independence over $\mathbb{Q}$ of the set $\{\log p\colon p \in \mathcal{P}\}$, where $\mathcal{P}$ denotes the set of all prime numbers. The most complicated case is that of algebraic irrational α. In this case, there is no precise information on the linear independence of the set $L(\alpha)$. We know only a very deep theorem of Cassels which asserts that at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linear independent over $\mathbb{Q}$. The latter theorem allows to prove a limit theorem for weakly convergent probability measures with explicitly given limit measure.

The paper [7] is devoted to joint limit theorems for periodic Hurwitz zeta-functions. Let $\zeta(s,\alpha_1;\mathfrak{a}_1),\dots,\zeta(s,\alpha_r;\mathfrak{a}_r)$ be a collection of periodic Hurwitz zeta-functions. In [7], two cases of the parameters $\alpha_1,\dots,\alpha_r$ were discussed. The first case is of algebraically independent $\alpha_1,\dots,\alpha_r$. Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle on the complex plane, and $\Omega_1 = \prod_{m=0}^{\infty} \gamma_m$, where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. Then Ω_1 is a compact topological group. Define

$$\underline{\Omega}_1 = \prod_{j=1}^{r} \Omega_{1j},$$

were $\Omega_{1j} = \Omega_1$ for $j = 1,\dots,r$. Then $\underline{\Omega}_1$ is also a compact topological group. Therefore, on $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1))$ ($\mathcal{B}(S)$ denotes the class of Borel sets of the space S) the probability Haar measure $\underline{m}_{1H}$ exists, and we obtain the probability space $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1), \underline{m}_{1H})$. Denote by $\underline{\omega}_1 = (\omega_{11},\dots,\omega_{1r})$ the elements of the group $\underline{\Omega}_1$, and, on the probability space $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1), \underline{m}_{1H})$, define $\mathbb{C}^r$-valued random element $\underline{\zeta}_1(\underline{\sigma},\underline{\alpha},\underline{\omega}_1;\underline{\mathfrak{a}})$ by the formula

$$\underline{\zeta}_1(\underline{\sigma},\underline{\alpha},\underline{\omega}_1;\underline{\mathfrak{a}}) = \big(\zeta_1(\sigma_1,\alpha_1,\omega_{11};\mathfrak{a}_1),\dots,\zeta_1(\sigma_r,\alpha_r,\omega_{1r};\mathfrak{a}_r)\big),$$

where $\underline{\sigma} = (\sigma_1,\dots,\sigma_r)$, $\underline{\alpha} = (\alpha_1,\dots,\alpha_r)$ and $\underline{\mathfrak{a}} = (\mathfrak{a}_1,\dots,\mathfrak{a}_r)$, and

$$\zeta_1(\sigma_j,\alpha_j,\omega_{1j};\mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_{1j}(m)}{(m+\alpha_j)^{\sigma_j}}, \quad \sigma_j > \frac{1}{2},\ j = 1,\dots,r.$$

Here $\mathfrak{a}_j = \{a_{mj}\colon m \in \mathbb{N}_0\}$, $j = 1,\dots,r$, are periodic sequences of complex numbers, and $\omega_{1j}(m)$ denotes the projection of $\omega_{1j} \in \Omega_{1j}$ to the coordinate space γ_m. Let $P_{\underline{\zeta}_1}$ be the distribution of the random element $\underline{\zeta}_1(\underline{\sigma},\underline{\alpha},\underline{\omega}_1;\underline{\mathfrak{a}})$ and $\zeta(\underline{\sigma}+\mathrm{i}t,\underline{\alpha};\underline{\mathfrak{a}}) = (\zeta(\sigma_1+\mathrm{i}t_1,\alpha_1;\mathfrak{a}_1),\dots,\zeta(\sigma_r+\mathrm{i}t_r,\alpha_r;\mathfrak{a}_r))$. Then the first theorem of [7] is the following statement.

Theorem 1. *Suppose that* $\min_{1\le j\le r}\sigma_j > \frac{1}{2}$, *and that the numbers* $\alpha_1,\dots,\alpha_r$ *are algebraically independent over* $\mathbb{Q}$. *Then*

$$\frac{1}{T}\operatorname{meas}\big\{t \in [0,T]\colon \underline{\zeta}(\underline{\sigma}+\mathrm{i}t,\underline{\alpha};\underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(\mathbb{C}^r\big),$$

converges weakly to $P_{\underline{\zeta}_1}$ *as* $T \to \infty$.

Here and in the sequel, meas{A} stands for the Lebesque measure of a measurable set $A \subset \mathbb{R}$. The second joint limit theorem of [7] deals with rational parameters $\alpha_1, \ldots, \alpha_r$. In this case, we use $\Omega_2 = \prod_{p \in \mathcal{P}} \gamma_p$, where $\gamma_p = \gamma$ for all primes p. The torus Ω_2 is also a compact topological group, and we obtain a new probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, where m_{2H} is the probability Haar measure on $(\Omega_2, \mathcal{B}(\Omega_2))$. Denote by $\omega_2(p)$ the projection of $\omega_2 \in \Omega_2$ to the coordinate space $\gamma_p, p \in \mathbb{P}$, and extend the function $\omega_2(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_2(m) = \prod_{p^l \| m} \omega_2^l(p), \quad m \in \mathbb{N}.$$

Suppose that $\alpha_j = a_j/q_j$, $0 < a_j < q_j$, $(a_j, q_j) = 1$, $j = 1, \ldots, r$, and, on the probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, define the $\mathbb{C}^r$-valued random element $\underline{\zeta}_2(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathfrak{a}})$ by the formula

$$\underline{\zeta}_2(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathfrak{a}}) = \big(\zeta_2(\sigma_1, \alpha_1, \omega_2; \mathfrak{a}_1), \ldots, \zeta_2(\sigma_r, \alpha_r, \omega_2; \mathfrak{a}_r)\big),$$

where, for $\sigma_j > \frac{1}{2}$,

$$\zeta_2(\sigma_j, \alpha_j, \omega_2; \mathfrak{a}_j) = \omega_2(q_j) q_j^{\sigma_j} \sum_{\substack{m=1 \\ m \equiv a_j (\mathrm{mod} q_j)}}^{\infty} \frac{a_{(m-a_j)/q_j} \omega_2(m)}{m^{\sigma_j}}, \quad j = 1, \ldots, r.$$

Let $P_{\underline{\zeta}_2}$ be the distribution of the random element $\underline{\zeta}_2(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathfrak{a}})$. Then the second theorem of [7] is of the form.

Theorem 2. *For $j = 1, \ldots, r$, suppose that $\alpha_j = \frac{a_j}{q_j}$, $0 < a_j < q_j$, $(a_j, q_j) = 1$, and that $\sigma_j > \frac{1}{2}$. Then*

$$\frac{1}{T} \operatorname{meas}\big\{t \in [0, T] \colon \underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(\mathbb{C}^r\big),$$

converges weakly to $P_{\underline{\zeta}_2}$ as $T \to \infty$.

The aim of this paper is to obtain a joint limit theorem for periodic Hurwitz zeta-functions with algebraic irrational parameters. The motivation for this is a possible application of limit theorems in the investigation of the universality for periodic Hurwitz zeta-functions.

For $j = 1, \ldots, r$, let $I(\alpha_j)$ be a maximal linearly independent over $\mathbb{Q}$ subset of the set $L(\alpha_j)$. Suppose that $L(\alpha_j) \neq I(\alpha_j)$, and define $D(\alpha_j) = L(\alpha_j) \setminus I(\alpha_j)$. If $d_{jm} = \log(m + \alpha_j) \in D(\alpha_j)$, then the set $I(\alpha_j) \cup \{d_{jm}\}$ is already linearly dependent over $\mathbb{Q}$. Thus, there exist elements $i_{jm_1}, \ldots, i_{jm_n} \in I(\alpha_j)$ and $k_{j0}, \ldots, k_{jn} \in \mathbb{Z} \setminus \{0\}$ such that

$$d_{jm} = -\frac{k_{j1}}{k_{j0}} i_{jm_1} - \cdots - \frac{k_{jn}}{k_{j0}} i_{jm_n}.$$

From this we find that

$$m + \alpha_j = (m_1 + \alpha_j)^{-\frac{k_{j1}}{k_{j0}}} \cdots (m_n + \alpha_j)^{-\frac{k_{jn}}{k_{j0}}}. \tag{1.1}$$

Define the sets

$$\mathcal{M}(\alpha_j) = \bigl\{m \in \mathbb{N}_0\colon \log(m+\alpha_j) \in I(\alpha_j)\bigr\}$$

and

$$\mathcal{N}(\alpha_j) = \bigl\{m \in \mathbb{N}_0\colon \log(m+\alpha_j) \in D(\alpha_j)\bigr\}.$$

Now let

$$\Omega_{3j} = \prod_{m\in\mathcal{M}(\alpha_j)} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathcal{M}(\alpha_j)$, and $\underline{\Omega}_3 = \prod_{j=1}^{r} \Omega_{3j}$. Then $\underline{\Omega}_3$ is a compact topological Abelian group. Therefore, on $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3))$, the probability Haar measure $\underline{m}_{3H}$ can be defined, and we obtain the probability space $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3), \underline{m}_{3H})$. Denote by $\omega_{3j}(m)$ the projection of $\omega_{3j} \in \Omega_{3j}$ to γ_m, $m \in \mathcal{M}(\alpha_j)$, and extend the function $\omega_{3j}(m)$ to the set $\mathbb{N}_0$ by the formula

$$\omega_{3j}(m) = \omega_{3j}(m_1)^{-\frac{k_{j1}}{k_{j0}}} \cdots \omega_{3j}(m_n)^{-\frac{k_{jn}}{k_{j0}}}, \quad m \in \mathcal{N}(\alpha_j),$$

if equality (1.1) holds. Here the principal values of roots are taken. Denote by $\underline{\omega}_3 = (\omega_{31}, \dots, \omega_{3r})$ the elements of the group $\underline{\Omega}_3$, and, on the probability space $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3), \underline{m}_{3H})$, define the $\mathbb{C}^r$-valued random element $\underline{\zeta}_3(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}})$ by the formula

$$\underline{\zeta}_3(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}}) = \bigl(\zeta_3(\sigma_1, \alpha_1, \omega_{31}; \mathfrak{a}_1), \dots, \zeta_3(\sigma_r, \alpha_r, \omega_{3r}; \mathfrak{a}_r)\bigr),$$

where, for $\sigma_j > \frac{1}{2}$,

$$\zeta_3(\sigma_j, \alpha_j, \omega_{3j}; \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_{3j}(m)}{(m+\alpha_j)^{\sigma_j}}, \quad j = 1, \dots, r.$$

Denote by $P_{\underline{\zeta}_3}$ the distribution of the random element $\underline{\zeta}_3(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}})$. The main result of the paper is the following theorem.

Theorem 3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraic irrational, the set $\bigcup_{j=1}^{r} I(\alpha_j)$ is linearly independent over $\mathbb{Q}$, and that* $\min_{1\le j\le r} \sigma_j > \frac{1}{2}$. *Then*

$$P_T(A) \overset{def}{=} \frac{1}{T}\,\mathrm{meas}\bigl\{t \in [0,T]\colon \underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(\mathbb{C}^r\bigr),$$

converges weakly to $P_{\underline{\zeta}_3}$ as $T \to \infty$.

Theorem 3 is the first attempt to obtain probabilistic limit theorems used in proofs of universality theorems for zeta-functions. On the other hand, Theorem 3 characterizes the asymptotic behaviour of a collection of periodic Hurwitz zeta-functions with algebraic irrational parameters. This is a motivation of the paper.

2 A Limit Theorem on the Torus $\underline{\Omega}_3$

In this section, we consider the weak convergence of

$$Q_T(A) = \frac{1}{T}\operatorname{meas}\Big\{t \in [0,T]\colon \Big(\big((m+\alpha_1)^{-\mathrm{i}t}\colon m \in \mathcal{M}(\alpha_1)\big), \dots, \\ \big((m+\alpha_r)^{-\mathrm{i}t}\colon m \in \mathcal{M}(\alpha_r)\big)\Big) \in A\Big\}, \quad A \in \mathcal{B}(\underline{\Omega}_3).$$

Theorem 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ satisfy the hypotheses of Theorem 3. Then Q_T converges weakly to the Haar measure $\underline{m}_{3H}$ as $T \to \infty$.*

Proof. The dual group of $\underline{\Omega}_3$ is isomorphic to

$$\mathcal{D} \overset{def}{=} \Big(\bigoplus_{m\in\mathcal{M}(\alpha_1)} \mathbb{Z}_m\Big) \oplus \cdots \oplus \Big(\bigoplus_{m\in\mathcal{M}(\alpha_r)} \mathbb{Z}_m\Big),$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha_j)$, $j = 1, \dots, r$. An element $\underline{k} = (k_{1m}\colon m \in \mathcal{M}(\alpha_1), \dots, k_{rm}\colon m \in \mathcal{M}(\alpha_r))$ of $\mathcal{D}$, where only a finite number of integers k_{jm}, $j = 1, \dots, r$, are distinct from zero, acts on $\underline{\Omega}_3$ by

$$\underline{x} \to \underline{x}^{\underline{k}} = \prod_{m\in\mathcal{M}(\alpha_1)} x_{1m}^{k_{1m}} \cdots \prod_{m\in\mathcal{M}(\alpha_r)} x_{rm}^{k_{rm}},$$

where $\underline{x} = ((x_{1m}\colon m \in \mathcal{M}(\alpha_1)), \dots, (x_{rm}\colon m \in \mathcal{M}(\alpha_r))) \in \underline{\Omega}_3$. Therefore, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is of the form

$$\begin{aligned} g_T(\underline{k}) &= \int_{\underline{\Omega}_3} \Big(\prod_{m\in\mathcal{M}(\alpha_1)} x_{1m}^{k_{1m}} \cdots \prod_{m\in\mathcal{M}(\alpha_r)} x_{rm}^{k_{rm}}\Big)\, \mathrm{d}Q_T \\ &= \frac{1}{T}\int_0^T \Big(\prod_{m\in\mathcal{M}(\alpha_1)} (m+\alpha_1)^{-\mathrm{i}tk_{1m}} \cdots \prod_{m\in\mathcal{M}(\alpha_r)} (m+\alpha_r)^{-\mathrm{i}tk_{rm}}\Big)\, \mathrm{d}t \\ &= \frac{1}{T}\int_0^T \exp\Big\{-\mathrm{i}t\Big(\sum_{m\in\mathcal{M}(\alpha_1)} k_{1m}\log(m+\alpha_1) + \cdots \\ &\quad + \sum_{m\in\mathcal{M}(\alpha_r)} k_{rm}\log(m+\alpha_r)\Big)\Big\}\, \mathrm{d}t, \end{aligned} \tag{2.1}$$

where only a finite number of integers k_{jm}, $j = 1, \dots, r$, are distinct from zero. Since the set $\bigcup_{j=1}^{r} I(\alpha_j)$ is linearly independent over $\mathbb{Q}$, we have that

$$l(\underline{k}) \overset{def}{=} \sum_{m\in\mathcal{M}(\alpha_1)} k_{1m}\log(m+\alpha_1) + \cdots + \sum_{m\in\mathcal{M}(\alpha_r)} k_{rm}\log(m+\alpha_r) = 0$$

if and only if $\underline{k} = \underline{0}$. Therefore, after integration in (2.1), we find that

$$g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ \frac{\exp\{-\mathrm{i}Tl(\underline{k})\}-1}{-\mathrm{i}Tl(\underline{k})} & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Thus,

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and it follows from a continuity theorem for probability measures on compact groups (see, for example, Theorem 1.4.2 from [3]) that the measure Q_T converges weakly to $\underline{m}_{3H}$ as $T \to \infty$. □

3 Limit Theorems for Absolutely Convergent Series

For fixed $\hat{\sigma} > \frac{1}{2}$, and $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, let

$$v(m,n,\alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\hat{\sigma}}\right\}, \quad j = 1,\dots,r.$$

Define

$$\zeta_n(s,\alpha_j;\mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} v(m,n,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r.$$

Then the series for $\zeta_n(s,\alpha_j;\mathfrak{a}_j)$ converges absolutely for $\sigma > \frac{1}{2}$ independently on the arithmetical nature of α_j [4]. For $A \in \mathcal{B}(\mathbb{C}^r)$, we set

$$P_{T,n}(A) = \frac{1}{T}\,\mathrm{meas}\big\{t \in [0,T] \colon \underline{\zeta}_n(\underline{\sigma}+\mathrm{i}t,\underline{\alpha};\underline{\mathfrak{a}}) \in A\big\},$$

where $\underline{\zeta}_n(\underline{\sigma}+\mathrm{i}t,\underline{\alpha};\underline{\mathfrak{a}}) = \big(\zeta_n(\sigma_1+\mathrm{i}t,\alpha_1;\mathfrak{a}_1),\dots,\zeta_n(\sigma_r+\mathrm{i}t,\alpha_r;\mathfrak{a}_r)\big)$. Moreover, for $\underline{\omega}_3 = (\omega_{31},\dots,\omega_{3r}) \in \underline{\Omega}_3$, let

$$\zeta_n(s,\alpha_j,\omega_{3j};\mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_{3j}(m) v(m,n,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r.$$

Obviously, the series for $\zeta_n(s,\alpha_j,\omega_{3j};\mathfrak{a}_j)$ also converges absolutely for $\sigma > \frac{1}{2}$. Let $\underline{\zeta}_n(\underline{\sigma}+\mathrm{i}t,\underline{\alpha},\underline{\omega}_3;\underline{\mathfrak{a}}) = \big(\zeta_n(\sigma_1+\mathrm{i}t,\alpha_1,\underline{\omega}_{31};\mathfrak{a}_1),\dots,\zeta_n(\sigma_r+\mathrm{i}t,\alpha_r,\omega_{3r};\mathfrak{a}_r)\big)$, and, for $A \in \mathcal{B}(\mathbb{C}^r)$ and a fixed $\underline{\hat{\omega}}_3 = (\hat{\omega}_{31},\dots,\hat{\omega}_{3r})$,

$$\hat{P}_{T,n}(A) = \frac{1}{T}\,\mathrm{meas}\big\{t \in [0,T] \colon \underline{\zeta}_n(\underline{\sigma}+\mathrm{i}t,\underline{\alpha},\underline{\hat{\omega}}_3;\underline{\mathfrak{a}}) \in A\big\}.$$

Theorem 5. *Suppose that* $\min_{1\le j\le r}\sigma_j > \frac{1}{2}$*. Then, on* $(\mathbb{C}^r,\mathcal{B}(\mathbb{C}^r))$ *, there exists a probability measure* P_n *such that the measures* $P_{T,n}$ *and* $\hat{P}_{T,n}$ *both converge weakly to* P_n *as* $T \to \infty$.

Proof. Define a function $h_n : \underline{\Omega}_3 \to \mathbb{C}^r$ by the formula

$$h_n(\underline{\omega}_3) = \left(\sum_{m=0}^{\infty} \frac{a_{m1}\omega_{31}(m)v(m,n,\alpha_1)}{(m+\alpha_1)^{\sigma_1}},\dots,\sum_{m=0}^{\infty} \frac{a_{mr}\omega_{3r}(m)v(m,n,\alpha_r)}{(m+\alpha_r)^{\sigma_r}}\right).$$

Since the series in the definition of h_n converge absolutely, the function h_n is continuous, moreover,

$$\begin{aligned} &h_n\big(\big((m+\alpha_1)^{-\mathrm{i}t} \colon m \in \mathcal{M}(\alpha_1)\big),\dots,\big((m+\alpha_r)^{-\mathrm{i}t} \colon m \in \mathcal{M}(\alpha_r)\big)\big) \\ &\quad = \big(\zeta_n(\sigma_1+\mathrm{i}t,\alpha_1;\mathfrak{a}_1),\dots,\zeta_n(\sigma_r+\mathrm{i}t,\alpha_r;\mathfrak{a}_r)\big) = \underline{\zeta}_n(\underline{\sigma}+\mathrm{i}t,\underline{\alpha};\underline{\mathfrak{a}}). \end{aligned}$$

Hence, we have that

$$P_{T,n}(A) = Q_T h_n^{-1}(A), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Therefore, Theorem 4 together with Theorem 5.1 from [1] show that the measure $P_{T,n}$ converges weakly to $\underline{m}_{3H} h_n^{-1}$ as $T \to \infty$.

It remains to prove that the measure $\hat{P}_{T,n}$ also converges weakly to $\underline{m}_{3H} h_n^{-1}$ as $T \to \infty$. Let a function $h : \underline{\Omega}_3 \to \underline{\Omega}_3$ be given by the formula $h(\underline{\omega}_3) = \underline{\omega}_3 \hat{\omega}_3$. Then we have that

$$\begin{aligned}
&h_n\big(h\big(\big(\big((m+\alpha_1)^{-\mathrm{i}t} \colon m \in \mathcal{M}(\alpha_1)\big), \ldots, \big((m+\alpha_r)^{-\mathrm{i}t} \colon m \in \mathcal{M}(\alpha_r)\big)\big)\big)\big) \\
&\quad = \big(\zeta_n(\sigma_1 + \mathrm{i}t, \alpha_1, \omega_{31}; \mathfrak{a}_1), \ldots, \zeta_n(\underline{\sigma}_r + \mathrm{i}t, \alpha_r, \omega_{3r}; \mathfrak{a}_r)\big) \\
&\quad = \underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, \underline{\hat{\omega}}_3; \underline{\mathfrak{a}}).
\end{aligned}$$

Thus, the above arguments show that the measure $\hat{P}_{T,n}$ converges weakly to $\underline{m}_{3H}(h_n h)^{-1}$ as $T \to \infty$. However, the invariance of the Haar measure $\underline{m}_{3H}$ implies the equality $\underline{m}_{3H}(h_n h)^{-1} = (\underline{m}_{3H} h^{-1}) h_n^{-1} = \underline{m}_{3H} h_n^{-1}$. □

4 Approximation in the Mean

In this section, we approximate $\underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, ; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, ; \underline{\mathfrak{a}})$, and $\underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}})$. We use the Euclidean metric ρ in $\mathbb{C}^r$. Let $\underline{z}_j = (z_{j1}, \ldots, z_{jr}) \in \mathbb{C}^r$, $j = 1, 2$, and

$$\rho(\underline{z}_1, \underline{z}_2) = \left(\sum_{j=1}^{r} |z_{1j} - z_{2j}|^2 \right)^{\frac{1}{2}}.$$

Lemma 1. *Suppose that* $\min_{1 \le j \le r} \sigma_j > \frac{1}{2}$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, ; \underline{\mathfrak{a}}), \underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, ; \underline{\mathfrak{a}})\big)\, \mathrm{d}t = 0.$$

Proof. By Lemma 6 from [6], we have that, for every $j = 1, \ldots, r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \big|\zeta(\sigma_j + \mathrm{i}t, \alpha_j; \mathfrak{a}_j) - \zeta_n(\sigma_j + \mathrm{i}t, \alpha_j, ; \mathfrak{a}_j)\big|\, \mathrm{d}t = 0.$$

Since

$$\rho(\underline{z}_1, \underline{z}_2) \le \sum_{j=1}^{r} |z_{1j} - z_{2j}|, \tag{4.1}$$

this proves the lemma. □

Lemma 2. *Suppose that* $\min_{1 \le j \le r} \sigma_j > \frac{1}{2}$. *Then, for almost all* $\underline{\omega}_3 \in \underline{\Omega}_3$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}}), \underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}})\big)\, \mathrm{d}t = 0.$$

Proof. By Lemma 15 from [6], for almost all $\omega_{3j} \in \Omega_{3j}$ and every $j = 1, \dots, r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta(\sigma_j + \mathrm{i}t, \alpha_j, \omega_{3j}; \mathfrak{a}_j) - \zeta_n(\sigma_j + \mathrm{i}t, \alpha_j, \omega_{3j}; \mathfrak{a}_j)\right| \mathrm{d}t = 0. \quad (4.2)$$

Let $\hat{\Omega}_{3j}$ be the subset of Ω_{3j} whose elements satisfy the later relation. Then we have that $m_{3jH}(\hat{\Omega}_{3j}) = 1$, where m_{3jH} is the Haar measure on $(\Omega_{3j}, \mathcal{B}(\Omega_{3j}))$, $j = 1, \dots, r$. Let $\underline{\hat{\Omega}}_3 = \hat{\Omega}_{31} \times \cdots \times \hat{\Omega}_{3r}$. Since the Haar measure $\underline{m}_{3H}$ is the product of the measures $m_{31H}, \dots, m_{3rH}$, we find that $\underline{m}_{3H}(\underline{\hat{\Omega}}_3) = 1$. Therefore, the assertion of the lemma follows from inequality (4.1) and relation (4.2). □

5 Limit Theorems for $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}})$ and $\underline{\zeta}_3(s, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}})$

For $A \in \mathcal{B}(\mathbb{C}^r)$, define one more probability measure

$$\hat{P}_T(A) = \frac{1}{T} \mathrm{meas}\bigl\{t \in [0, T] \colon \underline{\zeta}_3(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}, \underline{\omega}_3; \underline{\mathfrak{a}}) \in A\bigr\}.$$

Theorem 6. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ satisfy the hypotheses of Theorem 3, and that $\min_{1\le j\le r} \sigma_j > \frac{1}{2}$. Then, on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$, there exists a probability measure P such that the measures P_T and $\hat{P}_T$ both converge weakly to P as $T \to \infty$.*

Proof. Let θ be a random variable defined on a certain probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and uniformly distributed on the interval $[0, 1]$. Define

$$\underline{X}_{T,n} = \underline{X}_{T,n}(\underline{\sigma}) = \bigl(X_{T,n,1}(\sigma_1), \dots, X_{T,n,r}(\sigma_r)\bigr) = \underline{\zeta}(\underline{\sigma} + \mathrm{i}T\theta, \underline{\alpha}; \underline{\mathfrak{a}})$$

which is a $\mathbb{C}^r$-valued random element on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then we have, in view of Theorem 5, that

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n, \quad (5.1)$$

where $\underline{X}_n = \underline{X}_n(\underline{\sigma}) = (X_{n,1}(\sigma_1), \dots, X_{n,r}(\sigma_r))$ is the $\mathbb{C}^r$-valued random element which distribution is the limit measure P_n in Theorem 5, and, as usual, $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Since the series for $\zeta_n(\sigma_j + \mathrm{i}t, \alpha_j; \mathfrak{a}_j)$ converges absolutely, the properties of Dirichlet series imply

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta_n(\sigma_j + \mathrm{i}t, \alpha_j; \mathfrak{a}_j)\right|^2 \mathrm{d}t = \sum_{m=0}^{\infty} \frac{|a_{mj}|^2 v^2(m, n, \alpha_j)}{(m + \alpha_j)^{2\sigma_j}} \le \sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m + \alpha_j)^{2\sigma_j}},$$

for all $n \in \mathbb{N}$, $j = 1, \dots, r$. Hence, for $j = 1, \dots, r$,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta_n(\sigma_j + \mathrm{i}t, \alpha_j; \mathfrak{a}_j)\right| \mathrm{d}t \le R_j, \quad (5.2)$$

where $R_j = \Big(\sum\limits_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m+\alpha_j)^{2\sigma_j}} \Big)^{\frac{1}{2}} < \infty$. Now let $\varepsilon > 0$ be arbitrary number, and $M_j = M_j(\varepsilon) = R_j r \varepsilon^{-1}$, $j = 1, \dots, r$. Then, in view of (5.2),

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{P}\big(\big|X_{T,n,j}(\sigma_j)\big| > M_j \text{ for at least one } j = 1,\dots,r\big) \\
&\quad \le \sum_{j=1}^{r} \limsup_{T\to\infty} \mathbb{P}\big(\big|X_{T,n,j}(\sigma_j)\big| > M_j\big) \\
&\quad \le \sum_{j=1}^{r} \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\big\{t \in [0,T] \colon \big|\zeta_n(\sigma_j + \mathrm{i}t, \alpha_j; \mathfrak{a}_j)\big| \ge M_j\big\} \\
&\quad \le \sum_{j=1}^{r} \frac{1}{M_j} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \big|\zeta_n(\sigma_j + \mathrm{i}t, \alpha_j; \mathfrak{a}_j)\big| \,\mathrm{d}t \le \sum_{j=1}^{r} R_j/M_j = \varepsilon.
\end{aligned}$$

This together with (5.1) implies

$$\mathbb{P}\big(\big|X_{n,j}(\sigma_j)\big| > M_j \text{ for at least one } j = 1,\dots,r\big) \le \varepsilon. \tag{5.3}$$

for all $n \in \mathbb{N}$. Now define $K_\varepsilon^r = \big\{\underline{z} \in \mathbb{C}^r \colon |z_j| \le M_j,\ j = 1,\dots,r\big\}$. Then K_ε is a bounded closed set, thus it is a compact set on $\mathbb{C}^r$. Moreover, by (5.3),

$$\mathbb{P}\big(\underline{X}_n \in K_\varepsilon^r\big) \ge 1 - \varepsilon,$$

or, by the definition of $\underline{X}_n$,

$$P_n\big(K_\varepsilon^r\big) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{P_n \colon n \in \mathbb{N}\}$ is tight, and, by the Prokhorov theorem, see, for example, [1], it is relatively compact. Therefore, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that the measure P_{n_k} converges weakly to a certain probability measure P on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $k \to \infty$. In other words,

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{5.4}$$

Now define

$$\underline{X}_T = \underline{X}_T(\underline{\sigma}) = \underline{\zeta}(\underline{\sigma} + \mathrm{i}T\theta, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then $\underline{X}_T$ is a $\mathbb{C}^r$-valued random element on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Using Lemma 1.1, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\big\{t \in [0,T] \colon \rho\big(\underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}; \underline{\mathfrak{a}})\big) \ge \varepsilon\big\} \\
&\quad \le \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \rho\big(\underline{\zeta}(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(\underline{\sigma} + \mathrm{i}t, \underline{\alpha}; \underline{\mathfrak{a}})\big) \,\mathrm{d}t = 0.
\end{aligned}$$

This and the definitions of the random elements $\underline{X}_{T,n}$ and $\underline{X}_T$ imply

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\rho(\underline{X}_T, \underline{X}_{T,n}) \ge \varepsilon\big) = 0. \tag{5.5}$$

The relations (5.1), (5.4) and (5.5) show that the hypotheses of Theorem 4.2 from [1] are satisfied. Therefore,

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{5.6}$$

and this shows that the measure P_T converges weakly to P as $T \to \infty$. Moreover, in virtue of (5.6) we have that the measure P is independent of the choice of the subsequence P_{n_k}. Thus, the relative compactness of the family $\{P_n\}$ yields the relation

$$\underline{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{5.7}$$

Reasoning similarly to the case of the measure P_T and using Theorem 5, Lemma 2 and (5.7), we obtain without difficulty that the measure $\hat{P}_T$ also converges weakly to P as $T \to \infty$. □

In view of Theorem 6, for proving Theorem 3, it remains to identify the measure P in (5.6).

6 Proof of Theorem 3

We start with one statement from ergodic theory. Let $\underline{a}_t = \{((m+\alpha_1)^{-\mathrm{i}t} \colon m \in \mathcal{M}(\alpha_1)), \dots, ((m+\alpha_r)^{-\mathrm{i}t} \colon m \in \mathcal{M}(\alpha_r))\}$, $t \in \mathbb{R}$. Define the one-parameter family $\{\Phi_t \colon t \in \mathbb{R}\}$ of transformation of $\underline{\Omega}_3$ by the formula $\Phi_t(\underline{\omega}_3) = \underline{a}_t\underline{\omega}_3$, $\underline{\omega}_3 \in \underline{\Omega}_3$. Then $\{\Phi_t \colon t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on the group $\underline{\Omega}_3$.

Lemma 3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ satisfy the hypotheses of Theorem 3. Then the one-parameter group $\{\Phi_t \colon t \in \mathbb{R}\}$ is ergodic.*

Proof. Let χ be a character of the group $\underline{\Omega}_3$. We have seen in the proof of Theorem 4 that

$$\chi(\underline{\omega}_3) = \prod_{m\in\mathcal{M}(\alpha_1)} \omega_{31}^{k_{1m}}(m) \dots \prod_{m\in\mathcal{M}(\alpha_r)} \omega_{3r}^{k_{rm}}(m), \quad \underline{\omega}_3 = (\omega_{31}, \dots, \omega_{3r}) \in \underline{\Omega}_3,$$

where only a finite number of integers k_{jm}, $j = 1, \dots, r$, are distinct from zero. First suppose that χ is a non-trivial character. Then we have that

$$\begin{aligned}\chi(\underline{a}_t) &= \prod_{m\in\mathcal{M}(\alpha_1)} (m+\alpha_1)^{-\mathrm{i}tk_{1m}} \dots \prod_{m\in\mathcal{M}(\alpha_r)} (m+\alpha_r)^{-\mathrm{i}tk_{rm}} \\ &= \exp\Bigg\{-\mathrm{i}t\Bigg(\sum_{m\in\mathcal{M}(\alpha_1)} k_{1m}\log(m+\alpha_1) + \dots + \sum_{m\in\mathcal{M}(\alpha_r)} k_{rm}\log(m+\alpha_r)\Bigg)\Bigg\}.\end{aligned}$$

Using the linear independence of the set $\bigcup_{j=1}^{r} I(\alpha_j)$, hence, we find that there exists $\tau_0 \in \mathbb{R} \setminus \{0\}$ such that $\chi(\underline{a}_{\tau_0}) = 1$. The further proof is standard, see, for example, [5]. □

Proof of Theorem 3. We take a fixed continuity set A of the limit measure P in Theorem 6. Then, using the equivalent of the weak convergence of probability measures in terms of continuity sets, see Theorem 2.1 from [1], we have that

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\bigl\{t\in[0,T]\colon \underline{\zeta}_3(\underline{\sigma}+\mathrm{i}t,\underline{\alpha},\underline{\omega}_3;\underline{\mathfrak{a}})\in A\bigr\}=P(A). \tag{6.1}$$

Let $\hat{\theta}$ be a random variable on $(\underline{\Omega}_3,\mathcal{B}(\underline{\Omega}_3),\underline{m}_{3H})$ given by

$$\hat{\theta}(\underline{\omega}_3)=\begin{cases}1 & \text{if } \underline{\zeta}_3(\underline{\sigma},\underline{\alpha},\underline{\omega}_3;\underline{\mathfrak{a}})\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Then, obviously, the expectation $\mathbb{E}(\hat{\theta})$ of $\hat{\theta}$ equals

$$\int_{\underline{\Omega}_3}\hat{\theta}\,\mathrm{d}\underline{m}_{3H}=\underline{m}_{3H}\bigl(\underline{\omega}_3\in\underline{\Omega}_3\colon \underline{\zeta}_3(\underline{\sigma},\underline{\alpha},\underline{\omega}_3;\underline{\mathfrak{a}})\in A\bigr)=P_{\underline{\zeta}_3}(A). \tag{6.2}$$

From Lemma 3, the ergodicity of the random process $\hat{\theta}(\Phi_t(\underline{\omega}_3))$ follows. Therefore, an application of the classical Birkhoff–Khintchine theorem, see, for example, [2], shows that, for almost all $\underline{\omega}_3\in\underline{\Omega}_3$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\hat{\theta}\bigl(\Phi_t(\underline{\omega}_3)\bigr)\,\mathrm{d}t=\mathbb{E}(\hat{\theta}). \tag{6.3}$$

On the other hand, by the definitions of $\hat{\theta}$ and Φ_t, we have that

$$\frac{1}{T}\int_0^T\hat{\theta}\bigl(\Phi_t(\underline{\omega}_3)\bigr)\,\mathrm{d}t=\frac{1}{T}\operatorname{meas}\bigl\{t\in[0,T]\colon \underline{\zeta}_3(\underline{\sigma}+\mathrm{i}t,\underline{\alpha},\underline{\omega}_3;\underline{\mathfrak{a}})\in A\bigr\}.$$

This, (6.2) and (6.3) imply, for almost all $\underline{\omega}_3\in\underline{\Omega}_3$, the equality

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\bigl\{t\in[0,T]\colon \underline{\zeta}_3(\underline{\sigma}+\mathrm{i}t,\underline{\alpha},\underline{\omega}_3;\underline{\mathfrak{a}})\in A\bigr\}=P_{\underline{\zeta}_3}(A).$$

Hence, taking into account (6.1), we obtain that $P(A)=P_{\underline{\zeta}_3}(A)$. Since A was an arbitrary continuity set of P, the latter relation is true for all continuity sets of P, and this shows that $P(A)=P_{\underline{\zeta}_3}(A)$ for all $A\in\mathcal{B}(\mathbb{C}^r)$. □

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