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Asymptotics of a solution to the time-periodic heat equation set in domains with corner points

Laike periodinio šilumos laidumo uždavinio sprendinio asimptotika srityse su kampiniu tašku

Master thesis

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Introduction

Many theoretical studies in applied mathematics focus on analysis of various differential equations, though it is impossible to define a universal method for solving all of them. A good example is the heat equation, for which there is no general way to find solutions without any further conditions given. The heat equation is of fundamental importance in diverse scientific fields such as theory of partial differential equations (PDEs), physics, probability theory and financial mathematics. It is a prototypical parabolic PDE. In case the heat equation is provided along with some specific conditions, it may be an easy task to find a solution. However, the more generalized solution is on demand, the more complex solving becomes and some peculiar strategies have to be used. The principal step in this paper is to reduce our problem to a sequence of equations for which essential insights can be applied from results for elliptic differential equations.

Since the theory for solving parabolic differential equations is poorly developed and on the contrary the theory for elliptic differential equations is well-developed, this thesis uses results obtained by S.A. Nazarov, B.A. Plamenevskij in the monograph "Elliptic boundary value problems in domains with piecewise smooth boundaries" and by V. Maz'ya, S. Nazarov, B. Plamenevskij in the monograph "Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains". Not only are these results important as a general scheme for analysing PDEs, but also provide specific clues about the terms in asymptotic representation formulas.

In the first chapter we define a problem for the heat equation in a bounded domain $\Omega \subset R^2$ with a corner point. We also lay out the essential idea to use Fourier series as a key allowing us to transform parabolic problem to a sequence of problems involving elliptic differential equations. In the second chapter we introduce the notation of Sobolev spaces. The principle of constructing asymptotic representation in Ω is examined in two simpler examples. In the third chapter we begin a thorough analysis of previously obtained sequence of elliptic problems aiming to attain expressions and estimates of asymptotic terms. The final chapter consists of research on dependence between time-periodicity of asymptotic terms and time-periodicity of heat equation's right-hand side function f. One of the main results states the necessary space of function f for asymptotic terms to converge in time.

Chapter 1

Statement of the problem

1.1 Problem for the heat equation

Let $K_\alpha = \{x \in \mathbb{R}^2 : r > 0, \varphi \in (0, \alpha)\}$ be a sector with the opening angle $\alpha \in (0, 2\pi)$, where (r, φ) indicates polar coordinates with the pole at the origin O. We denote by Ω a bounded domain in $\mathbb{R}^2$ such that inside the disk $B_d = \{x \in \mathbb{R}^2 : |x| < d\}$ the domain Ω coincides with the sector K_α. We also define d such that $\Omega \cap \{x \in \mathbb{R}^2 : 0 < r < d, \varphi \in (0, \alpha)\}$ is a finite sector (see Figure 1).

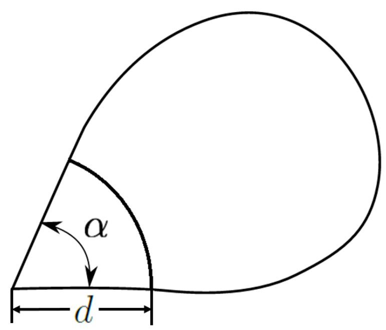

Figure 1

Let $\partial\Omega$ be the boundary of domain Ω. Let us assume that the origin O belongs to $\partial\Omega$ and the contour $\partial\Omega$ is smooth outside any neighborhood of O. We will refer to Ω as a bounded domain with a corner point.

We shall now consider in the domain Ω a time-periodic problem for the heat equation:

$$\begin{cases} u_t - \Delta u &= f, & x \in \Omega \times [0, 2\pi), \\ u &= 0, & x \in \partial\Omega \times [0, 2\pi), \\ u(x, 0) &= u(x, 2\pi), & x \in \Omega. \end{cases} \tag{1.1}$$

where f is also 2π-periodic in time. The 2π-periodicity is chosen for simplicity and any other periodicity can be reduced to the following case.

1.2 Reduction of a parabolic problem to a sequence of elliptic problems

Assume that $f \in L_2(0, 2\pi; L_2(\Omega))$. In such a case the function f can be rewritten in the following form as a Fourier series:

$$f(x,t) = \sum_{k=0}^{\infty} (f_{ck}(x)\cos(kt) + f_{sk}(x)\sin(kt)), \tag{1.2}$$

where $f_{ck} = \int\limits_0^{2\pi} f(x,t)\cos(kt)dt$ and $f_{sk} = \int\limits_0^{2\pi} f(x,t)\sin(kt)dt$, for $k = 0, 1, 2, ...$
Let us emphasize that the series (1.2) converges in $L_2(\Omega \times (0, 2\pi))$. We seek a formal solution of problem (1.1) in the form

$$u(x,t) = \sum_{n=0}^{\infty} (u_{ck}(x)\cos(kt) + u_{sk}(x)\sin(kt)). \tag{1.3}$$

Substituting relations (1.2), (1.3) into equation (1.1) and collecting terms in $\cos(kt)$ and $\sin(kt)$ separately, for every $k = 0, 1, 2, ...$ we obtain the following boundary value problem:

$$\left\{\begin{array}{rcll} -\Delta u_{ck}(x) + k u_{sk}(x) & = & f_{ck}(x), & x \in \Omega, \\ -\Delta u_{sk}(x) - k u_{ck}(x) & = & f_{sk}(x), & x \in \Omega, \\ u_{ck}(x) = u_{sk}(x) & = & 0, & x \in \partial\Omega. \end{array}\right. \tag{1.4}$$

A thorough examination of asymptotics of solutions to problem (1.4) will allow us to investigate the properties of a formal solution, satisfying the original problem (1.1) and having the form (1.3).

In the following Chapter we present certain weighted spaces and attain asymptotic representations of solutions to problems that are simpler than (1.4). In Chapter 3 we return to solving (1.4) with the right-hand side f_{ck}, f_{sk} decaying in the neighborhood of corner point and describe the asymptotic behavior of corresponding solutions u_{ck}, u_{sk}. The final chapter is dedicated to investigation of convergence of the series (1.3) in weighted spaces.

Chapter 2

Weighted function spaces. Asymptotics of solutions to related elliptic problems

2.1 Weighted Sobolev spaces

Before introducing the asymptotic representations of solutions to previously obtained equations (1.4), in this section we shall highlight the main ideas of constructing these formulas by explaining less complicated analogous cases, since the purpose of some terms might not be clear at first sight.

In the first place we specify weighted Sobolev spaces $V^l_\gamma(\Omega)$ of functions having generalized derivatives up to order l and the finite norm defined as follows:

$$\|u\|_{V^l_\gamma(\Omega)} = \left(\int\limits_\Omega \sum_{|\kappa|\le l} r^{2(\gamma-l+|\kappa|)} |\partial_x^\kappa u(x)|^2 dx \right)^{\frac{1}{2}},$$

here the weight r is equal to the distance between the origin O and $x = (x_1, x_2) \in \mathbb{R}^2$, $\kappa = (\kappa_1, \kappa_2)$ denotes a multi-index such that $\kappa_1, \kappa_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $|\kappa| = \kappa_1 + \kappa_2$.

In the following, we will concentrate on the cases where $l = 0$ and $l = 2$. The corresponding norms are defined by the formulas

$$\|u\|_{V^0_\gamma(\Omega)} = \left(\int\limits_\Omega r^{2\gamma} |u(x)|^2 dx \right)^{\frac{1}{2}},$$

$$\|u\|_{V^2_\gamma(\Omega)} = \left(\int\limits_\Omega r^{2(\gamma-2)} |u(x)|^2 dx + r^{2(\gamma-1)} \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i} \right|^2 dx + r^{2\gamma} \sum_{i,j=1}^{2} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 dx \right)^{\frac{1}{2}}.$$

Let us describe the behavior of functions from these spaces. Assume for simplicity that $\Omega = K_\alpha = \{(r, \varphi) : 0 < r < R, 0 < \varphi < \alpha\}$. Then $u = r^d$ will be an element of the space $V^0_\gamma(\Omega)$

if the following integral is finite:

$$\int\limits_{K_\alpha} r^{2\gamma}|u|^2 dx.$$

It is easy to see that

$$\int\limits_{K_\alpha} r^{2\gamma}|u|^2 dx = \int\limits_0^R \int\limits_0^\alpha r^{2\gamma}|r^d|^2 r dr d\varphi = c\, r^{2(\gamma+d+1)}\Big|_0^R < \infty$$

if $\gamma + d + 1 > 0$, or alternatively, $d > -\gamma - 1$. Analogously, we can verify that the function $u = r^d$ belongs to $V_\gamma^2(\Omega)$ if $d > -\gamma + 1$.

2.2 Examples of asymptotics of solutions to some elliptic problems

The theory concerning elliptic problems set in domains with piecewise smooth boundaries was developed in numerous works of V. Kondratyev, V. Maz'ya and his collaborators. Many results concerning asymptotic behavior of solutions to elliptic problems in domains with piecewise smooth boundaries were collected in the monographs [3], [4], [5]. In this Section we present several examples from the book [3].

Let us consider the problem

$$\begin{cases} -\Delta u(x) + nu(x) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega, \end{cases} \tag{2.1}$$

where n is a constant equal either to zero or unit, $\Omega \subset \mathbb{R}^2$ denotes a domain with a corner point (see Section 1.1).

Example 1.

In the first example we take n = 0 and thus obtain a Dirichlet problem for Laplace operator in plane domain with a corner point at the boundary:

$$\begin{cases} -\Delta u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{cases} \tag{2.2}$$

The following theorems concerning existence and uniqueness of solution and its asymptotic representation were proved in Sections 1.3.1 - 1.3.3, [3].

Theorem 1. *(see Theorem 1.3.2 in [3])*

Suppose that $|\beta - 1| < \pi/\alpha$, $f \in V^0_\beta(\Omega)$. *Then there exists a unique solution* $u \in V^2_\beta(\Omega)$ *of problem* (2.2) *and the following estimate holds:*

$$\|u\|_{V^2_\beta(\Omega)} \leq c\|f\|_{V^0_\beta(\Omega)}.$$

Theorem 2. *(see Theorem 1.3.7 and Theorem 1.3.8 in [3])*

Let $f \in V^0_\gamma(\Omega)$, $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right), m \in \mathbb{N}$ [1]. *Then the solution* u *of* (2.2) *from the space* $V^2_\beta(\Omega)$ *with* $|\beta - 1| < \pi/\alpha$ *possesses the following representation:*

$$u(x) = \chi(r) \sum_{k=1}^{m} \frac{c_k}{\sqrt{k\pi}}\, r^{k\pi/\alpha} \, \sin\left(\frac{k\pi\varphi}{\alpha}\right) + \omega(x), \tag{2.3}$$

where $\omega \in V^2_\gamma(\Omega)$, χ *is a cut-off function from* $C^\infty[0,\infty)$ *which is equal to 1 for* $r \leq d/2$ *and equal to 0 for* $r > d$. *The constants* c_k *are defined by*

[1] Let us emphasize that by taking larger m we require a higher decay rate, as $r \to 0$, from the right-hand side function f of (2.2)

$$c_k = \int_\Omega f\eta_k dx,\ k = 1, \ldots, m,$$

where

$$\eta_k = \chi(r)U_{-k}(x) + z_k(x),$$

$$U_{-k}(x) = \tfrac{1}{\sqrt{k\pi}}\ r^{-k\pi/\alpha}\ \sin(\tfrac{k\pi\varphi}{\alpha}),$$

$z_k \in V_1^2(\Omega)$ *is a solution of the problem*

$$\begin{cases} -\Delta z_k(x) &= \Delta\left(\chi(r)U_{-k}(x)\right), & x \in \bar{\Omega}\setminus 0, \\ z_k(x) &= 0, & x \in \partial\Omega \setminus 0. \end{cases}$$

Remark 1. *According to Remark 1.3.20 in [3], representation* (2.3) *is valid if* π/α *is not a rational number. Otherwise the asymptotic expansion of solution* $u \in V_\gamma^2(\Omega)$ *shall contain additional terms of type* $\log r$*. In the following we will avoid this type of generalization and assume that* $\pi/\alpha \notin \mathbb{Q}$*.*

According to Theorem 2, asymptotic representation of the solution $u \in V_\beta^2(\Omega), |\beta - 1| < \pi/\alpha$, to (2.2) consists of the following terms: $\hat{u}_1 = \frac{c_1}{\sqrt{\pi}}\ r^{\pi/\alpha}\sin(\frac{\pi\varphi}{\alpha})$, $\hat{u}_2 = \frac{c_2}{\sqrt{2\pi}}\ r^{2\pi/\alpha}\sin(\frac{2\pi\varphi}{\alpha})$, ... , $\hat{u}_m = \frac{c_m}{\sqrt{m\pi}}\ r^{m\pi/\alpha}\sin(\frac{m\pi\varphi}{\alpha})$ and ω, which belongs to $V_\gamma^2(\Omega)$. The functions $\hat{u}_k,\ k = 1, \ldots, m$, are solutions of the homogeneous problem (2.2). Let us recall that the function $u = r^d$ belongs to $V_\gamma^2(\Omega)$ only with $d > -\gamma + 1$. Since in our case

$$\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right),$$

then

$$-\gamma + 1 \in \left(\frac{m\pi}{\alpha}, \frac{(m+1)\pi}{\alpha}\right).$$

Obviously, none of the functions $\hat{u}_k,\ k = 1, \ldots, m$, belong to the space $V_\gamma^2(\Omega)$, where $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$. According to Theorem 2, ω is an element of $V_\gamma^2(\Omega)$. Therefore, it decays faster than $r^{m\pi/\alpha}$, as $r \to 0$.

As one can see from (2.3), the accuracy of asymptotics is determined by m. By taking larger m we obtain an asymptotic representation having more members and thus more accurate.

Example 2. *(see Section 1.3.6 in [3])*

In the second example we take n = 1:

$$\begin{cases} \Delta u - u &= f, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega \end{cases} \tag{2.4}$$

The first equation of this problem has an additional term on the left-hand side in comparison to problem (2.2). Due to the presence of this term, asymptotic representation of a solution is more complex. Below we will describe in details the main points of constructing asymptotic terms for the solution.

We start with the theorem concerning solvability of (2.4) in weighted Sobolev spaces.

Theorem 3. *(see Lemma 1.3.13, Theorems 1.3.14 and 1.3.18 in [3])*
Let $|\beta - 1| < \pi/\alpha$ *and* $f \in V^0_\beta(\Omega)$*. Then there exists a unique solution* $u \in V^2_\beta(\Omega)$ *of problem* (2.4)*, and the following estimate holds:*

$$\|u\|_{V^2_\beta(\Omega)} \leq c\|f\|_{V^0_\beta(\Omega)}.$$

Furthermore, assuming that f *in* (2.4) *has the form*

$$f = \chi(r) \sum_{n=0}^{N} r^{\mu_n - 2} P_n(\varphi, \log r) + \widetilde{f}(x),$$

where $1 - \beta < \mu_0 \leq \mu_q \leq \ldots \leq \mu_N \leq m\pi/\alpha$*,* $P_n(\theta, t)$ *is a polynomial in* t *whose coefficients are smooth functions of* φ *and* $\widetilde{f} \in V^0_\gamma(\Omega)$ *for some* $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$*, the following asymptotic formula for solution* u *holds:*

$$u = \chi(r) \sum_{n=0}^{N} r^{\mu_n} Q_n(\varphi, \log r) + \chi(r) \sum_{q=1}^{m} \frac{c_q}{\sqrt{q\pi}} r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + \tilde{u} \qquad (2.5)$$

Here $\tilde{u} \in V^2_\gamma(\Omega)$*,* $c_1, \ldots, c_m$ *are certain constants, and* $Q_n(\varphi, t)$ *is a polynomial in* t *of degree* κ_n*, where* $\kappa_n = \deg P_n$ *if* $\mu_n \neq k\pi/\alpha$ *for* $k = 1, \ldots, q$ *and* $\kappa_n = 1 + \deg P_n$ *otherwise.*

Remark 2. *In Example 2 we will analyse* f *of general form and will only use the existence and estimate of solution from Theorem 3, while representation* (2.5) *will be regarded in later sections.*

Let us assume that f belongs to a class of functions $V^0_\gamma(\Omega)$, where $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$, $m \in \mathbb{N}$, i.e. f decays faster than $r^{m\pi/\alpha - 2}$, as $r \to 0$. Since $V^0_\gamma \subset V^0_\beta$, we have, according to Theorem 3, for $f \in V^0_\gamma$ a solution u from $V^2_\beta(\Omega)$, with $|\beta - 1| < \pi/\alpha$. We look for the asymptotic representation of this solution in the form

$$u(x) = \chi(r) \left(\sum_{q=1}^{m} c_q \, r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + \sum_{j=1}^{M_1} u_j^{(1)}(x) + \ldots + \sum_{j=1}^{M_m} u_j^{(m)}(x) \right) + \omega(x), \qquad (2.6)$$

where $w(x) \in V^2_\gamma(\Omega)$. For $i = 1, \ldots, m$ the summation bounds M_i satisfy the inclusion

$$M_i \in \left(\frac{1}{2}\left(1 - \gamma - \frac{i\pi}{\alpha}\right) - 1, \frac{1}{2}\left(1 - \gamma - \frac{i\pi}{\alpha}\right) \right].^2 \qquad (2.7)$$

[2] If we take, for example, $\alpha = \pi/2$, $m = 4$, then $\gamma \in (-9, -7)$. Let us take $\gamma = -8$. Then, according to formula (2.6), $M_i \in (7/2 - i, 9/2 - i)$, i.e. $M_1 = 3, M_2 = 2, M_3 = 1, M_4 = 0$.

The formula for M_i will be explained later in this example. We aim to find an asymptotic representation in the corner point, therefore we will assume for a moment that we have an infinite corner K_α (see Section 1.1) and will take $\chi \equiv 1$. After we find the asymptotic representation of solution in the corner K_α, we will multiply it by the cut-off function χ in order to obtain the asymptotic representation of solution to the problem set in the whole domain.

Let us explain in more detailed way the procedure of constructing terms $\sum\limits_{j=1}^{M_i} u_j^{(i)}$ in representation (2.6). First of all let us notice that the functions $r^{q\pi/\alpha}\sin\left(\frac{q\pi\varphi}{\alpha}\right)$ satisfy the Laplace equation [3]

$$\Delta\left(r^{q\pi/\alpha}\sin\left(\frac{q\pi\varphi}{\alpha}\right)\right) = 0,\ q = 1,\ldots,m. \tag{2.8}$$

Now we investigate the construction of terms $\sum\limits_{j=1}^{M_1} u_j^{(1)}(x)$. By taking $M_1 = 1$ in (2.6), we get the following:

$$u(x) = \sum_{q=1}^{m} c_q r^{q\pi/\alpha}\sin\left(\frac{q\pi\varphi}{\alpha}\right) + u_1^{(1)} + \omega(x). \tag{2.9}$$

Employing (2.9) into (2.4), taking into account (2.8) and leaving only the term $\Delta u_1^{(1)}$ on the left-hand side we obtain

$$\Delta u_1^{(1)} = c_1 r^{\pi/\alpha}\sin\left(\frac{\pi\varphi}{\alpha}\right) + \sum_{q=2}^{m} c_q r^{q\pi/\alpha}\sin\left(\frac{q\pi\varphi}{\alpha}\right) + u_1^{(1)} + \omega + f - \Delta\omega.$$

Since $c_1 r^{\pi/\alpha}\sin\left(\frac{\pi\varphi}{\alpha}\right)$ is the term that approaches zero slower than any other term of the right-hand side, as $r \to 0$, it will thus determine the asymptotic behavior of $u_1^{(1)}$. Cancelling the rest of the terms, we obtain the following problem:

$$\begin{cases} \Delta u_1^{(1)} &= c_1 r^{\pi/\alpha}\sin\left(\frac{\pi\varphi}{\alpha}\right), & \textit{in } K_\alpha, \\ u_1^{(1)} &= 0, & \textit{on } \partial K_\alpha. \end{cases} \tag{2.10}$$

Looking for the function $u_1^{(1)}$ in the form

$$u_1^{(1)} = r^{\pi/\alpha+2}U_1^{(1)}(\varphi),$$

and substituting this expression into (2.10), we get for the function $U_1^{(1)}(\varphi)$ a boundary value problem for the second order differential equation

$$\begin{cases} \frac{\partial^2 U_1^{(1)}(\varphi)}{\partial\varphi^2} + \left(\frac{\pi}{\alpha}+2\right)^2 U_1^{(1)}(\varphi) = c_1\sin\left(\frac{\pi\varphi}{\alpha}\right),\ \varphi \in (0,\alpha), \\ U_1^{(1)}(0) = 0,\ U_1^{(1)}(\varphi) = 0. \end{cases}$$

By solving it we obtain $U_1^{(1)} = \frac{c_1\alpha}{4(\pi+\alpha)}\sin\left(\frac{\pi\varphi}{\alpha}\right)$ and finally deduce that

$$u_1^{(1)} = \frac{c_1\alpha}{4(\pi+\alpha)}\, r^{\pi/\alpha+2}\sin\left(\frac{\pi\varphi}{\alpha}\right). \tag{2.11}$$

[3] The Laplace operator in polar coordinates is given by $\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2}$

If $\pi/\alpha + 2 > -\gamma + 1$, then the function $u_1^{(1)} \in V_\gamma^2(\Omega)$ and $u_1^{(1)}$ is represented by ω. Consequently, we do not include it in the asymptotic representation formula as a separate member. In the other case, when $\pi/\alpha + 2 \leq -\gamma + 1$, $u_1^{(1)} \notin V_\gamma^2(\Omega)$ and thus $u_1^{(1)}$ has to be included into the asymptotic representation as a separate term. Assume, that $\pi/\alpha + 2 \leq -\gamma + 1$ and take $M_1 = 2$. Expression (2.6) now takes the following form

$$u(x) = \sum_{q=1}^{m} c_q r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + u_1^{(1)} + u_2^{(1)} + \omega(x), \tag{2.12}$$

with $u_1^{(1)}$ defined by (2.11). Substituting (2.12) into (2.4) delivers

$$\Delta u_2^{(1)} = \sum_{q=1}^{k} c_q r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + u_1^{(1)} + u_2^{(1)} + \omega - \Delta u_1^{(1)} - \Delta\omega + f. \tag{2.13}$$

Since $\Delta u_1^{(1)} = c_1 r^{\pi/\alpha} \sin\left(\frac{\pi\varphi}{\alpha}\right)$, two terms on the right-hand side of (2.13) cancel and we reduce this equation to

$$\Delta u_2^{(1)} = \sum_{q=2}^{m} c_q r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + u_1^{(1)} + u_2^{(1)} + \omega - \Delta\omega + f. \tag{2.14}$$

Let us assume that in the right-hand side of (2.14) the function $u_1^{(1)}$ is the term which approaches zero at a slowest rate, as $r \to 0$. Analogously, we obtain for the function $u_2^{(1)}$ the problem

$$\begin{cases} \Delta u_2^{(1)} &= \frac{c_1\alpha}{4(\pi+\alpha)} r^{\pi/\alpha+2} \sin\left(\frac{\pi\varphi}{\alpha}\right), & \textit{in } K_\alpha, \\ u_2^{(1)} &= 0, & \textit{on } \partial K_\alpha. \end{cases}$$

Seeking $u_2^{(1)}$ in the form $r^{\pi/\alpha+4}\ U_2^{(1)}(\varphi)$, we obtain a boundary value problem for the function $U_2^{(1)}(\varphi)$:

$$\begin{cases} \frac{\partial^2 U_2^{(1)}}{\partial\varphi^2}(\varphi) + (\frac{\pi}{\alpha}+4)^2\ U_2^{(1)}(\varphi) = \frac{c_1\alpha}{4(\pi+\alpha)}\ r^{\pi/\alpha+2} \sin\left(\frac{\pi\varphi}{\alpha}\right),\ \varphi \in (0,\alpha), \\ U_2^{(1)}(0) = 0,\ U_2^{(1)}(\alpha) = 0, \end{cases}$$

from which we get $u_2^{(1)} = \frac{c_1\alpha}{4(\pi+\alpha)} \frac{\alpha}{8(\pi+2\alpha)}\ r^{\pi/\alpha+4} \sin\left(\frac{\pi\varphi}{\alpha}\right)$. If $\pi/\alpha + 4 > -\gamma + 1$, then $u_2^{(1)} \in V_\gamma^2(\Omega)$ and we incorporate this term into ω. Then $M_1 = 1$. Otherwise, if $\pi/\alpha + 4 \leq -\gamma + 1$, i.e. $u_2^{(1)} \notin V_\gamma^2(\Omega)$, we include this function into asymptotic representation (2.6) as a separate term and continue the above procedure to construct functions $u_3^{(1)} \sim r^{\pi/\alpha+6}$, $u_4^{(1)} \sim r^{\pi/\alpha+8}$, ... until we reach, after a finite number of steps, the situation when $u_{M_1}^{(1)} \sim r^{\pi/\alpha+2M_1}$ is such that $u_{M_1}^{(1)} \notin V_\gamma^2(\Omega)$, while $u_{M_1+1}^{(1)} \sim r^{\pi/\alpha+2(M_1+1)}$ belongs to $V_\gamma^2(\Omega)$. This condition is satisfied if, on one hand, $\pi/\alpha+2(M_1+1) > -\gamma+1$, on the other hand, $\pi/\alpha+2M_1 \leq -\gamma+1$. These conditions are equivalent to, respectively, conditions $M_1 > \frac{1}{2}\left(1-\gamma-\frac{\pi}{\alpha}\right) - 1$ and $M_1 \leq \frac{1}{2}\left(1-\gamma-\frac{\pi}{\alpha}\right)$. Therefore, M_1 is an integer from the interval

$$M_1 \in \left(\frac{1}{2}\left(1-\gamma-\frac{\pi}{\alpha}\right) - 1, \frac{1}{2}\left(1-\gamma-\frac{\pi}{\alpha}\right)\right]. \tag{2.15}$$

Detailed computations yield that the functions $u_1^{(1)}, \dots, u_{M_1}^{(1)}$ are defined by the formulas

$$u_j^{(1)} = \frac{c_1 \; r^{\pi/\alpha+2j} \; \alpha^j}{j! \; 4^j \prod\limits_{i=1}^{j} (\alpha + i\pi)} \sin\left(\frac{\pi\varphi}{\alpha}\right).$$

Now we can repeat the same procedure for the terms $\sum\limits_{j=1}^{M_2} u_j^{(2)}, \dots, \sum\limits_{j=1}^{M_m} u_j^{(m)}$. In the same way we obtain for $j = 1, \dots, M_q, \; q = 1, \dots, m$

$$u_j^{(q)} = \frac{c_q \; r^{q\pi/\alpha+2j} \; \alpha^j}{j! \; 4^j \prod\limits_{i=1}^{j} (\alpha + i\pi)} \sin\left(\frac{q\pi\varphi}{\alpha}\right). \tag{2.16}$$

We wish to emphasize once again that these functions, namely $u_1^{(q)}, \dots, u_{M_q}^{(q)}, \; q = 1, \dots, m$, do not belong to the space $V_\gamma^2(\Omega)$, while the functions $u_{M_q+1}^{(q)}(\Omega)$, $q = 1, \dots, m$, should be elements of $V_\gamma^2(\Omega)$. Therefore, considering the fact that the integral in the norm of $V_\gamma^2(\Omega)$ has to converge for a function to belong to $V_\gamma^2(\Omega)$, the bounds of $M_i, i = 1, \dots, m$, follow from this reasoning. Arguing in the same way as above, we find that for $i = 1, \dots, m$

$$M_i \in \left(\frac{1}{2}\left(1 - \gamma - \frac{i\pi}{\alpha}\right) - 1, \frac{1}{2}\left(1 - \gamma - \frac{i\pi}{\alpha}\right)\right].$$

Now we will explain the procedure for determining constants c_q. First of all, we introduce functions ξ_q, satisfying the homogeneous problem

$$\begin{cases} \Delta\xi_q - \xi_q &= 0, \quad x \in \Omega, \\ \xi_q &= 0, \quad x \in \partial\Omega. \end{cases} \tag{2.17}$$

We look for ξ_q in the form

$$\xi_q = \chi z_q^{(0)} + \tilde{z}_q, \tag{2.18}$$

where

$$z_q^{(0)} = r^{-q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right).$$

This function serves as a principal term to the asymptotics of ξ_q and satisfies the equation $\Delta z_q^{(0)} = 0$. Substituting (2.18) into (2.17) we get for $\tilde{z}_q$ the problem

$$\Delta\tilde{z}_q - \tilde{z}_q = -\Delta(\chi z_q^{(0)}) + \chi z_q^{(0)} := f_0. \tag{2.19}$$

Since the cut-off function satisfies $\chi = 1$ for $r \le d/2$, the terms $\Delta\chi$ and $\nabla\chi$ are equal to zero in the neighborhood of the corner point. Therefore, support of the function $\Delta(\chi z_q^{(0)})$ is detached from O and the right-hand side f_0 of (2.19) coincides in the neighborhood of O with the term $z_q^{(0)}$. If $z_q^{(0)} \in V_\beta^0(\Omega)$ with some β such that $|\beta - 1| < \pi/\alpha$, then we have a unique solution

$\tilde{z}_q \in V^2_\beta(\Omega)$ of problem (2.17). We can easily check that $z \sim r^{-q\pi/\alpha} \in V^2_\beta(\Omega)$ if $q\pi/\alpha < 2+\pi/\alpha$. In this case we have $\xi_q = \chi z^{(0)}_q + \tilde{z}_q$. Furthermore, we also deduce that $\tilde{z}_q \sim r^{-q\pi/\alpha+2}$. However, if $q\pi/\alpha \geq 2+\pi/\alpha$, then instead of (2.18) we set $\xi_q = \chi(z^{(0)}_q + z^{(1)}_q) + \tilde{z}_q$, where $z^{(1)}_q$ is a solution of the problem

$$\begin{cases} \Delta z^{(1)}_q(x) &= z^{(0)}_q(x), \quad x \in \Omega, \\ z^{(1)}_k(x) &= 0, \qquad\quad x \in \partial\Omega. \end{cases}$$

By solving it we obtain

$$z^{(1)}_q(x) = \tfrac{\alpha r^{2-q\pi/\alpha} \sin(q\pi\varphi/\alpha)}{4(\alpha-k\pi)}.$$

Then $\tilde{z}_q$ is a solution of the problem

$$\Delta\tilde{z}_q - \tilde{z}_q = \tilde{f}_q + \chi z^{(1)}_q := f_q,$$

where $\tilde{f}_q$ incorporates all terms with supports not containing the corner point of Ω.

Going further, in cases $q\pi/\alpha \geq 4+\pi/\alpha$ (i.e. $f_1 \in V^0_\beta(\Omega)$ with some $\beta \in (1-\frac{\pi}{\alpha}, 1+\frac{\pi}{\alpha})$, we set $\xi_q = \chi(z^{(0)}_q + z^{(1)}_q) + \tilde{z}_q$ with $\tilde{z}_q \in V^2_\beta(\Omega)$ being a solution to (2.17). Otherwise we repeat the described procedure. Let m be an index defined by $\frac{\pi(k-1)}{2\alpha} - 1 < m < \frac{\pi(k-1)}{2\alpha}$. We continue the procedure until we reach a right-hand side $f_m \in V^0_\gamma(\Omega)$, $|1-\beta| < \pi/\alpha$ and after that define the function ξ_q by the following relation

$$\xi_q(x) = \chi(x) \sum_{j=0}^{m} z^{(j)}_q(x) + \tilde{z}_q(x), \tag{2.20}$$

where

$$z^{(j)}_q(x) = \left(\frac{\alpha}{4}\right)^j r^{-q\pi/\alpha+2j} \left(j! \prod_{n=1}^{j} (n\alpha - q\pi)\right)^{-1} \sin\left(\frac{q\pi\varphi}{\alpha}\right) \tag{2.21}$$

and $\tilde{z}_q$ is a solution to (2.17) with the right-hand side

$$f_m = \chi(x) z^{(m)}_q(x) - 2\nabla\chi(x)\nabla \sum_{j=0}^{m} z^{(j)}_k(x) - \Delta\chi(x) \sum_{j=0}^{m} z^{(j)}_k(x).$$[4]

Now we can derive expressions for the constants c_q in (2.6) and (2.16). We denote $\Omega_\delta = \Omega \setminus B_\delta$, where $B_\delta = \{x \in R^2 : |x| \leq \delta\}$. Multiplying $(2.4)_1$ by ξ_q and integrating over Ω_δ delivers

$$\int_{\Omega_\delta} f\xi_q dx = \int_{\Omega_\delta} (\Delta u - u)\xi_q dx. \tag{2.22}$$

We now apply integration by parts for the right-hand side of (2.22) and thus obtain

$$\int_{\Omega} ((\Delta u - u)\xi_q - (\Delta\xi_q - \xi_q)u)\, dx = \int_{\partial\Omega} \left(\xi_q \frac{\partial u}{\partial n} - u\frac{\partial \xi_q}{\partial n}\right) dS.$$

[4]Formulas (2.19)-(2.21) were derived in Section 1.3.6 in [3]

Since $\Delta\xi_q - \xi_q = 0$, we get

$$\int\limits_{\Omega_\delta} f\xi_q dx = \int\limits_{\partial\Omega_\delta} \left(\xi_q \frac{\partial u}{\partial n} - u\frac{\partial \xi_q}{\partial n}\right) dS.$$

$\partial\Omega = (\partial B_\delta \cap \Omega) \cup (\partial\Omega_\delta \setminus B_\delta)$ and the functions u and ξ_q equal zero in $\partial\Omega_\delta \setminus B_\delta$, therefore

$$\int\limits_{\Omega_\delta} f\xi_q dx = \int\limits_{\partial B_\delta \cap \Omega} \left(\xi_q \frac{\partial u}{\partial n} - u\frac{\partial \xi_q}{\partial n}\right) dS. \tag{2.23}$$

Converting from Cartesian coordinates to polar coordinates, also adding into account that $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ and $r = \delta$ in $\partial B_\delta \cap \Omega$ delivers

$$\int\limits_{\Omega_\delta} f\xi_k dx = \int\limits_0^\alpha \left(u\frac{\partial \xi_k}{\partial r} - \xi_k \frac{\partial u}{\partial r}\right) r \Bigg|_{r=\delta} d\varphi. \tag{2.24}$$

The following equation holds for the terms $\sin\left(\frac{q\pi\varphi}{\alpha}\right)$, with $q, j \in \mathbb{Z}$:

$$\int\limits_0^\alpha \sin\left(\frac{q\pi\varphi}{\alpha}\right) \sin\left(\frac{j\pi\varphi}{\alpha}\right) d\varphi = \begin{cases} 0, & j \neq q, \\ \alpha/2, & j = q, \end{cases} \tag{2.25}$$

therefore we obtain

$$\int\limits_{\Omega_\delta} f\xi_q dx = -q\pi c_q + \int\limits_0^\alpha F(\delta,\varphi)\ d\varphi, \tag{2.26}$$

where the term $F(\delta, \varphi)$ [5] stands for various multiplication products, each equivalent to some δ^m with a positive integer power of m. It was shown in Section 1.3.6 in [3] that $\int\limits_0^\alpha F(\delta,\varphi)\ d\varphi \xrightarrow{\delta \to 0} 0$. Therefore, as $\delta \to 0$, (2.26) tends to a limit

$$c_q = -\frac{1}{q\pi} \int\limits_\Omega f\xi_q dx,\ q = 1, \ldots, k.$$

[5] We will analyse the function $F(\delta, \varphi)$ more precisely in Section 3.4.

Chapter 3

Asymptotic representation of solutions to problems related to the heat equation

3.1 Elliptic systems

We now return to solving the sequence of problems for $k \in \mathbb{N}_0$:

$$\begin{cases} -\Delta u_{ck} + k u_{sk} &= f_{ck}, \qquad x \in \Omega, \\ -\Delta u_{sk} - k u_{ck} &= f_{sk}, \qquad x \in \Omega, \\ u_{ck} = 0, \ u_{sk} &= 0, \qquad x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let us assume that $\mathbf{f}_k = (f_{ck}, f_{sk}) \in L_2(\Omega)$. In a standard way (see, e.g., [6]) one can show that for every $k \in \mathbb{N}$ problem (3.1) has a unique generalized solution $\mathbf{u}_k = (u_{ck}, u_{sk}) \in W_2^1(\Omega)$ and derive the estimate

$$\|\mathbf{u}_k\|_{W_2^1(\Omega)} \leq c\|\mathbf{f}_k\|_{L_2(\Omega)}. \tag{3.2}$$

Moreover, for any positive constant d the inclusion $\mathbf{u}_k \in W_2^2(\Omega \setminus B_d)$ is valid and the estimate

$$\|\mathbf{u}_k\|_{W_2^2(\Omega\setminus B_d)} \leq C\left(\|\mathbf{f}_k\|_{L_2(\Omega\setminus B_{d/2})} + \|\mathbf{u}_k\|_{W_2^1(\Omega\setminus B_{d/2})}\right) \tag{3.3}$$

holds.

Following the ideas of Section 1.3.1, [3] we will show the inclusion $\mathbf{u}_k \in V_1^2(\Omega)$ and validity of the estimate

$$\|\mathbf{u}_k\|_{V_1^2(\Omega)} \leq C\|\mathbf{f}_k\|_{V_1^0(\Omega)}, \tag{3.4}$$

i.e., the inequality

$$\int\limits_{\Omega} |\mathbf{u}_k|^2 r^{-2} + |\nabla \mathbf{u}_k|^2 + \sum_{|\kappa|=2} |D_x^\kappa \mathbf{u}_k|^2 r^2 dx \le C \int\limits_{\Omega} |\mathbf{f}_k|^2 r^2 dx.$$

For sufficiently small d the intersection $\Omega \cap B_d$ coincides with the corner $K_d = \{(r, \varphi : 0 < r < d, 0 < \varphi < \alpha)\}$. Since $\mathbf{u}_k$ is equal to zero on $\partial\Omega \setminus \{O\}$, for any $r \in (0, d)$ Poincare's inequality implies

$$\int\limits_0^\alpha |u_{ck}(r,\varphi)|^2 d\varphi \le C \int\limits_0^\alpha \left| \frac{\partial}{\partial \varphi} u_{ck}(r,\varphi) \right|^2 d\varphi.$$

Using this we get

$$\int\limits_{\Omega\cap B_d} |u_{ck}(x)|^2 r^{-2} dx \le C \int\limits_{\Omega\cap B_d} \left| \frac{\partial}{\partial \varphi} u_{ck}(r,\varphi) \right|^2 r^{-2} d \le C \int\limits_{\Omega\cap B_d} |\nabla u_{ck}|^2 dx. \tag{3.5}$$

Analogous inequality is valid for the function u_{sk}, therefore the generalized solution $\mathbf{u}_k \in W_2^1(\Omega)$ satisfies

$$\int\limits_{\Omega} \left(|u_{ck}|^2 + |u_{sk}|^2 \right) r^{-2} dx + \int\limits_{\Omega} \left(|\nabla u_{ck}|^2 + |\nabla u_{sk}|^2 \right) dx \le C \int\limits_{\Omega} (|f_{ck}|^2 + |f_{sk}|^2) dx$$

and, consequently, belongs to the weighted space $V_0^1(\Omega)$.

Multiplying $(3.1)_1$ by u_{ck}, $(3.1)_2$ by u_{sk}, integrating over Ω and applying standard inequalities we get

$$\int\limits_{\Omega} \left(|\nabla u_{ck}|^2 + |\nabla u_{sk}|^2 \right) dx = \int\limits_{\Omega} (f_{ck} u_{ck} + f_{sk} u_{sk}) dx \le C_\varepsilon \int\limits_{\Omega} (|f_{ck}|^2 + |f_{sk}|^2) r^2 dx$$
$$+\varepsilon \int\limits_{\Omega} (|u_{ck}|^2 + |u_{sk}|^2) r^{-2} dx.$$

Using estimates of type (3.5) and taking suitable ε we conclude from the last inequality that

$$\int\limits_{\Omega} \left(|\nabla u_{ck}|^2 + |\nabla u_{sk}|^2 \right) dx \le C \int\limits_{\Omega} (|f_{ck}|^2 + |f_{sk}|^2) r^2 dx.$$

Multiplying $(3.1)_1$ by u_{sk}, $(3.1)_2$ by u_{ck} in a similar way we derive the estimate

$$\int\limits_{\Omega} \left(|u_{ck}|^2 + |u_{sk}|^2 \right) dx \le \frac{C}{k} \int\limits_{\Omega} (|f_{ck}|^2 + |f_{sk}|^2) r^2 dx.$$

Since the inequality $\int\limits_{\Omega} |v|^2 r^2 dx \le C \int\limits_{\Omega} |v|^2 dx$ is valid for any $v \in L_2(\Omega)$, we see from the two last estimates that

$$\int\limits_{\Omega} \left(|\nabla u_{ck}|^2 + |\nabla u_{sk}|^2 \right) r^2 dx + k \int\limits_{\Omega} \left(|u_{ck}|^2 + |u_{sk}|^2 \right) r^2 dx \le C \int\limits_{\Omega} (|f_{ck}|^2 + |f_{sk}|^2) r^2 dx. \tag{3.6}$$

Now we rearrange $(3.1)_1$ by moving the term ku_{sk} to the right-hand side of equation and treat u_{ck} as a solution to the problem

$$\begin{cases} -\Delta u_{ck} &=& f_{ck} + ku_{sk}, \quad x \in \Omega, \\ u_{ck} &=& 0, \qquad\qquad\; x \in \partial\Omega. \end{cases}$$

It is easy to see that the product χu_{ck} (a restriction of u_{ck} in the corner part of Ω) satisfies in sector K the following boundary value problem:

$$\begin{cases} -\Delta(\chi u_{ck}) &=& f_{ck} + ku_{sk} - 2\nabla\chi \cdot \nabla u_{ck} - u_{ck}\Delta\chi =: F_k, \quad x \in K, \\ \chi u_{ck} &=& 0, \quad x \in \partial K \setminus \{O\}. \end{cases}$$

Estimate (3.6) yields the inclusion $F_k \in V_1^0(K)$. Therefore, according to Theorem 1.2.1 in [3], χu_{ck} belongs to $V_1^2(K)$ and the estimate

$$\|\chi u_{ck}\|_{V_1^2(K)} \leq C\|F_k\|_{V_1^0(K)}$$

holds. Taking into account the structure of F_k we get that

$$\begin{aligned} \|F_k\|_{V_1^0(K)} \leq C \left(\|f_{ck}\|_{V_1^0(K)} + k\|\chi u_{sk}\|_{V_1^0(K)} + \|\nabla\chi \cdot \nabla u_{ck}\|_{V_1^0(K)} + \|(\Delta\chi)u_{ck}\|_{V_1^0(K)}\right) \\ \leq C\left(\|f_{ck}\|_{V_1^0(\Omega)} + \|f_{sk}\|_{V_1^0(\Omega)}\right). \end{aligned}$$

Here we exploited estimate (3.6) and the fact that, due to the cut-off function χ, supports of χu_{sk}, $\nabla\chi \cdot \nabla u_{ck}$ and $(\Delta\chi)u_{ck}$ are located in the intersection $K \cap \Omega$. Two last estimates yield inequality

$$\|\chi u_{ck}\|_{V_1^2(K)} \leq C\left(\|f_{ck}\|_{V_1^0(\Omega)} + \|f_{sk}\|_{V_1^0(\Omega)}\right). \tag{3.7}$$

Applying analogous procedure for $(3.1)_2$, we can see that inclusion $\chi u_{sk} \in V_1^0(K)$ is valid and the following estimate holds

$$\|\chi u_{sk}\|_{V_1^2(K)} \leq C\left(\|f_{ck}\|_{V_1^0(\Omega)} + \|f_{sk}\|_{V_1^0(\Omega)}\right). \tag{3.8}$$

Let us notice that the norms $\|\cdot\|_{L_2(\widehat{\Omega})}$ and $\|\cdot\|_{V_\beta^0(\widehat{\Omega})}$ are equivalent for any $\beta \in \mathbb{R}$ if $\widehat{\Omega}$ is a bounded domain detached from the origin O, for example, if $\widehat{\Omega} = \Omega \setminus B_d$. Therefore we have from (3.3)

$$\|\mathbf{u}_k\|_{V_1^2(\Omega\setminus B_d)} \leq c\left(\|\mathbf{f}_k\|_{V_1^0(\Omega\setminus B_{d/2})} + \|\mathbf{u}_k\|_{V_1^0(\Omega\setminus B_{d/2})} + \|\nabla\mathbf{u}_k\|_{V_1^0(\Omega\setminus B_{d/2})}\right).$$

This estimate together with inequalities (3.7), (3.8) and (3.6) implies the estimate (3.4).

3.2 Asymptotic expression

Assume now that $\mathbf{f}_k = (f_{ck}, f_{sk})$ in problem (3.1) belongs to the space $V^0_\gamma(\Omega)$, where $\gamma \in (1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha})$ for some $m \in \mathbb{N}$. It is obvious that $\gamma < 1$. Consequently $V^0_\gamma(\Omega) \subset V^0_1(\Omega)$ and, according to results of the previous Section, there exists a unique solution $\mathbf{u}_k = (u_{ck}, u_{sk}) \in V^2_1(\Omega)$ of (3.1). Following the scheme presented in Section 1.3.6 of [3], where problem (2.4) was considered, one may obtain the asymptotic representation of the solution of (3.1). Since this problem has a structure similar to problem (2.4), the representation $\mathbf{u}_k$ shall have a form similar to (2.5). However, asymptotic expansion of $\mathbf{u}_k$ shall depend on the parameter k explicitly. This dependence becomes very important when one analyses convergence of the series (1.3) to the solution of the time-periodic problem (1.1). Therefore we will present below the most important steps of construction of asymptotic formula and will emphasize derivation of estimates which depend on k explicitly.

As in the previous Section, one shall rearrange system (3.1) and consider functions u_{ck} and u_{sk} as solutions to problems

$$\begin{cases} -\Delta u_{ck} &= F_{ck}, \quad x \in \Omega, \\ \quad u_{ck} &= 0, \quad\ \ x \in \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta u_{sk} &= F_{sk}, \quad x \in \Omega, \\ \quad u_{sk} &= 0, \quad\ \ x \in \partial\Omega, \end{cases} \tag{3.9}$$

$$F_{ck} = f_{ck} + k u_{sk}, \qquad F_{sk} = f_{sk} - k u_{ck}.$$

Since $u_{ck}, u_{sk} \in V^2_1(\Omega) \subset V^0_{-1}(\Omega)$, the functions F_{ck} and F_{sk} belong to $V^0_{\beta_1}(\Omega)$, for $\beta_1 = \max\{-1, \gamma\}$.

First we consider the case $-1 \le \gamma$, i.e., $\beta_1 = \gamma$. Taking into account the assumption that $1 - \frac{(m+1)\pi}{\alpha} < \gamma < 1 - \frac{m\pi}{\alpha}$, we apply Theorem 2 and conclude that u_{ck} and u_{sk} admit representations

$$u_{ck} = \chi \sum_{q=1}^{m} c^{(q)}_{ck} r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + \widetilde{u}_{ck}, \qquad u_{sk} = \chi \sum_{q=1}^{m} c^{(q)}_{sk} r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) + \widetilde{u}_{sk}. \tag{3.10}$$

Here functions $\widetilde{u}_{ck}$, $\widetilde{u}_{sk}$ belong to the space $V^2_\gamma(\Omega)$ and satisfy estimates

$$\|\widetilde{u}_{ck}\|_{V^2_\gamma(\Omega)} \le C\|f_{ck} + k u_{sk}\|_{V^0_\gamma(\Omega)}, \quad \|\widetilde{u}_{sk}\|_{V^2_\gamma(\Omega)} \le C\|f_{sk} - k u_{ck}\|_{V^0_\gamma(\Omega)}.$$

Since $-1 \le \gamma < 1$ and (3.4) holds, we have the following inequalities

$$\|\mathbf{u}_k\|_{V^0_\gamma(\Omega)} \le C\|\mathbf{u}_k\|_{V^0_{-1}(\Omega)} \le C\|\mathbf{f}_k\|_{V^0_1(\Omega)} \le C\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)}.$$

Hence for the function $\widetilde{\mathbf{u}}_k = (\widetilde{u}_{ck}, \widetilde{u}_{sk})$ the estimate

$$\|\widetilde{\mathbf{u}}_k\|_{V^2_\gamma(\Omega)} \le C(1+k)\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)} \tag{3.11}$$

is valid. In this case, construction of asymptotic representation is completed.

Let us now consider the case $\gamma < -1$, i.e., when the function $\mathbf{F}_k = (F_{ck}, F_{sk})$ in (3.9) belongs to $V^0_{-1}(\Omega)$. The further step depends on whether the interval $[-1, 1)$ contains any of numbers $\lambda_q = 1 - q\pi/\alpha$, $q = 1, \ldots, m$, or not. We will start with the first situation.

If $\lambda_1, \ldots, \lambda_{Q_1} \in [-1, 1)$, then applying Theorem 2 we obtain asymptotic representations

$$u_{ck} = \chi \sum_{q=1}^{Q_1} u^{(q0)}_{ck} + \widetilde{u}^{(1)}_{ck}, \quad u_{sk} = \chi \sum_{q=1}^{Q_1} u^{(q0)}_{sk} + \widetilde{u}^{(1)}_{sk}. \tag{3.12}$$

Here

$$u^{(q0)}_{ck} = c^{(q)}_{ck} r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \quad u^{(q0)}_{sk} = c^{(q)}_{sk} r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \tag{3.13}$$

while $\widetilde{\mathbf{u}}^{(1)}_k = (\widetilde{u}^{(1)}_{ck}, \widetilde{u}^{(1)}_{sk})$ belongs to $V^2_{-1}(\Omega)$ and satisfies inequalities

$$\|\widetilde{\mathbf{u}}^{(1)}_k\|_{V^2_{-1}(\Omega)} \le C\|\mathbf{F}_k\|_{V^0_{-1}(\Omega)} \le Ck\|\mathbf{f}_k\|_{V^0_{-1}(\Omega)} \le Ck\|\mathbf{f}_k\|_{V^0_{\gamma}(\Omega)}. \tag{3.14}$$

Notice that the numbers $\lambda_1, \ldots, \lambda_{Q_1}$ belong to the interval $[-1, 1)$ if $Q_1 = \max\{q \in \mathbb{N} : q \le 2\alpha/\pi\}$, i.e., if $Q_1 = [2\alpha/\pi]$ ($[z]$ denotes an integer part of a number z).

If $\lambda_1 < -1$, the function $\mathbf{u}_k$ itself belongs to the space $V^2_{-1}(\Omega)$ and the estimate

$$\|\mathbf{u}_k\|_{V^2_{-1}(\Omega)} \le Ck\|\mathbf{f}_k\|_{V^0_{\gamma}(\Omega)} \tag{3.15}$$

holds. This estimate yields inequality $\|\mathbf{u}_k\|_{V^0_{-3}(\Omega)} \le Ck\|\mathbf{f}_k\|_{V^0_{\gamma}(\Omega)}$. Consequently F_{ck} and F_{sk} in (3.9) belong to $V^0_{\beta_2}(\Omega)$ for $\beta_2 = \max\{-3, \gamma\}$. If $\lambda_1 < -3$ (in this case also $\gamma < -3$) we get, in the same way as above, inclusion $\mathbf{u}_k \in V^2_{-3}(\Omega)$ and estimates

$$\|\mathbf{u}_k\|_{V^2_{-3}(\Omega)} \le C\left(\|\mathbf{f}_k\|_{V^0_{-3}(\Omega)} + k\|\mathbf{u}_k\|_{V^0_{-3}(\Omega)}\right) \le Ck^2\|\mathbf{f}_k\|_{V^0_{\gamma}(\Omega)}. \tag{3.16}$$

If $|\lambda_1|$ is sufficiently large, one can carry on this procedure and obtain, step by step, inclusions $\mathbf{u}_k \in V^2_{1-2j}(\Omega)$, $j = 3, 4, \ldots, n_1$, and the corresponding estimates

$$\|\mathbf{u}_k\|_{V^2_{1-2j}(\Omega)} \le Ck^j\|\mathbf{f}_k\|_{V^0_{\gamma}(\Omega)}$$

until such $n_1 \in \mathbb{N}$ that $\lambda_1 \in (1 - 2(n_1 + 1), 1 - 2n_1)$ is reached. In this case the estimate

$$\|\mathbf{u}_k\|_{V^2_{1-2n_1}(\Omega)} \le Ck^{n_1}\|\mathbf{f}_k\|_{V^0_{1-2n_1}(\Omega)}$$

is valid and it implies the following inequality

$$\|\mathbf{u}_k\|_{V^0_{1-2(n_1+1)}(\Omega)} \le Ck^{n_1}\|\mathbf{f}_k\|_{V^0_{1-2n_1}(\Omega)}.$$

Consequently, $\mathbf{u}_k$ belongs to $V^0_{1-2(n_1+1)}(\Omega)$ and (u_{ck}, u_{sk}) may be treated as a solution to (3.9) with $\mathbf{F}_k = (F_{ck}, F_{sk}) \in V^0_{1-2(n_1+1)}(\Omega)$. Since $\lambda_1 = 1 - \pi/\alpha$ is contained in the interval $(1 - 2(n_1+1), 1-2n_1)$, Theorem 2 leads to the following asymptotic representations:

$$u_{ck} = \chi u^{(10)}_{ck} + \widetilde{u}^{(1)}_{ck}, \quad u_{sk} = \chi u^{(10)}_{sk} + \widetilde{u}^{(1)}_{sk}, \tag{3.17}$$

in which $\widetilde{u}^{(1)}_{ck}$ and $\widetilde{u}^{(1)}_{ck}$ belong to $V^2_{1-2(n_1+1)}(\Omega)$ and satisfy

$$\|\widetilde{\mathbf{u}}^{(1)}_k\|_{V^2_{1-2(n_1+1)}(\Omega)} \le Ck^{n_1+1}\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)}. \tag{3.18}$$

The next step is to obtain asymptotic representations of $\widetilde{u}^{(1)}_{ck}$, $\widetilde{u}^{(1)}_{sk}$. Below we consider the case of representation (3.12), while the case of (3.17) can be analysed in the same way with obvious modifications.

Substituting (3.12) into (3.9) we obtain the following problem for $\widetilde{u}^{(1)}_{ck}$:

$$\begin{cases} -\Delta\widetilde{u}^{(1)}_{ck} &= F^{(1)}_{ck}, \quad x \in \Omega, \\ \widetilde{u}^{(1)}_{ck} &= 0, \qquad x \in \partial\Omega, \end{cases} \tag{3.19}$$

$$F^{(1)}_{ck} = f_{ck} + k\widetilde{u}^{(1)}_{sk} - \Delta\left(\chi\left(u^{(10)}_{ck} + \dots + u^{(Q_10)}_{ck}\right)\right) + \chi k\left(u^{(10)}_{sk} + \dots + u^{(Q_10)}_{sk}\right). \tag{3.20}$$

Let us examine the terms in $F^{(1)}_{ck}$. Estimate (3.14) gives the following inequality

$$\|\widetilde{u}^{(1)}_{sk}\|_{V^0_{-3}(\Omega)} \le Ck\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)}.$$

Therefore $f_{ck} + k\widetilde{u}^{(1)}_{sk}$ belongs to $V_{\beta_1}(\Omega)$, where $\beta_2 = \max\{-3, \gamma\}$. Functions $u^{(q0)}_{ck}$ are harmonic in sector K, therefore

$$\Delta\left(\chi\left(u^{(10)}_{ck} + \dots + u^{(Q_10)}_{ck}\right)\right) = 2\nabla\chi\cdot\nabla\left(u^{(10)}_{ck} + \dots + u^{(Q_10)}_{ck}\right) + (\Delta\chi)\left(u^{(10)}_{ck} + \dots + u^{(Q_10)}_{ck}\right).$$

The cut-off function $\chi = \chi(r)$ is constant in the vicinity of corner point O and for large r, therefore the third term on the right-hand side of (3.20) has a compact support detached from the origin and, consequently, is a smooth function. We recall that the function r^d is an element of the space $V^0_\beta(\Omega)$ only if $d > -\beta - 1$. Since $-1 \le \lambda_{Q_1} < \ldots < \lambda_1$, i.e., $Q_1\pi/\alpha \le 2$, neither of functions $u^{(10)}_{sk}, \ldots u^{(Q_10)}_{sk}$ belong to $V^0_{-3}(\Omega)$. In the case $\beta_2 = \gamma$, i.e., when $-3 < \gamma$, there may be such a number $Q^*_1 \le Q_1$ that the functions $u^{(q0)}_{sk}$, $q = 1, \ldots, Q^*_1$ do not belong to $V^0_\gamma(\Omega)$, while $u^{(q0)}_{sk} \in V^0_\gamma(\Omega)$ for $q = Q^*_1 + 1, \ldots, Q_1$. Taking all this into account we decompose $F^{(1)}_{ck}$ as

$$F^{(1)}_{ck} = \chi k\left(u^{(10)}_{sk} + \dots + u^{(Q^*_10)}_{sk}\right) + \widetilde{F}^{(1)}_{ck}, \tag{3.21}$$

where $\widetilde{F}_{ck}^{(1)}$ denotes the part of (3.20) belonging to $V^0_{\beta_2}(\Omega)$. Notice that $Q_1^* = Q_1$ in the case $\gamma \leq -3$. Applying Theorem 3 to problem (3.19) with the right-hand side (3.21) we get the following expression of function $\widetilde{u}_{ck}^{(1)}$:

$$\widetilde{u}_{ck}^{(1)} = \chi \sum_{q=1}^{Q_1^*} u_{ck}^{(q1)} + \chi \sum_{q=Q_1^*+1}^{Q_2} u_{ck}^{(q0)} + \widetilde{u}_{ck}^{(2)}, \tag{3.22}$$

where

$$u_{ck}^{(q1)} = c_{ck}^{(q1)} r^{q\pi/\alpha+2} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \quad c_{ck}^{(q1)} = \frac{\alpha}{4q\pi + 4\alpha} k c_{sk}^{(q)}, \tag{3.23}$$

functions $u_{ck}^{(q0)}$ are given by (3.13) and $\widetilde{u}_{ck}^{(2)}$ is an element of $V^2_{\beta_2}(\Omega)$. The summation index Q_2 in (3.22) is such that $\lambda_{Q_1+1}, \dots, \lambda_{Q_2} \in (\beta_2, -1)$. Namely, if $\gamma \leq -3$, then $Q_2 = [4\alpha/\pi]$, and in the case $\beta_2 = \gamma$ one shall take $Q_2 = m$.

The following remarks concern construction of (3.22).

According to Theorem 3, the asymptotic expression of $\widetilde{u}_{ck}^{(1)}$ should contain, in general, the linear combination $d_{ck}^{(1)} u_{ck}^{(10)} + \dots + d_{ck}^{(Q_1^*)} u_{ck}^{(Q_1^* 0)}$. This combination is not present in (3.22) due to the fact that $\widetilde{u}_{ck}^{(1)}$ is an element of $V^2_{-1}(\Omega)$, while none of the functions $u_{ck}^{(10)}, \dots, u_{ck}^{(Q_1 0)}$ belong to this space.

Secondly, let us comment on construction of the first sum in (3.22), which corresponds to the first term of (3.21). For $q = 1, \dots, Q_1^*$, function $u_{ck}^{(q1)}$ satisfies the following problem set in sector K (see details in Section 1.3.5 of [3])

$$\left\{ \begin{array}{rcll} \Delta u_{ck}^{(q1)} & = & k u_{sk}^{(q0)}, & x \in K, \\ u_{ck}^{(q1)} & = & 0, & x \in \partial K \setminus O. \end{array} \right. \tag{3.24}$$

We look for solution of this problem in the form

$$u_{ck}^{(q1)} = r^{q\pi/\alpha+2} U_{ck}^{(q1)}(\varphi).$$

Substituting this expression into (3.24) and writing the Laplace operator in polar coordinates the problem is reduced to a boundary value problem for the ordinary differential equation

$$\frac{d}{d\varphi} U_{ck}^{(q1)}(\varphi) + \left(\frac{q\pi}{\alpha} + 2\right)^2 U_{ck}^{(q1)}(\varphi) = -k c_{sk}^{(1)} \sin\left(\frac{\pi\varphi}{\alpha}\right), \quad U_{ck}^{(q1)}(0) = U_{ck}^{(q1)}(\alpha) = 0. \tag{3.25}$$

The general solution of (3.25) is

$$U_{ck}^{(q1)} = A^{(q1)} \cos\left(\frac{q\pi}{\alpha} + 2\right)\varphi + B^{(q1)} \sin\left(\frac{q\pi}{\alpha} + 2\right)\varphi + C_k^{(q1)} \sin\left(\frac{q\pi\varphi}{\alpha}\right),$$

where $C_k^{(q1)} = k c_{sk}^{(q)} \alpha/(4q\pi + 4\alpha)$. If $q\pi + 2\alpha \neq z\pi$, $z \in \mathbb{Z}$, the homogeneous boundary conditions imply $A^{(q1)} = B^{(q1)} = 0$. Consequently, solution $u_{ck}^{(q1)}$ is given by (3.23). In the case $q\pi + 2\alpha =$

πz^* for some $z^* \in \mathbb{Z}$, the homogeneous boundary conditions are satisfied for any constant $B^{(q1)}$ and therefore, function $u_{ck}^{(q1)}$ takes the following form

$$u_{ck}^{(q1)} = B^{(q1)} r^{q\pi/\alpha+2} \sin\left(\frac{q\pi}{\alpha}+2\right)\varphi + C_k^{(q1)} r^{q\pi/\alpha+2} \sin\left(\frac{q\pi\varphi}{\alpha}\right).$$

The identity $q\pi + 2\alpha = \pi z^*$ is equivalent to $q\pi/\alpha + 2 = \pi z^*/\alpha$, hence the first term in the formula above is equal to

$$B^{(q1)} r^{z^*\pi/\alpha} \sin\left(\frac{z^*\pi\varphi}{\alpha}\right).$$

As we will see in the following, this term will be included in the asymptotic representation of u_{ck} as one of the functions $u_{ck}^{(l0)}$, $l = q+1, \ldots, m$. In this case the constant $B^{(q1)}$ is determined together with $c_{ck}^{(q)}$ (see Chapter 4 below).

Let us also notice that, according to Theorem 3, in the case $q\pi + 2\alpha = \pi z^*$ asymptotic representation of $\widetilde{u}_{ck}^{(1)}$ should contain certain logarithmic terms. These terms are not present in (3.22), since the right-hand side of equations in (3.24) satisfies certain compatibility conditions (see the proof of Lemma 1.3.13 and Remark 1.3.20 in [3] about the similar issue for problem (2.4)).

Next we derive the estimate of $\widetilde{u}_{ck}^{(2)}$ in space $V_{\beta_2}^2(\Omega)$. Substituting expression (3.22) into (3.19) we see that $\widetilde{u}_{ck}^{(2)}$ is a solution to

$$\begin{cases} -\Delta \widetilde{u}_{ck}^{(2)} &= F_{ck}^{(2)}, \quad x \in \Omega, \\ \widetilde{u}_{ck}^{(2)} &= 0, \qquad x \in \partial\Omega, \end{cases} \tag{3.26}$$

where

$$F_{ck}^{(2)} = f_{ck} - \Delta\left(\chi \sum_{q=1}^{Q_1^*} u_{ck}^{(q1)} + \chi \sum_{q=Q_1^*+1}^{Q_2} u_{ck}^{(q0)}\right) + \chi k \sum_{q=1}^{Q_1^*} u_{sk}^{(q1)} + k\widetilde{u}_{sk}^{(1)}.$$

Taking into account relations (3.24) we rewrite $F_{ck}^{(2)}$ as follows

$$F_{ck}^{(2)} = f_{ck} + k\widetilde{u}_{sk}^{(1)} - 2\nabla\chi\cdot\nabla\left(\sum_{q=1}^{Q_1^*} u_{ck}^{(q1)} + \sum_{q=Q_1^*+1}^{Q_2} u_{ck}^{(q0)}\right) - \Delta\chi\left(\sum_{q=1}^{Q_1^*} u_{ck}^{(q1)} + \sum_{q=Q_1^*+1}^{Q_2} u_{ck}^{(q0)}\right).$$

In the same way as in the case of $F_{ck}^{(1)}$, see (3.20), we get that $F_{ck}^{(2)}$ belongs to $V_{\beta_2}^0(\Omega)$ and derive the following

$$\|F_{ck}^{(2)}\|_{V_{\beta_2}^0(\Omega)} \le C\left((1+k^2)\|\mathbf{f}_{ck}\|_{V_{\beta_2}^0(\Omega)} + k\sum_{q=1}^{Q_1^*}|c_{sk}^{(q)}| + \sum_{q=Q_1^*+1}^{Q_2}|c_{sk}^{(q)}|\right).$$

Then, repeating considerations of Subsection 1.2 in [3], we obtain in sector K the estimate

$$\|\widetilde{u}_{ck}^{(2)}\|_{V_{\beta_2}^2(K)} \le \|F_{ck}^{(2)}\|_{V_{\beta_2}^0(K)}$$

and conclude that in the whole domain Ω the estimate

$$\|\widetilde{u}_{ck}^{(2)}\|_{V_{\beta_2}^2(\Omega)} \le C\left(k^2\|f_{ck}\|_{V_{\beta_2}^0(\Omega)} + k\sum_{q=1}^{Q_1^*}|c_{sk}^{(q)}| + \sum_{q=Q_1^*+1}^{Q_2}|c_{ck}^{(q)}|\right)$$

holds.

Analogously we derive the asymptotic representation of the function $\widetilde{u}_{sk}^{(1)}$ as in (3.22):

$$\widetilde{u}_{sk}^{(1)} = \chi\sum_{q=1}^{Q_1^*} u_{sk}^{(q1)} + \chi\sum_{q=Q_1^*+1}^{Q_2} u_{sk}^{(q0)} + \widetilde{u}_{wk}^{(2)}, \tag{3.27}$$

where

$$u_{sk}^{(q1)} = -\frac{kc_{ck}^{(q)}\alpha}{4q\pi+4\alpha} r^{q\pi/\alpha+2}\sin\left(\frac{q\pi\varphi}{\alpha}\right), \tag{3.28}$$

$\widetilde{u}_{sk}^{(2)} \in V_{\beta_2}^2(\Omega)$ and satisfies the estimate

$$\|\widetilde{u}_{sk}^{(2)}\|_{V_{\beta_2}^2(\Omega)} \le C\left(k^2\|f_{sk}\|_{V_{\beta_2}^0(\Omega)} + k\sum_{q=1}^{Q_1^*}|c_{ck}^{(q)}| + \sum_{q=Q_1^*+1}^{Q_2}|c_{sk}^{(q)}|\right).$$

Combining (3.12), (3.22) and (3.27) we have

$$u_{ck} = \chi\sum_{q=1}^{Q_2} u_{ck}^{(q0)} + \chi\sum_{q=1}^{Q_1^*} u_{ck}^{(q1)} + \widetilde{u}_{ck}^{(2)}, \quad u_{sk} = \chi\sum_{q=1}^{Q_2} u_{sk}^{(q0)} + \chi\sum_{q=1}^{Q_1^*} u_{sk}^{(q1)} + \widetilde{u}_{sk}^{(2)}. \tag{3.29}$$

Let us recall that in this formula $Q_2 = [4\alpha/\pi]$, the index $Q_1 = [2\alpha/\pi]$ in the case $\gamma \le -3$ and $Q_1 \le [2\alpha/\pi]$ if $-3 < \gamma$. Functions $u_{ck}^{(q0)}$, $u_{sk}^{(q0)}$ are defined by (3.13), while $u_{ck}^{(q1)}$ and $u_{sk}^{(q1)}$ are given by (3.23) and (3.28). The function $\widetilde{\mathbf{u}}_k^{(2)} = (\widetilde{u}_{ck}^{(2)}, \widetilde{u}_{sk}^{(2)})$ belongs to $V_{\beta_2}^2(\Omega)$ and satisfies the estimate

$$\|\widetilde{\mathbf{u}}_k^{(2)}\|_{V_{\beta_2}^2(\Omega)} \le C\left(k^2\|\mathbf{f}_k\|_{V_\gamma^0(\Omega)} + k\sum_{q=1}^{Q_1^*}(|c_{ck}^{(q)}| + |c_{sk}^{(q)}|) + \sum_{q=Q_1^*+1}^{Q_2}(|c_{ck}^{(q)}| + |c_{sk}^{(q)}|)\right). \tag{3.30}$$

If $\beta_2 = \gamma$, the construction of asymptotic expansion of $\mathbf{u}_k \in V_1^2(\Omega)$ is completed. In the case $\beta_2 = -3$, we continue by substituting expressions (3.29) into (3.9) and considering function $\widetilde{u}_{ck}^{(2)}$ as a solution of the problem

$$\begin{cases} -\Delta\widetilde{u}_{ck}^{(2)} &= F_{ck}^{(2)}, \quad x \in \Omega, \\ \widetilde{u}_{ck}^{(2)} &= 0, \qquad x \in \partial\Omega, \end{cases} \tag{3.31}$$

where

$$F_{ck}^{(2)} = f_{ck} - \Delta\left(\chi\sum_{q=1}^{Q_1} u_{ck}^{(q1)} + \chi\sum_{q=Q_1+1}^{Q_2} u_{ck}^{(q0)}\right) + \chi k\sum_{q=1}^{Q_2} u_{sk}^{(q1)} + k\widetilde{u}_{sk}^{(2)}.$$

Taking into account equations (3.23) which hold for $q = 1, \ldots, Q_1$, we see that

$$F_{ck}^{(2)} = f_{ck} - 2\nabla\chi\cdot\nabla\sum_{q=1}^{Q_2} u_{ck}^{(q1)} - (\Delta\chi)\sum_{q=1}^{Q_2} u_{ck}^{(q1)} + \chi k\sum_{q=Q_1+1}^{Q_2} u_{sk}^{(q1)} + k\widetilde{u}_{sk}^{(2)}.$$

Now we can split $F_{ck}^{(2)}$ into the sum of two terms

$$F_{ck}^{(2)} = \widetilde{F}_{ck}^{(2)} + \chi k \sum_{q=Q_1+1}^{Q_2} u_{sk}^{(q1)}. \tag{3.32}$$

The first one is a function from $V_{\beta_3}^0(\Omega)$, $\beta_3 = \max\{-5, \gamma\}$ while the second one is a combination of functions $u_{sk}^{(q1)} \notin V_{-5}^0(\Omega)$ for $q = Q_1 + 1, \dots, Q_2$. Applying Theorem 3 for problem (3.31) with (3.32), one obtains the following representation

$$\widetilde{u}_{ck}^{(2)} = \chi \sum_{q=Q_1+1}^{Q_2} u_{ck}^{(q2)} + \chi \sum_{q=Q_2+1}^{Q_3} u_{ck}^{(q0)} + \widetilde{u}_{ck}^{(3)}. \tag{3.33}$$

Here $\widetilde{u}_{ck}^{(3)} \in V_{\beta_3}^2(\Omega)$, $u_{ck}^{(q0)}$ are given by (3.13), while $u_{ck}^{(q2)}$ has a form

$$u_{ck}^{(q2)} = c_{ck}^{(q2)} r^{q\pi/\alpha+4} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \quad c_{ck}^{(q2)} = \frac{\alpha^2}{(4q\pi + 4\alpha)(8q\pi + 16\alpha)} k^2 c_{sk}^{(q)}.$$

More precisely, functions $u_{ck}^{(q2)}$, $q = Q_1 + 1, \dots, Q_2$, are solutions to the following problems

$$\begin{cases} \Delta u_{ck}^{(q2)} &= k u_{sk}^{(q1)}, \quad x \in K, \\ u_{ck}^{(q2)} &= 0, \qquad\quad x \in \partial K \setminus O. \end{cases}$$

The bounds for q in the second sum of (3.33) are such that the numbers $\lambda_{Q_2+1}, \dots, \lambda_{Q_3} \in (\beta_3, -3)$. Analogously one can derive the asymptotic expansion of the function $\widetilde{u}_{sk}^{(2)}$ and the corresponding estimates. In a similar way we continue the procedure described above to obtain the complete asymptotic representation of the solution $(u_{ck}, u_{sk}) \in V_1^2(\Omega)$ of the problem (3.1). Namely we construct step by step functions

$$\begin{aligned} u_{ck}^{(qj)} &= \sigma_c(q) \frac{c_{sk}^{(q)} k^j \alpha^j}{\prod_{i=1}^{j}(4qi\pi + (2i)^2\alpha)} r^{q\pi/\alpha+2j} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \\ u_{sk}^{(qj)} &= \sigma_s(q) \frac{c_{ck}^{(q)} k^j \alpha^j}{\prod_{i=1}^{j}(4qi\pi + (2i)^2\alpha)} r^{q\pi/\alpha+2j} \sin\left(\frac{q\pi\varphi}{\alpha}\right). \end{aligned} \tag{3.34}$$

In the expressions above $c_{ck}^{(q)}$, $c_{sk}^{(q)}$ are constants that will be determined below (see Section 4), while $\sigma_c(q)$ and $\sigma_s(q)$ are defined by formulas

$$\sigma_c(q) = \operatorname{sign}\left(\sin\frac{(2q+3)\pi}{4}\right), \quad \sigma_s(q) = \operatorname{sign}\left(\sin\frac{(2q+1)\pi}{4}\right), \tag{3.35}$$

i.e., the function $\sigma_c(q)$ for values $q = 1, 2, \dots$ generates the sequence of signs $+--++--++\dots$, while $\sigma_s(q)$ for $q = 1, 2, \dots$ produces $--++--++\dots$.

For every $q = 1, \dots, m$, one shall construct functions $u_{ck}^{(qj)}$, $u_{ck}^{(qj)}$, $j = 1, \dots, M_q$ where M_q is such that

$$u_{ck}^{(qM_q)}, u_{sk}^{(qM_q)} \in V_\gamma^0(\Omega). \tag{3.36}$$

Since $u_{ck}^{(qM_q)}, u_{sk}^{(qM_q)} \sim r^{q\pi/\alpha+2M_q}$ in the neighborhood of the corner point O, condition (3.36) is satisfied if $q\pi/\alpha + 2M_q \geq -\gamma - 1$, i.e., if

$$M_q > \frac{-1-\gamma}{2} - \frac{q\pi}{2\alpha}.$$

It is clear that condition (3.36) is satisfied if we set

$$M_q = \left[\frac{1-\gamma}{2} - \frac{q\pi}{2\alpha}\right]$$

where $[z]$ denotes an integer part of a number z. Furthermore, taking into account that $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$ we see that the inclusion (3.36) is guaranteed if

$$M_q = \left[\frac{(m+1-q)\pi}{2\alpha}\right]. \tag{3.37}$$

Considerations of this section may be summed up in the following statement.

Lemma 1. *Assume that $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$ and $\mathbf{f}_k = (f_{ck}, f_{sk})$ belongs to the space $V_\gamma^0(\Omega)$. In this case the solution $\mathbf{u}_k = (u_{ck}, u_{sk}) \in V_1^2(\Omega)$ of problem (3.1) can be represented in the form*

$$u_{ck} = \chi \sum_{q=1}^{m} \sum_{j=0}^{M_q} u_{ck}^{(qj)} + \tilde{u}_{ck}, \quad u_{sk} = \chi \sum_{q=1}^{m} \sum_{j=0}^{M_q} u_{sk}^{(qj)} + \tilde{u}_{sk}. \tag{3.38}$$

Here $\tilde{\mathbf{u}}_k = (\tilde{u}_{ck}, \tilde{u}_{sk}) \in V_\gamma^2(\Omega)$, functions $u_{ck}^{(q0)}$, $u_{sk}^{(q0)}$ and $u_{ck}^{(qj)}$, $u_{sk}^{(qj)}$, $j = 1, \ldots, M_q$, are defined in (3.13) and (3.34), respectively, while the summation bounds M_q are given by (3.37).

Moreover, repeating the same steps as in derivation of the estimate (3.30), one can obtain the following inequality

$$\|\tilde{\mathbf{u}}_k\|_{V_\gamma^2(\Omega)} \leq C \left(k^n \|\mathbf{f}_k\|_{V_\gamma^0(\Omega)} + \sum_{i=1}^{n} k^{n-i} \sum_{q=Q_{i-1}}^{Q_i} \left(|c_{ck}^{(q)}| + |c_{sk}^{(q)}|\right) \right). \tag{3.39}$$

Here $n \in \mathbb{N}$ is such that $\gamma \in [1-2n, 1-2(n-1))$, while, $Q_0 = 1$, $Q_i = [2i\alpha/\pi]$, for $i = 1, \ldots, n-1$ and $Q_n = m$.

3.3 The adjoint problem

To determine the coefficients $c_{ck}^{(q)}, c_{sk}^{(q)}$ in (3.34), we will need to solve a problem formally adjoint to (3.1) (see, e.g., Subsections 1.3.2, 1.3.6 in [3]). Let us take $\eta_{ck}, \eta_{sk} \in C^\infty(\Omega)$. Multiplying $(3.1)_1$ by η_{ck}, $(3.1)_2$ by η_{sk} and integrating by parts twice, for $k \in \mathbb{N}_0$ we derive our adjoint problem

$$\begin{cases} -\Delta\eta_{ck} - k\eta_{sk} &= 0, \qquad x \in \Omega, \\ -\Delta\eta_{sk} + k\eta_{ck} &= 0, \qquad x \in \Omega, \\ \eta_{ck} = 0, \ \eta_{sk} &= 0, \qquad x \in \partial\Omega. \end{cases} \tag{3.40}$$

We look for $\eta_{ck}^{(q)}, \eta_{sk}^{(q)}$ in the form

$$\eta_{ck}^{(q)} = \chi\eta_{ck}^{(q0)} + \tilde{\eta}_{ck}^{(q)}, \quad \eta_{sk}^{(q)} = \chi\eta_{sk}^{(q0)} + \tilde{\eta}_{sk}^{(q)}, \tag{3.41}$$

where χ is a cut-off function defined in Theorem 2. Let the main asymptotic terms of $\eta_{ck}^{(q)}, \eta_{sk}^{(q)}$ be given by

$$\eta_{ck}^{(q0)} = \eta_{sk}^{(q0)} = r^{-q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right)$$

Substituting $\eta_{ck}^{(q)}, \eta_{sk}^{(q)}$ into (3.40) we get for $\tilde{\eta}_{ck}^{(q)}, \tilde{\eta}_{sk}^{(q)}$ the problem

$$\begin{cases} -\Delta\tilde{\eta}_{ck}^{(q)} - k\tilde{\eta}_{sk}^{(q)} &= \zeta_{ck}^{(q0)} + k\chi\eta_{sk}^{(q0)} := F_{ck}^{(q0)}, \qquad x \in \Omega, \\ -\Delta\tilde{\eta}_{sk}^{(q)} + k\tilde{\eta}_{ck}^{(q)} &= \zeta_{sk}^{(q0)} - k\chi\eta_{ck}^{(q0)} := F_{sk}^{(q0)}, \qquad x \in \Omega, \quad k = 0, 1, \ldots \\ \tilde{\eta}_{ck}^{(q)} = 0, \ \tilde{\eta}_{sk}^{(q)} &= 0, \qquad x \in \partial\Omega, \end{cases} \tag{3.42}$$

where

$$\zeta_{ck}^{(q0)} = 2\nabla\chi \cdot \nabla\eta_{ck}^{(q0)} + \eta_{ck}^{(q0)}\Delta\chi, \quad \zeta_{sk}^{(q0)} = 2\nabla\chi \cdot \nabla\eta_{sk}^{(q0)} + \eta_{sk}^{(q0)}\Delta\chi. \tag{3.43}$$

Due to the definition of the cut-off function χ we get that the supports of $\zeta_{ck}^{(q)}, \zeta_{sk}^{(q)}$ are detached from O, therefore the functions $F_{ck}^{(q0)}$ and $F_{sk}^{(q0)}$ in (3.42) coincide near the corner point O with the terms $k\eta_{sk}^{(q0)}$ and $-k\eta_{ck}^{(q0)}$, respectively. If

$$\frac{q\pi}{\alpha} < 1, \tag{3.44}$$

then $\eta_{ck}^{(q0)}$ and $\eta_{sk}^{(q0)}$ belong to $L_2(\Omega)$. This inclusion immediately follows from the inequality

$$\int\limits_{K_\alpha} |r^{-q\pi/\alpha}|^2 dx = \int\limits_0^R \int\limits_0^\alpha |r^{-q\pi/\alpha}|^2 r dr d\varphi = c\, r^{-2q\pi/\alpha+2}\Big|_0^R < \infty,$$

which is valid if $-2q\pi/\alpha + 2 > 0$, i.e. $q\pi/\alpha < 1$. Then we get that $F_{ck}^{(q0)}$, $F_{sk}^{(q0)} \in L_2(\Omega)$ in problem (3.42), therefore it has a unique solution $\tilde{\eta}_{ck}^{(q)}, \tilde{\eta}_{sk}^{(q)}$ in the space $W_2^1(\Omega)$. Indeed, consider the corresponding homogeneous problem

$$\begin{cases} -\Delta\widetilde{\eta}_{ck}^{(q)} - k\widetilde{\eta}_{sk}^{(q)} &= 0, \qquad x \in \Omega, \\ -\Delta\widetilde{\eta}_{sk}^{(q)} + k\widetilde{\eta}_{ck}^{(q)} &= 0, \qquad x \in \Omega, \quad k = 0, 1, \dots \\ \widetilde{\eta}_{ck}^{(q)} = 0, \ \widetilde{\eta}_{sk}^{(q)} &= 0, \qquad x \in \partial\Omega. \end{cases} \tag{3.45}$$

Multiplying $(3.45)_1$ by $-\widetilde{\eta}_{sk}^{(q)}$, $(3.45)_2$ by $\widetilde{\eta}_{ck}^{(q)}$, integrating by parts once and summing the obtained relations we get that

$$k \int\limits_{K_\alpha} (|\widetilde{\eta}_{ck}^{(q)}|^2 + |\widetilde{\eta}_{sk}^{(q)}|^2) dx = 0.$$

Consequently, the elliptic homogeneous problem (3.45) has only a trivial solution. Using the Fredholm theorem, we get the existence of a unique generalized solution $\widetilde{\eta}_{ck}^{(q)}$, $\widetilde{\eta}_{sk}^{(q)} \in W_2^1(\Omega)$ for every $F_{ck}^{(q0)}, F_{sk}^{(q0)} \in L_2(\Omega)$ (see, for example, Theorem 4.4 in [1]).

If condition (3.44) is not satisfied, i.e. if $q\pi/\alpha \geq 1$, then instead of $\widetilde{\eta}_{ck}^{(q)}, \widetilde{\eta}_{sk}^{(q)}$ defined by (3.41) we consider the functions

$$\eta_{ck}^{(q)} = \chi\left(\eta_{ck}^{(q0)} + \eta_{ck}^{(q1)}\right) + \widetilde{\eta}_{ck}^{(q)}, \eta_{sk}^{(q)} = \chi\left(\eta_{sk}^{(q0)} + \eta_{sk}^{(q1)}\right) + \widetilde{\eta}_{sk}^{(q)},$$

where $\eta_{ck}^{(q1)}, \eta_{sk}^{(q1)}$ solves in corner K_α the equations

$$\Delta\eta_{ck}^{(q1)} = -k\eta_{sk}^{(q0)}, \ \Delta\eta_{sk}^{(q1)} = k\eta_{ck}^{(q0)}$$

and satisfies the homogeneous boundary conditions. We find that

$$\eta_{ck}^{(q1)} = -\frac{k\alpha}{4(\alpha - q\pi)} r^{-q\pi/\alpha+2} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \quad \eta_{sk}^{(q1)} = \frac{k\alpha}{4(\alpha - q\pi)} r^{-q\pi/\alpha+2} \sin\left(\frac{q\pi\varphi}{\alpha}\right).$$

The right-hand side of (3.42) in this case is, respectively, $\zeta_{ck}^{(q1)} + k\chi\eta_{sk}^{(q1)}$ and $\zeta_{sk}^{(q1)} - k\chi\eta_{ck}^{(q1)}$, where $\zeta_{ck}^{(q1)}$ and $\zeta_{sk}^{(q1)}$ are smooth functions, defined in an analogous way as (3.43). The functions $\widetilde{\eta}_{ck}^{(q)}, \widetilde{\eta}_{sk}^{(q)}$ now can be found by solving problem (3.42) with the right-hand side

$$F_{ck}^{(q1)} = \zeta_{ck}^{(q1)} + k\chi\eta_{sk}^{(q1)}, \ F_{sk}^{(q1)} = \zeta_{sk}^{(q1)} - k\chi\eta_{ck}^{(q1)},$$

which coincides in the neighborhood of O with $k\eta_{sk}^{(q1)}$ and $-k\eta_{ck}^{(q1)}$, respectively.

If $q\pi/\alpha < 3$, then $F_{ck}^{(q1)}, F_{sk}^{(q1)} \in L_2(\Omega)$ and we find a unique solution $\widetilde{\eta}_{ck}^{(q)}, \widetilde{\eta}_{sk}^{(q)} \in W_2^1(\Omega)$. Otherwise, we continue this procedure by constructing the functions

$$\begin{aligned} \eta_{ck}^{(qj)} &= \frac{\sigma_c(j) k^j \alpha^j}{\prod_{j=1}^q (4j^2\alpha - 4j\pi q)} r^{-q\pi/\alpha+2j} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \\ \eta_{sk}^{(qj)} &= \frac{\sigma_s(j) k^j \alpha^j}{\prod_{j=1}^q (4j^2\alpha - 4j\pi q)} r^{-q\pi/\alpha+2j} \sin\left(\frac{q\pi\varphi}{\alpha}\right), \end{aligned} \tag{3.46}$$

satisfying relations

$$\Delta\eta_{ck}^{(qj)} = -k\eta_{sk}^{(q,j-1)},\ \Delta\eta_{sk}^{(qj)} = k\eta_{ck}^{(q,j-1)}. \tag{3.47}$$

In (3.46) the constants $\sigma_c(j)$ and $\sigma_s(j)$ are defined by (3.35). Now we can define the functions $\eta_{ck}^{(q)}, \eta_{sk}^{(q)}$ as follows

$$\eta_{ck}^{(q)} = \chi\sum_{j=0}^{J_q}\eta_{ck}^{(qj)} + \widetilde{\eta}_{ck}^{(q)},\ \eta_{sk}^{(q)} = \chi\sum_{j=0}^{J_q}\eta_{sk}^{(qj)} + \widetilde{\eta}_{sk}^{(q)}, \tag{3.48}$$

there the functions $\widetilde{\eta}_{ck}^{(q)}, \widetilde{\eta}_{sk}^{(q)}$ are weak solutions in $W_2^1(\Omega)$ to the problem

$$\begin{cases} -\Delta\widetilde{\eta}_{ck}^{(q)} - k\widetilde{\eta}_{sk}^{(q)} = \zeta_{ck}^{(qJ_q)} + k\chi\eta_{sk}^{(qJ_q)} := F_{ck}^{(qJ_q)}, & x\in\Omega, \\ -\Delta\widetilde{\eta}_{sk}^{(q)} + k\widetilde{\eta}_{ck}^{(q)} = \zeta_{sk}^{(qJ_q)} - k\chi\eta_{ck}^{(qJ_q)} := F_{sk}^{(qJ_q)}, & x\in\Omega,\ ,k=0,1,\dots \\ \widetilde{\eta}_{ck}^{(q)} = 0,\ \widetilde{\eta}_{sk}^{(q)} = 0, & x\in\partial\Omega, \end{cases} \tag{3.49}$$

where

$$\zeta_{ck}^{(qJ_q)} = 2\nabla\chi\nabla\left(\sum_{j=1}^{J_q}\eta_{ck}^{(qj)}\right) + \Delta\chi\left(\sum_{j=1}^{J_q}\eta_{ck}^{(qj)}\right),\ \zeta_{sk}^{(qJ_q)} = 2\nabla\chi\nabla\left(\sum_{j=1}^{J_q}\eta_{sk}^{(qj)}\right) + \Delta\chi\left(\sum_{j=1}^{J_q}\eta_{sk}^{(qj)}\right). \tag{3.50}$$

The functions $\zeta_{ck}^{(qJ_q)}$ and $\zeta_{sk}^{(qJ_q)}$ belong to $C_0^\infty(\Omega)$ while the terms $k\chi\eta_{sk}^{(qJ_q)}$ and $k\chi\eta_{ck}^{(qJ_q)}$ contain multipliers $r^{2J_q-q\pi/\alpha}$. Let us take J_q in (3.48) such that $2J_q - q\pi/\alpha > -2$, i.e., assume that J_q is the smallest non-negative integer satisfying the inequality

$$J_q > \frac{q\pi}{2\alpha} - 1.$$

This requirement is equivalent to the condition

$$J_q = \left[\frac{q\pi}{2\alpha}\right]. \tag{3.51}$$

Assumption (3.51) ensures that the functions $\eta_{ck}^{(qJ_q)}$, $\eta_{sk}^{(qJ_q)}$ and, consequently, the data of (3.49), belong to the space $V_1^0(\Omega)$. Now in the same way as in Subsection 3.1, we can prove existence of the unique solution $\widetilde{\eta}_{ck}^{(q)}$, $\widetilde{\eta}_{ck}^{(q)} \in V_1^2(\Omega)$ to problem (3.49) and derive the estimate

$$\|\widetilde{\eta}_{ck}^{(q)}\|_{V_1^2(\Omega)} + \|\widetilde{\eta}_{sk}^{(q)}\|_{V_1^2(\Omega)} \le C\left(\|\zeta_{ck}^{(qJ_q)}\|_{V_1^0(\Omega)} + \|\zeta_{sk}^{(qJ_q)}\|_{V_1^0(\Omega)} + k\|\chi\eta_{ck}^{(qJ_q)}\|_{V_1^0(\Omega)} + k\|\chi\eta_{sk}^{(qJ_q)}\|_{V_1^0(\Omega)}\right).$$

Having in mind the structure of the functions $\zeta_{ck}^{(qJ_q)}$, $\zeta_{sk}^{(qJ_q)}$ and explicit dependence of $\eta_{ck}^{(qj)}$, $\eta_{sk}^{(qj)}$ on the parameter k (see (3.46)), we estimate the right-hand side of the last inequality as follows:

$$\|\zeta_{ck}^{(qJ_q)}\|_{V_1^0(\Omega)} + \|\zeta_{sk}^{(qJ_q)}\|_{V_1^0(\Omega)} \le Ck^{J_q}, \quad k\|\chi\eta_{ck}^{(qJ_q)}\|_{V_1^0(\Omega)} + k\|\chi\eta_{sk}^{(qJ_q)}\|_{V_1^0(\Omega)} \le Ck^{J_q+1}.$$

Taking into account (3.51), we conclude that

$$\|\widetilde{\eta}_{ck}^{(q)}\|_{V_1^2(\Omega)} + \|\widetilde{\eta}_{sk}^{(q)}\|_{V_1^2(\Omega)} \le Ck^{q\pi/2\alpha+1}. \tag{3.52}$$

Remark 3. *Expressions* (3.46) *contain* $\alpha - q\pi$, $2\alpha - q\pi$, $\dots$, $J_q\alpha - q\pi$ *in denominators. In general, some of these terms may be equal to zero. In such cases formulas* (3.46) *become meaningless (we refer the reader to Remark 1.3.20 in [3] where this issue is discussed in details). In our case none of the denominators are equal to zero, since the number* J_q *is bounded by* $q\pi/2\alpha$ *from above and, consequently,*

$$i\alpha - q\pi \leq -\frac{q\pi}{2} < 0, \quad \text{for any} \quad i = 1, \dots, J_q.$$

3.4 Representation and estimates of constants $c_{(ck)}^{(q)}, c_{(sk)}^{(q)}$

Using results from the adjoint problem (3.40), the representation and estimates of constants $c_{ck}^{(q)}, c_{sk}^{(q)}$ will be derived. We multiply $(3.1)_1$ by $\eta_{ck}^{(q)}$ and $(3.1)_2$ by $\eta_{sk}^{(q)}$, integrate by parts over the domain $\Omega_\delta = \Omega \setminus B_\delta$ and add, side by side, the two resulting relations. Since $\eta_{ck}^{(q)}$ and $\eta_{sk}^{(q)}$ satisfy (3.40), we find

$$\int\limits_{\Omega_\delta} \left(f_{ck}\eta_{ck}^{(q)} + f_{sk}\eta_{sk}^{(q)} \right) dx = \int\limits_{\partial\Omega_\delta} \left(u_{ck}\frac{\partial \eta_{ck}^{(q)}}{\partial n} - \eta_{ck}^{(q)}\frac{\partial u_{ck}}{\partial n} + u_{sk}\frac{\partial \eta_{sk}^{(q)}}{\partial n} - \eta_{sk}^{(q)}\frac{\partial u_{sk}}{\partial n} \right) dS.$$

We now separate $\partial\Omega_\delta$ into the union of $\partial B_\delta \cap \Omega$ and $\partial\Omega \setminus B_\delta$. In $\partial B_\delta \cap \Omega$ we have $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ and $r = \delta$. The functions $u_{ck}, u_{sk}, \eta_{ck}^{(q)}, \eta_{sk}^{(q)}$ equal zero in $\partial\Omega_\delta \setminus B_\delta$. Therefore, converting to polar coordinates we derive the following

$$\int\limits_{\Omega_\delta} \left(f_{ck}\eta_{ck}^{(q)} + f_{sk}\eta_{sk}^{(q)} \right) dx = \int\limits_0^\alpha r \left(\eta_{sk}^{(q)}\frac{\partial u_{sk}}{\partial r} - u_{sk}\frac{\partial \eta_{sk}^{(q)}}{\partial r} + \eta_{ck}^{(q)}\frac{\partial u_{ck}}{\partial r} - u_{ck}\frac{\partial \eta_{ck}^{(q)}}{\partial r} \right)\Bigg|_{r=\delta} d\varphi \tag{3.53}$$

If we substitute the asymptotic representations, perform differentiation and multiplication procedures, after a careful examination we see that each of the four terms in the right-hand side integral of (3.53) produces exactly one term not depending on δ, while the remaining terms possess a positive integer power of δ or depend on $\tilde{u}_{ck}(\delta), \tilde{u}_{sk}(\delta), \tilde{\eta}_{ck}(\delta), \tilde{\eta}_{sk}(\delta)$ which tend to zero as $\delta \to 0$ (we do not rewrite these terms due to the large extent of the formulae). Consequently we have

$$\int\limits_0^\alpha r\eta_{sk}^{(q)}\frac{\partial u_{sk}}{\partial r}\bigg|_{r=\delta} d\varphi = c_{sk}^{(q)}\frac{q\pi}{2} + o(\delta).$$

Treating in the same way the rest products under the integral sign on the right-hand side of (3.53) and passing to the limit $\delta \to 0$, we conclude that this integral converges to $(c_{ck}^{(q)} + c_{sk}^{(q)})\pi q$. Therefore, for every $k \in \mathbb{N}_0$ and $q = 1, \ldots, m$ we have the following relation

$$c_{ck}^{(q)} + c_{sk}^{(q)} = \frac{1}{\pi q}\int\limits_\Omega \left(f_{ck}\eta_{ck}^{(q)} + f_{sk}\eta_{sk}^{(q)} \right) dx. \tag{3.54}$$

Remark 4. *Let us instead take* $\eta_{ck}^{(q0)} = -r^{-q\pi/\alpha}\sin\left(\frac{q\pi\varphi}{\alpha}\right)$ *and* $\eta_{sk}^{(q0)} = r^{-q\pi/\alpha}\sin\left(\frac{q\pi\varphi}{\alpha}\right)$ *in* (3.48) *and thus construct corresponding functions* $\eta_{ck}^{*(q)}, \eta_{sk}^{*(q)}$ *as in the previous Section. Using* $\eta_{ck}^{*(q)}, \eta_{sk}^{*(q)}$ *as test functions in the procedure described above, we derive the relation, analogous to* (3.54)*, namely:*

$$-c_{ck}^{(q)} + c_{sk}^{(q)} = \frac{1}{\pi q}\int\limits_\Omega \left(f_{ck}\eta_{ck}^{*(q)} + f_{sk}\eta_{sk}^{*(q)} \right) dx. \tag{3.55}$$

Denoting the right-hand sides of (3.54) and (3.55) by $d_{ck}^{(q)}$ and $d_{sk}^{(q)}$, respectively, we see that

$$c_{ck}^{(q)} = \frac{1}{2}\left(d_{ck}^{(q)} - d_{sk}^{(q)} \right), \quad c_{sk}^{(q)} = \frac{1}{2}\left(d_{ck}^{(q)} + d_{sk}^{(q)} \right). \tag{3.56}$$

Let us now derive estimates of the constants $c_{ck}^{(q)}$, $c_{sk}^{(q)}$ in terms of functions f_{ck}, f_{sk} and parameter k. In the case when $(f_{ck}, f_{sk}) \in V_\gamma^0(\Omega)$, $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$, from (3.56) we have that

$$|c_{ck}^{(q)}| \leq \int\limits_\Omega |f_{ck}||\eta_{ck}^{(q)} - \eta_{ck}^{*(q)}| + |f_{sk}||\eta_{sk}^{(q)} - \eta_{sk}^{*(q)}|dx \leq \|f_{ck}\|_{V_\gamma^0(\Omega)} \left(\|\eta_{ck}^{(q)}\|_{V_{-\gamma}^0(\Omega)} + \|\eta_{ck}^{*(q)}\|_{V_{-\gamma}^0(\Omega)}\right)$$
$$+\|f_{sk}\|_{V_\gamma^0(\Omega)} \left(\|\eta_{sk}^{(q)}\|_{V_{-\gamma}^0(\Omega)} + \|\eta_{sk}^{*(q)}\|_{V_{-\gamma}^0(\Omega)}\right). \tag{3.57}$$

Let us recall that, for example, $\eta_{ck}^{(q)}$ has the following structure

$$\eta_{ck}^{(q)} = \chi \sum_{j=0}^{J_q} k^j r^{-q\pi/\alpha + 2j} V_{ck}^{(qj)} + \widetilde{\eta}_{ck}^{(q)} \tag{3.58}$$

where $V_{ck}^{(qj)}$, $j = 0, \ldots, J_q$, are smooth functions depending on φ only (see (3.46)). Therefore

$$\|\eta_{ck}^{(q)}\|_{V_{-\gamma}^0(\Omega)} \leq \sum_{j=0}^{J_q} k^j \|\chi r^{2j - q\pi/\alpha} V_{ck}^{(qj)}\|_{V_{-\gamma}^0(\Omega)} + \|\widetilde{\eta}_{ck}^{(q)}\|_{V_{-\gamma}^0(\Omega)}. \tag{3.59}$$

Since $-2\gamma \in \left(\frac{2m\pi}{\alpha} - 2, \frac{2(m+1)\pi}{\alpha} - 2\right)$, the quantities

$$\|\chi r^{2j-q\pi/\alpha} V_{ck}^{(qj)}\|_{V_{-\gamma}^0(\Omega)} = \left(\int\limits_\Omega |\chi r^{2j-q\pi/\alpha} V_{ck}^{(qj)}|^2 r^{-2\gamma} dx\right)^{1/2}$$

are finite for every $j = 0, \ldots, J_q$ and may be bounded by a constant C independent of k. Furthermore,

$$\|\widetilde{\eta}_{ck}^{(q)}\|_{V_{-\gamma}^0(\Omega)} = \left(\int\limits_\Omega |\widetilde{\eta}_{ck}^{(q)}(x)|^2 r^{-2\gamma} dx\right)^{1/2} = \left(\int\limits_\Omega |\widetilde{\eta}_{ck}^{(q)}(x)|^2 r^{-2} r^{2-2\gamma} dx\right)^{1/2} \leq C\|\widetilde{\eta}_{ck}^{(q)}\|_{V_{-1}^0(\Omega)} \leq Ck^{q\pi/2\alpha+1}.$$

Here we exploited the fact that $2 - 2\gamma > 0$ and the estimate $\|\widetilde{\eta}_{ck}^{(q)}\|_{V_{-1}^0(\Omega)} \leq Ck^{q\pi/2\alpha+1}$ which is a consequence of (3.52). Coming back to (3.59) and estimating J_q by $q\pi/2\alpha$ from above (see (3.51)) we conclude that

$$\|\eta_{ck}^{(q)}\|_{V_{-\gamma}^0(\Omega)} \leq Ck^{q\pi/2\alpha+1}.$$

Analogous estimates are valid for the functions $\eta_{sk}^{(q)}$, $\eta_{ck}^{*(q)}$, $\eta_{sk}^{*(q)}$. Therefore the estimate (3.57) can be rewritten as follows

$$|c_{ck}^{(q)}| \leq Ck^{q\pi/2\alpha+1} \left(\|f_{ck}\|_{V_\gamma^0(\Omega)} + \|f_{sk}\|_{V_\gamma^0(\Omega)}\right). \tag{3.60}$$

In the same way we derive the estimate

$$|c_{sk}^{(q)}| \leq Ck^{q\pi/2\alpha+1} \left(\|f_{ck}\|_{V_\gamma^0(\Omega)} + \|f_{sk}\|_{V_\gamma^0(\Omega)}\right). \tag{3.61}$$

Chapter 4

Asymptotic representation of a solution to the time-periodic problem

Let us consider again the time-period boundary value problem for the heat equation

$$\begin{cases} u_t - \Delta u &= f, & x \in \Omega \times [0, 2\pi), \\ u &= 0, & x \in \partial\Omega \times [0, 2\pi), \\ u(x,0) &= u(x, 2\pi), & x \in \Omega. \end{cases} \tag{4.1}$$

with the 2π-periodic in time function f which is represented in the following form

$$f(x,t) = \sum_{k=0}^{\infty} (f_{ck}(x)\cos(kt) + f_{sk}(x)\sin(kt)). \tag{4.2}$$

Assume that $f \in L_2(0, 2\pi; V^0_\gamma(\Omega))$ with $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$ for some $m \in \mathbb{N}$. Then for every $k = 0, 1, \ldots,$ the Fourier coefficients f_{ck}, f_{sk} belong to the space $V^0_\gamma(\Omega)$ and the following inequality holds:

$$\sum_{k=0}^{\infty} \left(||f_{ck}||^2_{V^0_\gamma(\Omega)} + ||f_{sk}||^2_{V^0_\gamma(\Omega)} \right) < \infty. \tag{4.3}$$

For each $k = 0, 1, \ldots$ we consider the problem

$$\begin{cases} -\Delta u_{ck}(x) + k u_{sk}(x) &= f_{ck}(x), & x \in \Omega, \\ -\Delta u_{sk}(x) - k u_{ck}(x) &= f_{sk}(x), & x \in \Omega, \\ u_{ck}(x) = 0, \ u_{sk}(x) &= 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

If $k = 0$, this system splits into two boundary value problems for the Poisson equation. Asymptotic representations (2.3) of the solutions u_{c0}, $u_{s0} \in V^2_\beta(\Omega)$, $|\beta - 1| < \pi/\alpha$ are obtained using Theorem 2. For the cases corresponding to $k > 0$, we use results presented in Chapter 3 and have the following representations of the functions $u_{ck}, u_{sk} \in V^2_\beta(\Omega)$:

$$u_{ck}(x) = \chi(r) \left(\sum_{q=1}^{m} c^{(q)}_{ck} \, r^{q\pi/\alpha} \, \sin\left(\frac{q\pi\varphi}{\alpha}\right) + \sum_{j=1}^{M_1} u^{(1j)}_{ck}(x) + \cdots + \sum_{j=1}^{M_m} u^{(mj)}_{ck}(x) \right) + \tilde{u}_{ck}(x), \tag{4.5}$$

$$u_{sk}(x) = \chi(r) \left(\sum_{q=1}^{m} c_{sk}^{(q)} \ r^{q\pi/\alpha} \ \sin\left(\frac{q\pi\varphi}{\alpha}\right) + \sum_{j=1}^{M_1} u_{sk}^{(1j)}(x) + \cdots + \sum_{j=1}^{M_m} u_{sk}^{(mj)}(x) \right) + \tilde{u}_{sk}(x). \quad (4.6)$$

Substituting (4.5) and (4.6) into

$$u(x,t) = \sum_{k=0}^{\infty} \left(u_{ck}(x)\cos(kt) + u_{sk}(x)\sin(kt) \right), \quad (4.7)$$

we get series which can be divided into terms of the three following types:

Type 1: $r^{q\pi/\alpha} \sin\left(\frac{q\pi\varphi}{\alpha}\right) \sum_{k=0}^{\infty} \left(c_{ck}^{(q)} \cos(kt) + c_{sk}^{(q)} \sin(kt) \right), \quad q = 1, \dots, m;$

Type 2: $r^{q\pi/\alpha+2j} \sin\left(\frac{q\pi\varphi}{\alpha}\right) \sum_{k=0}^{\infty} \left(\lambda_c^{(qj)} c_{ck}^{(q)} k^j \cos(kt) + \lambda_s^{(qj)} c_{sk}^{(q)} k^j \sin(kt) \right), q = 1, \dots, m, \ j = 1, \dots, M_q,$

$$\lambda_c^{(qj)} = \frac{\sigma_c(q)\alpha^j}{\prod_{i=1}^{q}(4qi\pi + (2i)^2\alpha)}, \qquad \lambda_s^{(qj)} = \frac{\sigma_c(q)\alpha^j}{\prod_{i=1}^{q}(4qi\pi + (2i)^2\alpha)}.$$

Type 3: $\sum_{k=0}^{\infty} \left(\tilde{u}_{ck}(x)\cos(kt) + \tilde{u}_{sk}(x)\sin(kt) \right).$

The terms of Type 1 and Type 2 consist of a part which depends on the spatial variables r and φ only, and a time dependent part determined by the series

$$\sum_{k=0}^{\infty} \left(c_{ck}^{(q)} \cos(kt) + c_{sk}^{(q)} \sin(kt) \right), \quad q = 1, \dots, m \quad (4.8)$$

in the terms of Type 1 and by the series

$$\sum_{k=0}^{\infty} \left(\lambda_c^{(qj)} c_{ck}^{(q)} k^j \cos(kt) + \lambda_s^{(qj)} c_{sk}^{(q)} k^j \sin(kt) \right), \quad q = 1, \dots, m, \ j = 1, \dots, M_q \quad (4.9)$$

in the terms of Type 2. We will show that under certain assumptions on the function $f = f(x,t)$ series (4.8) and (4.9) converge in $L_2(0, 2\pi)$.

From estimates (3.60), (3.61) we have for every $q = 1, \dots, m$:

$$\sum_{k=0}^{\infty} |c_{ck}^{(q)}|^2 + |c_{sk}^{(q)}|^2 \le C \sum_{k=0}^{\infty} k^{q\pi/\alpha+2} \left(\|f_{ck}\|^2_{V_\gamma^0(\Omega)} + \|f_{sk}\|^2_{V_\gamma^0(\Omega)} \right). \quad (4.10)$$

Assume that $f = f(x,t)$ has time derivatives $\partial^l f/\partial t^l \in L^2(0, 2\pi, V_\gamma^0(\Omega))$ up to the order N_1 for some $N_1 \in \mathbb{N}$[1]. This ensures that the following series, obtained after differentiating (4.2) with respect to t, converges:

$$\sum_{k=0}^{\infty} k^l \left(\|f_{ck}\|^2_{V_\gamma^0(\Omega)} + \|f_{sk}\|^2_{V_\gamma^0(\Omega)} \right) < \infty, \quad q = 1, \dots, N_1.$$

By taking N_1 satisfying the condition

$$N_1 = [m\pi/2\alpha + 1], \quad (4.11)$$

[1]Let us recall that the space of functions having time derivatives $\partial^l f/\partial t^l$ with values in $V_\gamma^0(\Omega)$ up to the order N_1 is denoted by the symbol $H^{N_1}(0, 2\pi, V_\gamma^0(\Omega))$. See, for example, [2].

we ensure that the series in (4.10) converge. Consequently we obtain the convergence of series (4.8) in $L_2(0,2\pi)$ for every $q=1,\dots,m$.

Let us examine convergence of the Type 2 series (4.9). Among them the series corresponding to $j=M_q$ contains the highest power of k, namely the term k^{M_q}. Since $\lambda_c^{(qM_q)}$ and $\lambda_s^{(qM_q)}$ are independent of k, we have, combining (3.37) and (3.60), (3.61)

$$\sum_{k=0}^{\infty}|\lambda_c^{(qM_q)}c_{ck}^{(q)}k^{M_q}|^2+|\lambda_s^{(qM_q)}c_{sk}^{(q)}k^{M_q}|^2\le C\sum_{k=0}^{\infty}k^{q\pi/\alpha+2}k^{(m+1-q)\pi/\alpha}\left(\|f_{ck}\|^2_{V^0_\gamma(\Omega)}+\|f_{sk}\|^2_{V^0_\gamma(\Omega)}\right).$$

The series on the right-hand side of the last inequality converges if $f\in H^{N_2}(0,2\pi;V^0_\gamma(\Omega))$, where

$$N_2=[(m+1)\pi/2\alpha+1]\,.$$

It remains to prove the convergence of Type 3 series. Estimates (3.39), (3.60) and (3.61) yield that

$$\|\tilde{\mathbf{u}}_k\|_{V^2_\gamma(\Omega)}\le C\left(k^n+\sum_{i=1}^{n}k^{n-i}\sum_{q=Q_{i-1}}^{Q_i}k^{q\pi/2\alpha+1}\right)\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)}.$$

Having in mind that $Q_i=[2i\alpha/\pi]$, $i=1,\dots,n-1$ and $Q_n=m$, we have

$$\sum_{i=1}^{n}k^{n-i}\sum_{q=Q_{i-1}}^{Q_i}k^{q\pi/2\alpha+1}\le C\sum_{i=1}^{n}k^{n-i}k^{Q_i\pi/2\alpha+1}\le C\left(\sum_{i=1}^{n-1}k^{n-i}k^{i+1}+k^{m\pi/2\alpha+1}\right)$$

Therefore,

$$\|\tilde{\mathbf{u}}_k\|_{V^2_\gamma(\Omega)}\le C\left(k^{n+1}+k^{m\pi/2\alpha+1}\right)\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)}. \tag{4.12}$$

Since $1-2n\le\gamma<1-2(n-1)$ (see Lemma 1) and $\gamma\in\left(1-\frac{(m+1)\pi}{\alpha},1-\frac{m\pi}{\alpha}\right)$, the inequality

$$n+1<\frac{1-\gamma}{2}+2<\frac{(m+1)\pi}{2\alpha}+2$$

holds. Consequently we derive, using (4.12), the estimate

$$\|\tilde{\mathbf{u}}_k\|_{V^2_\gamma(\Omega)}\le Ck^{(m+1)\pi/2\alpha+2}\|\mathbf{f}_k\|_{V^0_\gamma(\Omega)}. \tag{4.13}$$

Assume that f belongs to the space $H^{N_3}(0,2\pi;V^0_\gamma(\Omega))$, where

$$N_3=[(m+1)\pi/2\alpha+3]\,.$$

This assumption and estimate (4.13) guarantee convergence of the Type 3 series in the space $L_2(0,2\pi;V^2_\gamma(\Omega))$. Since we have the convergence of all types of series, the previous results can be summarized as the following statement.

Theorem 4.

Assume that a time-periodic function $f = f(x,t)$ *belongs to the space* $H^N(0, 2\pi; V^0_\gamma(\Omega))$ *with* $N = [(m+1)\pi/2\alpha + 3]$ *and* $\gamma \in \left(1 - \frac{(m+1)\pi}{\alpha}, 1 - \frac{m\pi}{\alpha}\right)$ *for some* $m \in \mathbb{N}$. *Then the main asymptotic part of a time-periodic solution to the problem* (4.1) *in the neighborhood of the corner point* O *admits the following representation:*

$$\sum_{q=1}^{m} r^{\frac{q\pi}{\alpha}} \sin\left(\frac{q\pi\varphi}{\alpha}\right) C^{(q)}(t) + \sum_{q=1}^{M_1} r^{\frac{\pi}{\alpha}+2j} \sin\left(\frac{\pi\varphi}{\alpha}\right) C^{(1q)}(t) + \ldots + \sum_{q=1}^{M_m} r^{\frac{m\pi}{\alpha}+2j} \sin\left(\frac{m\pi\varphi}{\alpha}\right) C^{(mq)}(t).$$

Here $M_q = \max\{n \in \mathbb{N} : n \leq \frac{(m-q+1)\pi}{2\alpha}\}$, *while the time-periodic functions* $C^{(q)}, C^{(q1)}, \ldots, C^{(qM_q)}$, $q = 1, \ldots, m$, *belong to the space* $L_2(0, 2\pi)$ *and are defined by the series* (4.8) *and* (4.9).

Summary

In this thesis "Asymptotics of a solution to the time-periodic heat equation set in domains with corner points" by Vytenis Šumskas, a time-periodic boundary value problem for the heat equation is considered. It is assumed that the problem is set in a two dimensional domain Ω having a corner point on the boundary. The time-periodic problem was reduced to a sequence of elliptic problems in the domain with a corner point. Asymptotic representations of solutions to these problems were obtained using methods proposed in numerous works of V. Maz'ya, S. Nazarov and B. Plamenevskij. Estimates of asymptotic terms were derived. These estimates together with representations of solutions to elliptic systems were used to determine the time-periodicity of solution to the time-periodic problem in a neighborhood of the corner point of domain Ω.

Santrauka

Vytenio Šumsko darbe „Laike periodinio šilumos laidumo uždavinio sprendinio asimptotika srityse su kampiniu tašku“ tiriamas laiko atžvilgiu periodinis kraštinis šilumos laidumo uždavinys. Uždavinys nagrinėjamas dvimatėje srityje Ω, turinčioje kampinį tašką. Darbe parabolinis uždavinys yra išskaidomas į elipsinių uždavinių seką, kurių tolimesnei analizei taikoma elipsinių uždavinių srityse su kampiniais taškais teorija, plačiai išvystyta *V. Maz'ya, S. Nazarov ir B. Plamenevskij* darbuose. Sukonstravus sprendinio asimptotinę išraišką kampinio taško aplinkoje, randami į asimptotiką įeinančių narių įverčiai. Šie rezultatai panaudojami uždavinio sprendinio periodiškumui laike tirti.

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