MATHEMATICAL MODELLING AND ANALYSIS
Volume 19 Number 1, February 2014, 52–65
http://dx.doi.org/10.3846/13926292.2014.893261

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

A Mixed Joint Universality Theorem for Zeta-Functions. II

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Received June 28, 2013; revised December 26, 2013; published online February 20, 2014

Abstract. In the paper, a joint universality theorem on the approximation of analytic functions for zeta-function of a normalized Hecke eigen cusp form and a collection of periodic Hurwitz zeta-functions with algebraically independent parameters is obtained.

Keywords: Hurwitz zeta-function, universality, zeta-function of certain cusp form.

AMS Subject Classification: 11M41.

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$, α, $0 < \alpha \le 1$, be a fixed parameter, and $s = \sigma + it$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and is analytically continued to an entire function if

$$a \stackrel{def}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l = 0,$$

while if $a \neq 0$, then $\zeta(s, \alpha; \mathfrak{a})$ is a meromorphic function having the unique simple pole at $s = 1$ with residue a.

In [4], a joint universality theorem for the Riemann zeta-function $\zeta(s)$ and a collection of periodic Hurwitz zeta-functions has been obtained. For $j = 1, \dots, r$ let α_j, $0 < \alpha_j \le 1$, be a fixed parameter, $l_j \in \mathbb{N}$, and, for $j = 1, \dots, r$, $l = 1, \dots, l_j$, let $\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period k_{jl}, and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$ denote the corresponding periodic Hurwitz zeta-function. Denote by k_j the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, and define

$$B_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j j1} & a_{k_j j2} & \dots & a_{k_j jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Let $D = \{s \in \mathbb{C} \colon \frac{1}{2} < \sigma < 1\}$, and let, for brevity,

$$\nu_T(\dots) = \frac{1}{T} \text{meas}\{\tau \in [0, T] \colon \dots\},$$

where $\text{meas} A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and in the place of dots a condition satisfied by τ is to be written. Then the main result of [4] is contained in the following theorem.

Theorem 1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, and that $\text{rank}(B_j) = l_j$, $j = 1, \dots, r$. For every $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let K_{jl} be a compact subset of the strip D with connected complement, and let $f_{jl}(s)$ be a continuous on K_{jl} function which is analytic in the interior of K_{jl}. Moreover, let K be a compact subset of the strip D with connected complement, and $f(s)$ be a continuous non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \nu_T \Big(\sup_{s \in K} \big|\zeta(s + i\tau) - f(s)\big| < \varepsilon,$$
$$\sup_{1 \le j \le r} \sup_{1 \le j \le l_j} \sup_{s \in K_{jl}} \big|\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)\big| < \varepsilon \Big) > 0.$$

A natural question arises if the Riemann zeta-function in Theorem 1 can be replaced by other zeta-functions which are universal in a certain strip?

Let F be a normalized Hecke eigen cusp form of weight κ for the full modular group, and let

$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi i m z}$$

be its Fourier series expansion. The zeta-function $\varphi(s, F)$ attached to the form F is defined, for $\sigma > \frac{\kappa+1}{2}$, by

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and is analytically continued to an entire function. Moreover, for $\sigma > \frac{\kappa+1}{2}$, the function $\varphi(s, F)$ has the Euler product over primes

$$\varphi(s, F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are complex conjugate numbers such that $\alpha(p)+\beta(p)=c(p)$, and

$$\left|\alpha(p)\right| < p^{\frac{\kappa-1}{2}}, \qquad \left|\beta(p)\right| \leq p^{\frac{\kappa-1}{2}}.$$

In [5], the universality of the function $\varphi(s,F)$ has been obtained. Let $D_\kappa = \{s \in \mathbb{C}\colon \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$.

Theorem 2. [5] *Let K be a compact subset of the strip D_κ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K, and analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K}\left|\varphi(s+i\tau,F)-f(s)\right| < \varepsilon\Big) > 0.$$

The main result of the present paper connects Theorems 1 and 2.

Theorem 3. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\operatorname{rank}(B_j)=l_j$, $j=1,\dots,r$. Let K_{jl} and f_{jl} be the same as in Theorem 1, and K and $f(s)$ be the same as in Theorem 2. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K}\left|\varphi(s+i\tau,F)-f(s)\right| < \varepsilon,$$
$$\sup_{1\leq j\leq r}\ \sup_{1\leq l\leq l_j}\ \sup_{s\in K_{jl}}\left|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl})-f_{jl}(s)\right| < \varepsilon\Big) > 0.$$

For the proof of Theorem 3, we apply the probabilistic approach based on a joint limit theorem in the space of analytic functions. Theorem 3 is the first result on the joint universality for zeta-functions which presents the universality property in two different strips D and D_κ. This is the novelty of the paper.

1 Functional Limit Theorems

For a region G on the complex plane, let us denote by $H(G)$ the space of analytic functions on G equipped with the topology of uniform convergence on compacta. Let

$$H^v(D_\kappa,D) = H(D_\kappa)\times\underbrace{H(D)\times\cdots\times H(D)}_{v_1}, \quad v_1=\sum_{j=1}^r l_j, \quad v=v_1+1.$$

For brevity, we set

$$\underline{\alpha}=(\alpha_1,\dots,\alpha_r), \quad \underline{\mathfrak{a}}=(\mathfrak{a}_{11},\dots,\mathfrak{a}_{1l_1},\dots,\mathfrak{a}_{r1},\dots,\mathfrak{a}_{rl_r})$$

and

$$\underline{\zeta}(\hat{s},s,\underline{\alpha};\underline{\mathfrak{a}},F) = \big(\varphi(\hat{s},F),\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,$$
$$\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})\big).$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. In this section, we consider the weak convergence of the probability measure

$$P_T(A) \stackrel{def}{=} \nu_T\big(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F) \in A\big), \quad A \in \mathcal{B}\big(H^v(D_\kappa, D)\big),$$

as $T \to \infty$. To state a limit theorem, we need some notation.

Denote by γ the unit circle on the complex plane, and define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the tori $\hat{\Omega}$ and Ω are compact topological Abelian groups. Therefore, on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ and $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measures $\hat{m}_H$ and m_H, respectively, can be defined. We obtain two probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$ and $(\Omega, \mathcal{B}(\Omega), m_H)$.

Furthermore, we put $\underline{\Omega} = \hat{\Omega} \times \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then the Tikhonov theorem implies again that $\underline{\Omega}$ is a compact topological Abelian group, and, similarly as above, we obtain one more probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where $\underline{m}_H$ is the probability Haar measure on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \Omega$ to the coordinate space γ_p, $p \in \mathcal{P}$, ($\mathcal{P}$ is the set of all prime numbers), and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. Let $\underline{\omega} = (\hat{\omega}, \omega_1, \dots, \omega_r)$ stand for elements of $\underline{\Omega}$. On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^v(D_\kappa, D)$-valued random element $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$ by the formula

$$\begin{aligned}\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) = \big(&\varphi(\hat{s}, \hat{\omega}, F), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \\ &\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),\end{aligned}$$

where

$$\varphi(\hat{s}, \hat{\omega}, F) = \prod_p \left(1 - \frac{\alpha(p)\hat{\omega}(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\hat{\omega}(p)}{p^s}\right)^{-1},$$

and

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r, \ l = 1, \dots, l_j.$$

Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$, i.e., for $A \in \mathcal{B}(H^v(D_\kappa, D))$,

$$P_{\underline{\zeta}}(A) = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega} \colon \underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\big).$$

Theorem 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then P_T converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

We divide the proof of Theorem 4 into several lemmas. Define

$$\begin{aligned}Q_T(A) \quad &= \nu_T\Big(\big((p^{-i\tau} : p \in \mathcal{P}\big), \big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \dots, \\ &\qquad \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)\Big) \in A\Big), \quad A \in \mathcal{B}(\underline{\Omega}).\end{aligned}$$

Lemma 1. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then Q_T converges weakly to the Haar measure $\underline{m}_H$ as $T\to\infty$.*

Proof of the lemma is given in [4], Lemma 1.

Let $\sigma_1>\frac12$ be a fixed number,

$$u_n(m)=\exp\left\{-(m/n)^{\sigma_1}\right\},\quad m,n\in\mathbb{N},$$
$$u_n(m,\alpha_j)=\exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1}\right\},\quad m\in\mathbb{N}_0,\ n\in\mathbb{N}.$$

Define

$$\varphi_n(\hat{s},F)=\sum_{m=1}^{\infty}\frac{c(m)u_n(m)}{m^{\hat{s}}},\qquad \zeta_n(s,\alpha_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}u_n(m,\alpha_j)}{(m+\alpha_j)^s},$$

$j=1,\dots,r$, $l=1,\dots,l_j$. By a standard method based on the application of the Mellin formula, it is obtained that the series for $\varphi_n(\hat{s},F)$ converges absolutely for $\operatorname{Re}\hat{s}>\frac{\kappa}{2}$, and the series for $\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})$ converges absolutely for $\sigma>\frac12$.

We extend the functions $\hat{\omega}(p)$ to the set $\mathbb{N}$ by the formula

$$\hat{\omega}(m)=\prod_{p^l\|m}\hat{\omega}^l(p),\quad m\in\mathbb{N},$$

where $p^l\parallel m$ means that $p^l\mid m$ but $p^{l+1}\nmid m$, and define

$$\varphi_n(\hat{s},\hat{\omega},F)=\sum_{m=1}^{\infty}\frac{c(m)\hat{\omega}(m)u_n(m)}{m^{\hat{s}}},$$
$$\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega_j(m)u_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\ l=1,\dots,l_j.$$

Clearly, the series for $\varphi_n(\hat{s},\hat{\omega},F)$ converges absolutely for $\operatorname{Re}\hat{s}>\frac{\kappa}{2}$, and the series for $\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})$ converges absolutely for $\sigma>\frac12$. For brevity, we set

$$\underline{\zeta}_n(\hat{s},s,\underline{\alpha};\underline{\mathfrak{a}},F)=\big(\varphi_n(\hat{s},F),\zeta_n(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta_n(s,\alpha_1;\mathfrak{a}_{1l_1}),\\ \dots,\zeta_n(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_r;\mathfrak{a}_{rl_r})\big)$$

and

$$\underline{\zeta}_n(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F)=\big(\varphi_n(\hat{s},\hat{\omega},F),\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{11}),\dots,\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\\ \zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})\big).$$

Now, on the space $(H^v(D_\kappa,D),\mathcal{B}(H^v(D_\kappa,D)))$, define two probability measures

$$P_{T,n}(A)=\nu_T\big(\underline{\zeta}_n(\hat{s}+i\tau,s+i\tau,\underline{\alpha};\underline{\mathfrak{a}},F)\in A\big),$$
$$P_{T,n,\underline{\omega}_0}(A)=\nu_T\big(\underline{\zeta}_n(\hat{s}+i\tau,s+i\tau,\underline{\alpha},\underline{\omega}_0;\underline{\mathfrak{a}},F)\in A\big),$$

where $\underline{\omega}_0=(\hat{\omega}_0,\omega_{10},\dots,\omega_{r0})$ is a fixed element of $\underline{\Omega}$.

Lemma 2. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures $P_{T,n}$ and $P_{T,n,\underline{\omega}_0}$ both converge weakly to the same probability measure P_n on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.*

Proof. We argue similarly to the proof of Lemma 2 from [4]. The absolute convergence of the series for $\varphi_n(\hat{s}, F)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ implies the continuity of the function $h_n : \underline{\Omega} \to H^v(D_\kappa, D)$ defined by the formula

$$h_n(\underline{\omega}) = \underline{\zeta}_n(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F).$$

Moreover, we have that

$$\begin{aligned} &h_n\big((p^{-i\tau} : p \in \mathcal{P}), \big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \ldots, \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)\big) \\ &\quad = \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F). \end{aligned}$$

Hence, $P_{T,n} = Q_T h_n^{-1}$. This, the continuity of the function h_n and Theorem 5.1 from [1] together with Lemma 1 show that the measure $P_{T,n}$ converges weakly to $P_n = \underline{m}_H h_n^{-1}$ as $T \to \infty$.

Let the function $g_n : \underline{\Omega} \to H^v(D_\kappa, D)$ be given by the formula $g_n(\underline{\omega}) = h_n(\underline{\omega}\,\underline{\omega}_0)$. Then the above arguments show that the measure $P_{T,n,\underline{\omega}_0}$ converges weakly to the measure $\underline{m}_H g_n^{-1}$ as $T \to \infty$. However, the invariance of the Haar measure $\underline{m}_H$ implies the equality $\underline{m}_H h_n^{-1} = \underline{m}_H g_n^{-1}$. This proves the lemma. □

For the proof of Theorem 4, we need to pass from $\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}})$ to $\underline{\zeta}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}})$. This procedure requires the metric on the space $H^v(D_\kappa, D)$ which induces its topology of uniform convergence on compacta. It is well known that there exists a sequence $\{\hat{K}_k : k \in \mathbb{N}\}$ of compact subsets of D_κ and a sequence of compact subsets of D such that $D_\kappa = \bigcup_{k=1}^\infty \hat{K}_k$ and $D = \bigcup_{k=1}^\infty K_k$. Moreover, the sets $\hat{K}_k$ and K_k can be chosen to satisfy $\hat{K}_k \subset \hat{K}_{k+1}$, $K_k \subset K_{k+1}$ for all $k \in \mathbb{N}$, and, for every compact $\hat{K} \subset D_\kappa$ and $K \subset D$, there exist $\hat{k}$ and k such that $\hat{K} \subset \hat{K}_{\hat{k}}$ and $K \subset K_k$. For $\hat{f}, \hat{g} \in H(D_\kappa)$, let

$$\hat{\rho}(\hat{f}, \hat{g}) = \sum_{k=1}^\infty 2^{-k} \frac{\sup_{s \in \hat{K}_k} |\hat{f}(s) - \hat{g}(s)|}{1 + \sup_{s \in \hat{K}_k} |\hat{f}(s) - \hat{g}(s)|}$$

and similarly, for $f, g \in H(D)$, let

$$\rho(f, g) = \sum_{k=1}^\infty 2^{-k} \frac{\sup_{s \in K_k} |f(s) - g(s)|}{1 + \sup_{s \in K_k} |f(s) - g(s)|}.$$

Then $\hat{\rho}$ and ρ are the metrics on $H(D_\kappa)$ and $H(D)$, respectively, which induce the topology of uniform convergence on compacta. For $\underline{f} = (\hat{f}, f_{11}, \ldots, f_{1l_1}, \ldots, f_{r1}, \ldots, f_{rl_r})$, $\underline{g} = (\hat{g}, g_{11}, \ldots, g_{1l_1}, \ldots, g_{r1}, \ldots, g_{rl_r}) \in H^v(D_\kappa, D)$, define

$$\rho_v(\underline{f}, \underline{g}) = \max\Big(\hat{\rho}(\hat{f}, \hat{g}), \max_{1 \le j \le r} \max_{1 \le l \le l_j} \rho(f_{jl}, g_{jl})\Big).$$

Then we have that ρ_v is a metric on $H^v(D_\kappa, D)$ inducing its topology.

Having the metric on $H^v(D_\kappa, D)$, we can approximate in the mean $\underline{\zeta}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, F)$ by $\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, F)$, and $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$ by $\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$.

Lemma 3. *The relation*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_v\big(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F), \underline{\zeta}_n(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F)\big)\,\mathrm{d}\tau=0$$

holds.

Proof. In [5], it was obtained that, for every compact subset $K \subset D_\kappa$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\big|\varphi(s+i\tau, F)-\varphi_n(s+i\tau, F)\big|\,\mathrm{d}\tau = 0.$$

Hence, we have that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \hat{\rho}\big(\varphi(\hat{s}+i\tau, F), \varphi_n(\hat{s}+i\tau, F)\big)\,\mathrm{d}\tau = 0. \tag{1.1}$$

Similarly, it follows from [6] that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \max_{1\le j\le r}\max_{1\le l\le l_j} \rho\big(\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}), \zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl})\big)\,\mathrm{d}\tau=0.$$

This, (1.1) and definition of the metric ρ_v prove the lemma. □

Lemma 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega} \in \underline{\Omega}$,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_v\big(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F), \underline{\zeta}_n(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)\big)\,\mathrm{d}\tau = 0.$$

Proof. In [5], it was proved that, for every compact subset $K \subset D_\kappa$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\big|\varphi(s+i\tau, \hat{\omega}, F)-\varphi_n(s+i\tau, \hat{\omega}, F)\big|\,\mathrm{d}\tau = 0$$

for almost all $\hat{\omega} \in \hat{\Omega}$. From this, we obtain that, for almost all $\hat{\omega} \in \hat{\Omega}$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \hat{\rho}\big(\varphi(\hat{s}+i\tau, \hat{\omega}, F), \varphi_n(\hat{s}+i\tau, \hat{\omega}, F)\big)\,\mathrm{d}\tau = 0. \tag{1.2}$$

Similarly, by [6], we have that, for almost all $(\omega_1, \dots, \omega_r) \in \Omega_1 \times \cdots \times \Omega_r$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \max_{1\le j\le r}\max_{1\le l\le l_j} \rho\big(\zeta(s+i\tau, \alpha_j, \omega_j; \mathfrak{a}_{jl}), \zeta_n(s+i\tau, \alpha_j, \omega_j; \mathfrak{a}_{jl})\big)\,\mathrm{d}\tau = 0. \tag{1.3}$$

Since the measure $\underline{m}_H$ is the product of the Haar measures on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$, and on $(\Omega_1 \times \cdots \times \Omega_r, \mathcal{B}(\Omega_1 \times \cdots \times \Omega_r))$, (1.2), (1.3) and the definition of the metric ρ_v imply, for almost all $\underline{\omega} \in \underline{\Omega}$, the equality of the lemma. □

For $\underline{\omega} \in \underline{\Omega}$, define one more probability measure

$$\widetilde{P}_T(A) \overset{def}{=} \nu_T\big(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\big), \quad A \in \mathcal{B}\big(H^v(D_\kappa, D)\big).$$

Lemma 5. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures P_T and $\widetilde{P}_T$ both converge weakly to the same probability measure P on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.*

Proof. Let θ be a random variable on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ which is uniformly distributed on $[0,1]$. On $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$, define the $H^v(D_\kappa, D)$-valued random element $\underline{X}_{T,n}$ by the formula

$$\underline{X}_{T,n}(\hat{s}, s) = \big(X_{T,n}(\hat{s}), X_{T,n,1,1}(s), \ldots, X_{T,n,1,l_1}(s), \ldots, X_{T,n,r,1}(s), \ldots, X_{T,n,r,l_r}(s)\big) = \underline{\zeta}_n(\hat{s} + i\theta T, s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}, F).$$

Then Lemma 2 implies the relation

$$\underline{X}_{T,n}(\hat{s}, s) \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n(\hat{s}, s), \tag{1.4}$$

where

$$\underline{X}_n(\hat{s}, s) = \big(X_n(\hat{s}), X_{n,1,1}(s), \ldots, X_{n,1,l_1}(s), \ldots, X_{n,r,1}(s), \ldots, X_{n,r,l_r}(s)\big)$$

is an $H^v(D_\kappa, D)$-valued random element with the distribution P_n in the notation of Lemma 2, and, as usual, $\xrightarrow{\mathcal{D}}$ means the convergence in distribution. We have mentioned above that the series for $\varphi_n(s, F)$ converges absolutely for $\sigma > \frac{\kappa}{2}$. Therefore, for $\sigma > \frac{\kappa}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big|\varphi_n(\sigma + it, F)\big|^2 \,\mathrm{d}t = \sum_{m=1}^{\infty} \frac{c^2(m) u_n^2(m)}{m^{2\sigma}} \le \sum_{m=1}^{\infty} \frac{c^2(m)}{m^{2\sigma}} < \infty \tag{1.5}$$

for all $n \in \mathbb{N}$, because of the Deligne [3] estimate

$$\big|c(m)\big| \ll m^{\frac{\kappa-1}{2}}.$$

Similarly, the absolute convergence of the series for $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ shows that, for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big|\zeta_n(\sigma + it, \alpha_j; \mathfrak{a}_{jl})\big|^2 \,\mathrm{d}t = \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2 u_n^2(m, \alpha_j)}{(m + \alpha_j)^{2\sigma}} \le \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m + \alpha_j)^{2\sigma}} < \infty \tag{1.6}$$

for all $n \in \mathbb{N}$. Now a simple application of the Cauchy integral formula and (1.5) lead to the inequality

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s \in \hat{K}_k} \big|\varphi_n(s + i\tau, F)\big| \,\mathrm{d}\tau \le \hat{C}_k \left(\sum_{m=1}^{\infty} \frac{c^2(m)}{m^{2\hat{\sigma}_k}} \right)^{\frac{1}{2}}, \quad n \in \mathbb{N} \tag{1.7}$$

with some $\hat{C}_k > 0$ and $\hat{\sigma}_k > \frac{\kappa}{2}$. Analogically, (1.6) shows that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_k} \left|\zeta_n(s+i\tau,\alpha_j;\mathfrak{a}_{jl})\right| \mathrm{d}\tau \le C_k\left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_k}}\right)^{\frac{1}{2}}, \quad n\in\mathbb{N} \tag{1.8}$$

with some $C_k > 0$ and $\sigma_k > \frac{1}{2}$. Here $\hat{K}_k$ and K_k are compact sets from the definition of the metric ρ_v.

We set

$$\hat{R}_k = \hat{C}_k\left(\sum_{m=1}^{\infty} \frac{c^2(m)}{m^{2\hat{\sigma}_k}}\right)^{\frac{1}{2}}, \qquad R_{jlk} = C_k\left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_k}}\right)^{\frac{1}{2}}.$$

Then, taking $\hat{M}_k = \hat{R}_k 2^{k+1}\varepsilon^{-1}$ and $M_{jlk} = R_{jlk}2^{v_1+k+1}\varepsilon^{-1}$, where $k\in\mathbb{N}$ and $\varepsilon > 0$ is an arbitrary number, we obtain by (1.7) and (1.8) that

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{P}\Big(\Big(\sup_{\hat{s}\in\hat{K}_k} \left|X_{T,n}(\hat{s})\right| > \hat{M}_k\Big) \vee \exists j,l\colon \Big(\sup_{s\in K_k} \left|X_{T,n,j,l}(s)\right| > M_{jlk}\Big)\Big) \\
&\quad\le \limsup_{T\to\infty} \mathbb{P}\Big(\sup_{\hat{s}\in\hat{K}_k} \left|X_{T,n}(\hat{s})\right| > \hat{M}_k\Big) \\
&\qquad + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{T\to\infty} \mathbb{P}\Big(\sup_{s\in K_k} \left|X_{T,n,j,l}(s)\right| > M_{jlk}\Big) \\
&\quad\le \frac{1}{\hat{M}_k} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{\hat{s}\in\hat{K}_k} \left|\varphi_n(\hat{s}+i\tau,F)\right| \mathrm{d}\tau \\
&\qquad + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{1}{M_{jlk}} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_k} \left|\zeta_n(s+i\tau,\alpha_j;\mathfrak{a}_{jl})\right| \mathrm{d}\tau \\
&\quad\le \frac{\hat{R}_k}{\hat{M}_k} + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{R_{jlk}}{M_{jlk}} = \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2^k}.
\end{aligned}$$

Using (1.4), hence, we deduce that, for all $n\in\mathbb{N}$,

$$\mathbb{P}\Big(\Big(\sup_{\hat{s}\in\hat{K}_k} \left|X_n(\hat{s})\right| > \hat{M}_k\Big) \vee \exists j,l\colon \Big(\sup_{s\in K_k} \left|X_{n,j,l}(s)\right| > M_{jlk}\Big)\Big) \le \frac{\varepsilon}{2^k}. \tag{1.9}$$

Define a set

$$\begin{aligned}
H_\varepsilon^v = \Big\{&(g, g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r}) \in H^v(D_\kappa,D)\colon \sup_{\hat{s}\in\hat{K}_k} \left|g(\hat{s})\right| \le \hat{M}_k, \\
&\sup_{s\in K_k} \left|g_{jl}(s)\right| \le M_{jlk},\ j=1,\dots,r,\ l=1,\dots,l_j,\ k\in\mathbb{N}\Big\}.
\end{aligned}$$

Then H_ε^v is a compact subset of the space $H^v(D_\kappa,D)$. Moreover, in view of (1.9),

$$\mathbb{P}\big(\underline{X}_n(\hat{s},s)\in H_\varepsilon^v\big) \ge 1-\varepsilon\sum_{k=1}^{\infty}\frac{1}{2^k} = 1-\varepsilon$$

for all $n \in \mathbb{N}$. Thus, by the definition of the random element $\underline{X}_n(\hat{s}, s)$,

$$P_n\big(H^v_\varepsilon\big) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight, and, by the Prokhorov theorem, it is relatively compact. Therefore, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $k \to \infty$. This can be written in the form

$$\underline{X}_{n_k}(\hat{s}, s) \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{1.10}$$

Define one more $H^v(D_\kappa, D)$-valued random element $\underline{X}_T(\hat{s}, s)$ by the formula

$$X_T(\hat{s}, s) = \underline{\zeta}(\hat{s} + i\theta T, s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}, F).$$

Then Lemma 3 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\rho_v\big(\underline{X}_T(\hat{s}, s), \underline{X}_{T,n}(\hat{s}, s)\big) \geq \varepsilon\big) \\
&\quad = \lim_{n\to\infty} \limsup_{T\to\infty} \nu_T\big(\rho_v\big(\underline{\zeta}(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F), \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\big) \geq \varepsilon\big) \\
&\quad \leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \rho_v\big(\underline{\zeta}(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F), \\
&\qquad \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F)\big)\, \mathrm{d}\tau = 0.
\end{aligned}$$

This, (1.4), (1.10) and Theorem 4.2 of [1] imply the relation

$$\underline{X}_T(\hat{s}, s) \xrightarrow[T\to\infty]{\mathcal{D}} P \tag{1.11}$$

and thus, P_T converges weakly to P as $T \to \infty$. The relation (1.11) also shows that the measure P is independent of the choice of the sequence $\{P_{n_k}\}$, and this yields the relation

$$\underline{X}_n(\hat{s}, s) \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{1.12}$$

It remains to show that the measure $\widetilde{P}_T$ also converges weakly to P as $T \to \infty$. We set

$$\begin{aligned}
\underline{\widetilde{X}}_{T,n}(\hat{s}, s) &= \underline{\zeta}_n(\hat{s} + i\theta T, s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F), \\
\underline{\widetilde{X}}_T(\hat{s}, s) &= \underline{\zeta}(\hat{s} + i\theta T, s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F).
\end{aligned}$$

Then the above arguments together with Lemmas 2 and 4, and relation (1.12) applied for the random elements $\underline{\widetilde{X}}_{T,n}(\hat{s}, s)$ and $\underline{\widetilde{X}}_T(\hat{s}, s)$ show that the measure $\widetilde{P}_T$ also converges weakly to P as $T \to \infty$. $\square$

In order to prove Theorem 4, it suffices to show that the limit measure P in Lemma 5 is the distribution of the random element $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$. Define $\underline{\Phi}_\tau(\underline{\omega}) = \underline{a}_\tau \underline{\omega}$, $\underline{\omega} \in \underline{\Omega}$, where $\underline{a}_\tau = \{(p^{-i\tau} : p \in \mathcal{P}), ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \ldots,$

$((m+\alpha_r)^{-i\tau}\colon m \in \mathbb{N}_0)\}$ for $\tau \in \mathbb{R}$. Then $\{\underline{\Phi}_\tau\colon \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on $\underline{\Omega}$. A set $A \in \mathcal{B}(\underline{\Omega})$ is called invariant with respect to this group if, for every $\tau \in \mathbb{R}$, the sets A and $\underline{\Phi}_\tau(A)$ may differ one from another only by $\underline{m}_H$-measure zero. The group $\{\underline{\Phi}_\tau\colon \tau \in \mathbb{R}\}$ is ergodic if its σ-field of invariant sets consists only of the sets having $\underline{m}_H$-measure zero or one.

Lemma 6. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the group $\{\underline{\Phi}_\tau\colon \tau \in \mathbb{R}\}$ is ergodic.*

Proof of the lemma is given in [7, Lemma 7].

Proof of Theorem 4. We fix a continuity set A of the limit measure P in Lemma 5. Then, using an equivalent of the weak convergence of probability measures in terms of continuity sets, Theorem 2.1 of [1], we have by Lemma 5 that

$$\lim_{T\to\infty} \nu_T \left(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right) = P(A). \tag{1.13}$$

On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the random variable $\xi(\underline{\omega})$ by the formula

$$\xi(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then the expectation

$$\mathbb{E}\xi = \underline{m}_H \left(\underline{\omega} \in \underline{\Omega}\colon \underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right) = P_{\underline{\zeta}}(A), \tag{1.14}$$

where $P_{\underline{\zeta}}$ is the distribution of the random element $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$. Lemma 6 implies the ergodicity of the process $\xi(\underline{\Phi}_\tau(\underline{\omega}))$. Therefore, by the Birkhoff–Khintchine theorem, see, for example, [2], for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi\left(\underline{\Phi}_\tau(\underline{\omega})\right) \mathrm{d}\tau = \mathbb{E}\xi. \tag{1.15}$$

However, by the the definitions of ξ and $\underline{\Phi}_\tau$, we have that

$$\frac{1}{T} \int_0^T \xi\left(\underline{\Phi}_\tau(\underline{\omega})\right) \mathrm{d}\tau = \nu_T \left(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right).$$

Therefore, taking into account (1.14) and (1.15), we obtain that

$$\lim_{T\to\infty} \nu_T \left(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right) = P_{\underline{\zeta}}(A).$$

This and (1.13) show that $P(A) = P_{\underline{\zeta}}(A)$. Since A is an arbitrary continuity set A of P, hence, $P(A) = P_{\underline{\zeta}}(A)$ for all continuity sets of P. Therefore, $P(A) = P_{\underline{\zeta}}(A)$ for all $A \in \mathcal{B}(\underline{H}^v(D_\kappa, D))$ because all continuity sets form a determining class, see [1]. This completes the proof of Theorem 4. □

2 The Support of the Measure $P_{\underline{\zeta}}$

For the proof of the Theorem 3, we need the support of the measure $P_{\underline{\zeta}}$. Since the space $H^v(D_\kappa, D)$ is separable, the support of $P_{\underline{\zeta}}$ is a minimal closed set $S_{P_{\underline{\zeta}}}$ of $H^v(D_\kappa, D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. The set $S_{P_{\underline{\zeta}}}$ consists of all points $\underline{g} \in H^v(D_\kappa, D)$ such that $P_{\underline{\zeta}}(G) > 0$ for every open neighbourhood G of $\underline{g}$.

Let

$$S_\kappa = \{g \in H(D_\kappa)\colon g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 5. *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(B_j) = l_j$, $j = 1 \dots, r$. *Then the support of $P_{\underline{\zeta}}$ is the set $S_\kappa \times H^{v_1}(D)$.*

Proof. By the definition,

$$H^v(D_\kappa, D) = H(D_\kappa) \times H^{v_1}(D).$$

Since the spaces $H(D_\kappa)$ and $H^{v_1}(D)$ are separable, it suffices [1] to consider $P_{\underline{\zeta}}(A)$ for $A = B \times C$, where $B \in \mathcal{B}(H(D_\kappa))$ and $C \in \mathcal{B}(H^{v_1}(D))$. The Haar measure $\underline{m}_H$ is the product of the Haar measures $\hat{m}_H$ and m_H^r on $(\hat{\Omega},\mathcal{B}(\hat{\Omega}))$ and $(\Omega_1 \times \cdots \times \Omega_r, \mathcal{B}(\Omega_1 \times \cdots \times \Omega_r))$, respectively. Therefore, we have that, for $A = B \times C \in \mathcal{B}(H^v(D_\kappa, D))$,

$$\begin{aligned} P_{\underline{\zeta}}(A) &= \underline{m}_H\big(\underline{\omega} \in \underline{\Omega}\colon \underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\big) \\ &= \underline{m}_H\big(\underline{\omega} \in \underline{\Omega}\colon \varphi(\hat{s}, \hat{\omega}, F) \in B, \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \\ &\quad \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in C\big) \\ &= \hat{m}_H\big(\hat{\omega} \in \hat{\Omega}\colon \varphi(\hat{s}, \hat{\omega}, F) \in B\big) \\ &\quad \times m_H^r\big((\omega_1, \dots, \omega_r) \in \Omega_1 \times \cdots \times \Omega_r\colon \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \\ &\quad \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in C\big). \end{aligned} \tag{2.1}$$

In [5], it was obtained that the support of the random element $\varphi(\hat{s}, \hat{\omega},\ F)$ is the set S_κ, i.e., S_κ is a minimal closed subset of $H(D_\kappa)$ such that

$$\hat{m}_H\big(\hat{\omega} \in \hat{\Omega}\colon \varphi(\hat{s}, \hat{\omega}, F) \in S_\kappa\big) = 1. \tag{2.2}$$

Also, in [6], it was proved that $H^{v_1}(D)$ is the support of the random element

$$\big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \\ \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),$$

i.e., $H^{v_1}(D)$ is a minimal closed set of $H^{v_1}(D)$ such that

$$m_H^r\big((\omega_1, \dots, \omega_r) \in \Omega_1 \times \cdots \times \Omega_r\colon \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \\ \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in H^{v_1}(D)\big) = 1. \tag{2.3}$$

Therefore, the theorem is a result of (2.1)–(2.3). □

3 Proof of Theorem 3

We start with the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 7. *Let K be a compact subset of the complex plane with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K}\bigl|f(s)-p(s)\bigr| < \varepsilon.$$

Proof is given in [8], [9].

Proof of Theorem 3. By Lemma 7, there exist polynomials $p(s)$ and $p_{jl}(s)$ such that

$$\sup_{s\in K}\bigl|f(s)-p(s)\bigr| < \frac{\varepsilon}{4} \tag{3.1}$$

and

$$\sup_{1\le j\le r}\ \sup_{1\le l\le l_j}\ \sup_{s\in K_{jl}}\bigl|f_{jl}(s)-p_{jl}(s)\bigr| < \frac{\varepsilon}{2}. \tag{3.2}$$

Since $f(s)\neq 0$ on K, $p(s)\neq 0$ on K as well if ε is small enough. Thus, on K we can define a continuous branch of $\log p(s)$ which will be analytic in the interior of K. Therefore, by Lemma 7, there exists a polynomial $q(s)$ such that

$$\sup_{s\in K}\bigl|p(s)-\mathrm{e}^{q(s)}\bigr| < \frac{\varepsilon}{4}.$$

This together with (3.1) shows that

$$\sup_{s\in K}\bigl|f(s)-\mathrm{e}^{q(s)}\bigr| < \frac{\varepsilon}{2}. \tag{3.3}$$

Define

$$G=\Biggl\{(g,g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r})\in H^v(D_\kappa,D)\colon \sup_{s\in K}\bigl|g(s)-\mathrm{e}^{q(s)}\bigr|<\frac{\varepsilon}{2},\ \sup_{1\le j\le r}\ \sup_{1\le l\le l_j}\ \sup_{s\in K_{jl}}\bigl|g_{jl}(s)-p_{jl}(s)\bigr|<\frac{\varepsilon}{2}\Biggr\}.$$

In view of Theorem 5, $\bigl(\mathrm{e}^{q(s)},p_{11}(s),\dots,p_{1l_1}(s),\dots,p_{r1}(s),\dots,p_{rl_r}(s)\bigr)$ is an element of the support of the measure $P_{\underline{\zeta}}$. Since the set G is open, hence, we have that $P_{\underline{\zeta}}(G)>0$. Therefore, by Theorem 4 and an equivalent of the weak convergence of probability measures in terms of open sets (Theorem 2.1 of [1]), we obtain that

$$\liminf_{T\to\infty}\nu_T\Biggl(\sup_{s\in K}\bigl|\varphi(s+i\tau,F)-\mathrm{e}^{q(s)}\bigr|<\frac{\varepsilon}{2},\ \sup_{1\le j\le r}\ \sup_{1\le l\le l_j}\ \sup_{s\in K_{jl}}\bigl|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl})-p_{jl}(s)\bigr|<\frac{\varepsilon}{2}\Biggr)>0.$$

Combining this with (3.2) and (3.3) completes the proof of the theorem. □

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