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Aukšto dažnio duomenų multifraktalumas ir volatilumo prognozavimas

Multifractality of High Frequency Data and Volatility Forecasting

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Santrauka

Šio darbo tikslas yra pabrėžti duomenų multi-fraktalumo savybės svarbą naudojant didesnio nei vienos dienos dažnio finansinius duomenis. Empiriniai rezultatai yra gauti pritaikant apibendrintą Hurst eksponentę ir multifraktalinę nutrendintų svyravimų analizę šiems finasiniams indeksams: Dow Jones industriniam vidurkiui, Australijos vertybinių popierių indeksui, Nikkei-225 ir NASDAQ-100. Multifraktalumas buvo aptiktas visų išvardintų indeksų kainose. Tam kad tinkamai panaudotumėm aukšto dažnio duomenis, buvo suskaičiuotas faktinis volatilumas ir pritaikytas Binominis Markovo pasikeitimų multifraktalinis modelis. Šio modelio prognozės galia buvo palyginta su gerai žinomu heterogeniniu auto-regresiniu modeliu. Binominis Markovo pasikeitimų multifraktalinis modelis parodė geresnes prognozes trumpiems prognozavimo horizontams. Tuo tarpu heterogeninis auto-regresinis modelis buvo pranašesnis prognozuojant daugiau nei 10 žingsnių į priekį.

Raktiniai žodžiai: multi-fraktalumas, volatilumas, aukšto dažnio duomenys, multifraktalinis modelis

Multifractality of High Frequency Data and Volatility Forecasting

Abstract

The purpose of this thesis is to emphasize the importance of multi-fractal concept by providing an empirical evidence using intra-day financial time series. The Multifractal Detrended Fluctuation analysis and the Generalized Hurst exponent methods were applied on the price indices of Dow Jones Industrial Average, Australian Securities Exchange, Nikkei-225 and NASDAQ-100. The presence of multi-scaling was detected on prices of each mentioned index. In order to employ high frequency data we calculated realized volatility and applied the Binomial Markov-Switching Multifractal model. The power of prediction accuracy was compared to well-established Heterogeneous Auto-Regressive model. The Binomial Markov-Switching Multifractal model showed better performance on the short horizons while Heterogeneous Auto-Regressive model surpassed the latter on long horizon.

Keywords: multi-fractality, volatility, high frequency data, multi-fractal model

Contents

Chapter 1

Introduction

The financial theory is trying to understand how do the financial markets work and to recognize a process generating the financial data using various methods. Empirical finding and result obtained so far helps getting more accurate forecasts by developing and adjusting existing models. This thesis studies the multi-fractality or so called multi-scaling property of time series. This property have been found in financial series relatively lately Di Matteo, Aste, and Dacorogna (2005).

This property is related to the natural feeling about complexity of markets and at the same time it is attractive because it allows us to describe the time series using scaling exponents. Before the explanation of what multi-fractality is, we will provide previous studies in the theory. At the beginning the theoretical work required the assumption of independent and Gaussian returns Fama (1965) and this soon had many applications in the modeling, e.g. the Capital Asset Pricing Model by Sharpe (1964) and the Black-Sholes model by Black and Scholes (1973). Later this assumption has been found contradicting by number of papers by overview of so-called stylized facts, e.g. Cont (2001). Also these facts should be captured by the model which tries to describe the market.

Firstly, the distribution of returns was found to be leptokurtic and it was explained by the fact that extreme events have higher probability than normal distribution would suggest. Volatility clustering is another known fact of the financial data. Indeed, the volatility of price returns is not constant but contrariwise it fluctuates differently within different periods. This fact is captured by the modeling of the second moment by family of the Autoregressive conditional heteroskedasticity (ARCH) models. ARCH model was proposed by Robert Engle in 1980s and it describes volatility conditionally on components of past volatility and returns which are expressed by volatility and white noise Engle (1982). Its generalized form (GARCH) is more popular and it also models volatility conditionally on past volatility, but the squared returns are introduced additionally T. Bollerslev (1986).

The main weakness of ARCH family models is that they can not explain behavior of volatility at different frequencies B. Mandelbrot and Fisher (1997). Lauren Calvet (2004) emphasize that there is a logical reason of this feature as long as the economical shocks have different duration and frequencies, e.g. change in political situation, business cycle, financial crises, etc. Also, the ARCH type models are not able to capture the property of multi-fractality in absolute moments of fluctuations. Multi-fractality of moments is a stylized fact which is the main subject of our thesis.

In order to generate data with all the mentioned properties above multi-fractal models have been introduced recently. In general multi-fractal process are retrieved on the priciple of iteration. Lauren Calvet (2004) formulated the Markov-Switching Multifractal (MSM) model based on the idea that volatility is composed of shocks which have different frequency and duration. Basically the MSM model was an improvement of the earlier Multifractal Model of Asset Returns formulated by B. Mandelbrot and Fisher (1997). In order to capture the volatility, MSM model contains several variables which are based on switching regimes with the values coming from the same distribution. The product of these variables then describes the volatility. The frequency of the volatility components is determined by regime switching probability.

1.1 Purpose of the thesis

The aim of this thesis is to investigate the theoretical framework behind the multifractal detection methods and discrete time Markov-Switching Multifractal (MSM) model, and apply it in order to forecast the realized volatility. We will provide basic theory on multifractality and its detection methods, which robustness will be investigated. The MSM model itself is a pure regime-switching model, using a probability transition matrix, multiple frequencies and arbitrarily many states.

By using 5 minutes closed price data from aggregated indices, we will conduct an empirical study of the performance of the binomial MSM model for realized volatility. More specifically the MSM model will be compared to leading realized volatilty forecasting HAR model, both in- and out-of-sample comparisons will be conducted in very short horizons such as one day and at longer forecast horizons, such as 5 to 20 business days.

In short this thesis will try to answer the following research questions:

- Is there any evidence that methods to detect multi-fractality are robust as long as they have no asymptotic theory developed?

- Do any of selected intra-day datasets have multi-fractal trace?
- How does the MSM model performs to predict realized volatility in comparison with more established model?

1.2 Structure of the thesis

The thesis core is consisted of three chapters which cover the theoretical framework of the thesis, the methodology and a representation of the empirical results. The thesis is ended with a chapter of concluding remarks of the study. Thus, a detail structure of the thesis is listed below

- Chapter 2 - The aim of this chapter is to provide the theoretical framework which will be used throughout the thesis. Firstly, we review some fractal and multifractal concepts necessary for understanding what is multifractal process. Secondly, we present selected methods to detect whether the process is multi-fractal or not. Following this, we turn to the one of the main subject of the thesis, the Markov switching multi-fractal (MSM) model. In this chapter we also review alternative Heterogeneous Auto-Regressive (HAR) model for realized volatility.
- Chapter 3 - In this chapter we start by introducing the data, which is analyzed in the empirical section. Thereafter we present the tests on robustness of multi-fractality detection methods. The results of these methods are showed in the separate section. Finally we discuss and compare prediction accuracy between MSM and HAR model.
- Chapter 4 - The final chapter summarizes and concludes on the main findings of the thesis.

Chapter 2

Theoretical Framework

2.1 Fractals and Multifractals

The aim of this section is to provide a broader picture of fractal analysis and main definitions related to fractality. The fractal theory was described by Mandelbrot (1982) in order to study the roughness of surfaces. Mandelbrot was studying rough and complex patterns what is the contrast to well known Euclidean measures. Self-similarity, which definition will be provided later, is the main characterization of fractals. In short we can say that if an object is self-similar then the whole view is similar to its scaled down parts. This lets us to observe same or at least similar patterns at every scale. Visually, self-similarity which can be found in nature and can be understood very intuitively. Self-similar patterns are observed in the branches of a tree (branches can be divided to smaller ones following the same rule again and again), snowflakes, coastlines and others phenomenons B. Mandelbrot and Hudson (2007).

A dimension of fractals is characterized by non-interger number as opposed to well known Euclidean geometric objects which have an integer number of dimension (e.g. a straight line has a dimension equal to one and a plane is the two dimensional object) Cornelis (2004). From the perspective of fractals an object is analysed at different scales in order to detect the similarities. These similarities can be described in a form of a power law or so called scaling law. We present the definition of scaling law based on Kantelhardt (2008).

Definition 2.1. (Scaling law) A power law with a scaling exponent (e.g. α) describing the behaviour of a quantity F (e.g. fluctuation, spectral power) as function of a scale parameter s (e.g. time scale, frequency) at least asymptotically: $F(s) \sim s^{\alpha}$. The symbol $\sim$ denotes proportionality and asymptotically it holds $\alpha = \lim_{s\to\infty} \frac{\log F(s)}{\log s}$. The power law should be valid for a large range of s values, e.g. at least for one order of magnitude.

A system which can be described by a scaling law with a non-integer scaling exponent is called fractal system. Fractals appear to be very complex usually, however they have its simplicity which lies directly in the fact that they can be described by the unique scaling exponent. In contrast, there are more complicated systems called multi-fractal systems whose scaling properties cannot be characterized by a single number but needs to be described by a function of scaling exponents Kantelhardt (2008). Note that the definition of (multi)fractal system does not state that the scaling law must hold for all possible scales s and that is why we have to be carefull not to work with scales exceeding a maximum scale when estimating scaling exponent Kantelhardt (2008). Di Matteo (2007) says that there are two approaches of estimating scaling exponent and they are sometimes mixed together. The first approach, which is not the object of our interest in this thesis, is to use different length of time intervals (e.g. daily, weekly, monthly) and analyse how the distribution is changing (for self-similar or so called self-affine object, the distribution remains the same), e.g. for financial data which is the main object of our research it holds that the distribution of returns have thinner tails by increasing time intervals. Another possibility to estimate scaling exponent is observe a behavior of a process using sample moments of increments while changing scale parameter (e.g. time interval) and look for a dependence of scaling law. In this research we use the second approach and therefore in the next section we define the multifractal process using the sample moments of increments perspective.

2.2 Multifractality

Prior to a discussion of multifractal models, we want to explore some of the ideas of multifractality and provide basic definitions. In general a fractal is a geometrical object which has fractional dimension

$$D = \lim_{\epsilon \to 0} \frac{N(\epsilon)}{\frac{1}{\epsilon}}$$

where $N(\epsilon)$ is the number of "boxes" needed to cover the object as a function of measure size ϵ, which goes to zero. One very important property of fractals is self similarity, which means that the object is similar at any scale, but is not identical. Authors often mention the branches of a tree which are similar to each other ignoring the size, therefore each branch is also unique.

In his 1963 publication Benot Mandelbrot proposed that the distribution of returns should be self-similar i.e. invariant to changes in the time scale.

Definition 2.2. A stochastic process $X(t)$ that satisfies (sign $\stackrel{d}{=}$ below means equality of distributions)

$$\{X(ct_1), ..., X(ct_k)\} \stackrel{d}{=} \{c^H X(t_1), ..., c^H X(t_k)\}$$

for some $H > 0$ and all $c, k, t_1, ..., t_k \geq 0$ is called self similar or self-affine. The number H is the self-similarity index or scaling exponent, of the process $X(t)$.

As it was already mentioned, multifractal process is defined based on scaling properties of moments of its increments. The definition works with discrete time because of the nature of data we will be working with in the empirical part. The definition of the multifractal process which we use in this thesis is directly used in methods which can detect multifractal behavior of a process.

We present definition through the scaling of moments from B. Mandelbrot and Fisher (1997)

Definition 2.3. (Multifractal Process). A stochastic process $X(t)$ is called multifractal if it has stationary increments and satisfies the scaling law

$$\mathbb{E}(|X(t)|^q) \sim c(q)t^{\tau(q)+1}$$

for all $t \in \mathcal{T}$, $q \in \mathcal{Q}$ where $\mathcal{T}$ and $\mathcal{Q}$ are real line intervals. $\mathcal{T}$ and $\mathcal{Q}$ have positive length with $0 \in \mathcal{T}$, $[0, 1] \subseteq \mathcal{Q}$, τ and c are functions with domain $\mathcal{Q}$.

The function τ is called the scaling function. This function has an intercept $\tau(0) = -1$, moreover, the function is concave (for the proof see B. Mandelbrot and Fisher (1997)). The research of scaling function is a main subject of multifractality detection methods which will be introduced in the next section. The definition claims that the multiscaling is able to reflect the behavour of various size of process fluctuations because their weight in the mean changes by changing q. The higher is q, the more weight is put on large fluctuations Kantelhardt (2008). If $\tau(q)$ is a linear function of q we say that the processes is uniscaling and is identified by a single exponent. For this reason all sizes of fluctuations must be ruled by the same scaling law. In addition to that, for self-affine processes it holds (B. Mandelbrot and Fisher (1997)):

$$\tau(q) = qH - 1$$

. The H measure is the famous Hurst exponent which will be defined soon. Usually this relationship is often generalized in the empirical analysis and so-called generalized Hurst exponent $H(q)$ is used

$$\tau(q) = qH(q) - 1 \tag{2.1}$$

The second moment is in the main subject of many academics for the reason it relates to scaling of the autocorrelation function and it helps us to detect long memory in a process by this way. It can be defined with reference to Di Matteo (2007):

Definition 2.4. (Hurst exponent). $X(t)$ is a Gaussian random function for which $\mathbb{E}(X(t)) = 0$ and $\mathbb{E}(X^2(t)) = 1$ and $C(\Delta t)$ is its auto-correlation function

$$C(\Delta t) = \mathbb{E}|X(t)X(t + \Delta t))$$

If the auto-correlation function has the following behavior $C(\Delta t) \sim |\Delta t|^{-\beta}$ as $\Delta t \to +\infty$ where $0 < \beta \leq 2$. Then

$$H = 1 - \frac{\beta}{2}$$

is called the Hurst exponent.

The value of the Hurst exponent could indicate if the process is correlated or not. The case when Hurst exponent is equal to 0.5 tells us that the increments of a process are independent or short-term correlated. When $0.5 < H < 1$ the process is persistent. In other words we can say that the process has a long memory or long-term correlated. For the latter case it is more likely that a positive change in price will be followed by another positive and negative one by another negative change. Also long-term correlated process has the autocorrelation function which decays very slowly. When $0 < H < 0.5$ the process is called antipersistent. For such a process, it is more likely that an increment will be followed by an opposite sign increment than with a positive one Di Matteo (2007).

Up to now we discussed the global scaling properties of a process. For completness, we also provide a definition of the Hölder exponent. Having this exponent one can capture local scaling properties of a process. Following the Di Matteo (2007):

Definition 2.5. (Hölder exponent). Let $X(t)$ be a stochastic process. The Hölder exponent $\alpha(t)$ is defined by the relation:

$$c_t(\Delta t)^{\alpha(t)} \sim \mathbb{E}|X(t + \Delta t) - X(t)|$$

as $\Delta t \to 0$. Here $\alpha(t)$ and c_t are, respectively, the local Hölder exponent and the prefactor at t.

At a given point t the Hölder exponent describes a roughness of a process. The rule is: having the lower value of the Hölder exponent the rougher a process is. It was already mentioned that unifractals are defined by a single value of scaling exponent at every point of the process. While for the multifractal processes the Hölder exponents is not unique but it is a continuum of local scales Di Matteo (2007).

There is another way to characterize multifractals using a so called multifractal spectrum if the scaling function $\tau(q)$ is defined. Legendre transformation of the scaling function can be applied in order to obtain multifractal spectrum (for the derivation see Riedi (1999)). The width of multifractal spectrum is an indicator whether the process is multiscaling (at $\tau'(q) = \alpha$). The wider is the difference between maximum and minimum values of α the more likely process is multifractal:

$$f(\alpha) = q\alpha - \tau(q)$$

or alternatively using the generalized Hurst exponent (2.1)

$$\alpha = H(q) + qH'(q) \quad \textit{and} \quad f(\alpha) = q[\alpha - H(q)] + 1$$

Before presenting methods which can detect multifractality, we will introduce two types of multifractality source Kantelhardt (2008). To be more precise multifractalty of time series can be caused by these two occurrences:

- Firstly, multifractality can be caused by the data being drawn from a heavy-tailed probability distribution.
- Secondly, by the long-term correlations of small and large fluctuations.

Very straightforward and well known way to detect the source of multifractality is to shuffle the data randomly and apply the multifractal test. If a data do not show multifractality after the shuffling but showed before, the source is long-term correlations. If the shuffling did no impact on the data then the distribution is the main reason for multifractality. Also it can be the other case then multifratality of the data will be just weakened. This means that both sources of multifractality exists in the data Kantelhardt and Zschiegner (2002).

2.3 Multifractality detection methods

The previous section explained the multifractal property of the time series and described how scaling function defines it. This section presents two methods which will be used to conduct the multifractal analysis in the the empirical part of this research. The first of the methods is the Generalized Hurst exponent (GHE) method which measure the multifractality by using the increments of the time series. Although the method is simple and easily applied, unfortunately it can not be used for the nonstationary time series. For this reason the Multifractal detrended Fluctuation (MF-DFA) analysis is introduced in order to detect the level of multifractality of the price indices presented in the following chapter. The procedure, which is deployed relatively simple for both of them, will be presented shortly by describing correct usage step by step. The finite sample properties of each of them will be discussed as well. As usual, they has its strengths and weaknesses and we want to implement the complex analysis of multifractality by employing both of them. The purpose of performing the multifractal analysis is to find out if we are dealing with monofractality or multifractality in the intra-day time series by estimating the scaling function described in the previous section.

2.3.1 Generalized Hurst Exponent

The first selected method in our research is the Generalized Hurst Exponent. The word "generalized" stamds for the fact that this method is capable to detect not only monofractal process, but multifractal as well. The provided GHE procedure is based on Di Matteo, Aste, and Dacorogna (2005):

- For the given a time series of prices P_t, calculate the logarithmic returns $r_t = \log P_t - \log P_{t-1}$ where $t = 1, ..., T$
- Calculate statistics for which the properties of fractality will be studied:

$$K_q(\Delta t) = \frac{\sum_{t=0}^{T-\Delta t} |r(t+\Delta t) - r(t)|^q}{\sum_{t=0}^{T-\Delta t} |r(t)|^q}$$

- Repeat the previous step with different values of Δt . The minimum value for Δt is one unit of time of price series
- Analyze log-log plots of Δt versus $K_q(\Delta t)$ for different values of q. If there are any traces of fractality in the time series, then the statistics $K_q(\Delta t)$ has scaling behavior:

$$K_q(\Delta t) \sim c\Delta t^{qH(q)}$$

- In order to estimate the Hurst exponent $H(q)$, a simple linear regression can be applied on logarithm of both sides of scaling behavior.

2.3.2 Multifractal Detrended Fluctuation Analysis

The Multifractal Detrended Fluctuation Analysis (MF-DFA) was selected as a second method in order to disregard the nonstationarities in the analyzed time series. It is a generalization of the Detrended Fluctuation Analysis that is capable to detect monofractality only. The Detrended Fluctuation Analysis is appropriate method for measuring the scaling properties of time series and its generalization was derived to estimate the generalized Hurst exponent Kantelhardt and Zschiegner (2002). The provided MF-DFA procedure is based on Kantelhardt (2008):

- Calculate the so called profile in the first step :

$$Y(t) = \sum_{k=1}^{t} \left(r(k) - \frac{1}{T} \sum_{k=1}^{T} r(k) \right), \quad t = 1, ..., T.$$

- Divide the profle $Y(t)$ into T_s equal segments which do not overlap:

$$T_s = \left\lfloor \frac{T}{s} \right\rfloor$$

- Estimate the local trend $y_{\nu,s}(i)$ for each of the $\nu = 1, ..., T_s$ segments by fitting the polynomial order m (order can be chosen freely, e.g. linear, quadratic, cubic, etc.). Then calculate the variance:

$$F^2(s,\nu) = \frac{1}{s} \sum_{i=1}^{s} \{Y[(\nu-1)+i] - y_\nu(i)\}^2, \quad \nu = 1, ..., T_s$$

- Average the variance over all segments in order to obtain the q-th order fluctuation function:

$$F_q(s) = \left(\frac{1}{N_s} \sum_{\nu=1}^{N_s} [F^2(s,\nu)]^{q/2} \right)^{1/q}$$

where $q \in \mathbb{R}/\{0\}$. For $q = 0$ adjusted fluctuation function is introduced:

$$F_0(s) = exp\left(\frac{1}{2N_s} \sum_{\nu=1}^{N_s} ln[F^2(s,\nu)] \right)$$

- Repeat the previous steps for various lengths s to retrieve the rule how the function $F_q(s)$ depends on s.
- Analyze log-log plots of s versus $F_q(s)$ for different values of q.If there are any traces of fractality in the time series, then:

$$F_q(s) \sim s^{H(q)}$$

- In order to estimate the Hurst exponent $H(q)$, a simple linear regression can be applied on logarithm of both sides of scaling behavior.

2.4 Volatility forecasting models

In the following section we briefly discuss the general concept of volatility which will provide the background for the subsequent sections on volatility forecasting models. Also the estimator of realized volatility will be presented which concept will be implemented into standard Binomial Markov-Switching Multifractal model in order to compare forecasting accuracy to well-known Heterogeneous Auto-Regressive model of realized volatility.

2.4.1 The concept of volatility

This section uses the discussion based on the Andersen (2006) of the volatility modeling. In the most cases the models of volatility takes the following definition of financial returns over the points in time $t = 1, ..., T$:

$$r_t = \mu_t + \sigma_t \epsilon_t \tag{2.2}$$

where $r_t = ln(P_t/P_{t-1})$, P_t is the asset price, $\mu_t = \mathbb{E}_{t-1}[r_t]$ is the time $t-1$ conditional mean of the returns, $\sigma_t^2 = Var_{t-1}[r_t]$ is the conditional volatility or variance process and ϵ_t is the error component which is an independently and identically distributed with mean zero and variance equal to one. In this thesis we consider the error term to be distributed according to standard normal. Also we will use so called 'centered' returns which can be modeled as

$$r_t = \sigma_t \epsilon_t \tag{2.3}$$

It is important to note that the volatility σ_t^2 is unobservable, i.e. it is a latent process. However Andersen (2006) showed that so called realized volatilty (RV) obtained from high frequency (intra-day) data is an consistent estimator of the actual volatility at period t. According to authors the empirical estimation of volatility can be efficiently obtained via RV:

$$RV_t = \sum_{j=1}^{\frac{1}{\Delta}} r^2(t-1+j\cdot\Delta, \Delta) = \sum_{j=1}^{\frac{1}{\Delta}} (\ln P(t-1+j\cdot\Delta) - \ln P(t-1+j\cdot\Delta-\Delta))^2 \quad (2.4)$$

with $0 < \Delta < 1$ and $\frac{1}{\Delta}$ integer.

T. Andersen and Bollerslev (2001) also showed that returns standardized to the realized volatility are distributed normally.

2.4.2 The Markov-Switching Multifractal (MSM) model

In this part of the thesis, the discrete-time version of the main model called the Markov-Switching Multifractal (MSM) is presented. This model was firstly proposed by Lauren Calvet (2004). The MSM model has intuitive economic interpretation, since it employs the fact that financial markets are driven by a number of economic units with different levels of persistence. The model is quite simple to understand and only requires only four parameters in the binomial version.

Lets take a financial series P_t and centered log returns defined in the previous section. The MSM model considers a market which is driven by a first-order Markov state vector with $\bar{k}$ volatility components:

$$M_t = (M_{1,t}, M_{2,t}, ..., M_{\bar{k},t}) \in R_+^{\bar{k}}$$

The volatility is estimated by multiplying together the random first-order Markov components. For simplicity it is assumed that the components of M_t have the identical marginal distribution, but may vary at different frequencies. Lets assume that the volatility state vector contains observations up to date $t-1$. For each $k \in 1, ..., \bar{k}$, the multiplier of the next period $M_{k,t}$ is taken from some fixed distribution M with probability γ_k, and otherwise is equal to the previous value: $M_{k,t} = M_{k,t-1}$. The dynamics of $M_{k,t}$ may be stated as

- $M_{k,t}$ Drawn from distribution M with probability γ_k
- $M_{k,t} = M_{k,t-1}$ with probability $1 - \gamma_k$,

it is important that these outcomes are independent across k and t. The distribution of M has a positive support and mean equal to one: $M > 0$ and $E[M] = 1$ Lauren Calvet (2004).

The multipliers $M_{k,t}$ are persistent and positive, also they satisfy $\mathbb{E}[M_{k,t}] = 1$ under the assumptions mentioned above. In addition to that the multipliers are different only by their probabilities of transition k, but not by the marginal distribution M. The components of the different frequencies are independent: to be more precise, the variables $M_{k,t}$ and $M_{k',t'}$ are independent when k is not equal to k'.

According to the MSM model the volatility process σ_t in (2.3) is defined by the product of $\overline{k}$ volatility components and a scale factor $\overline{\sigma}$, which is a positive constant:

$$\sigma_t = \sigma(M_t) = \overline{\sigma}\left(\prod_{i=1}^{k} M_{i,t}\right)^{1/2} \tag{2.5}$$

because of the fact that multipliers are statistically independent, the parameter $\overline{\sigma}$ coincides with the unconditional standard deviation of the r_t.

Now let $RV_t^{(d)}$ be an estimate of daily realized variance. In order to apply the MSM model for $RV_t^{(d)}$, we only consider the volatility specification in (2.5):

$$RV_t^{(d)} = RV^{(d)} \prod_{i=1}^{k} M_{i,t} \tag{2.6}$$

By using $\mathbb{E}[M_{i,t}] = 1$ one can derive the equality $\mathbb{E}[RV_t^{(d)}] = RV^{(d)}$. Therefore, we may obtain an estimate of the scaling factor $RV^{(d)}$ as

$$\hat{RV}^{(d)} = \tfrac{1}{T}\sum_{t=1}^{T} RV_t^{(d)} \tag{2.7}$$

The transition probabilities $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{\overline{k}})$ of each Markov component was proposed by Laurent Calvet and Fisher (2001) as follows:

$$\gamma_k = 1 - (1-\gamma_1)^{b^{k-1}}, \quad where \quad \gamma_1 \in (0,1), \quad b \in (1,\infty) \tag{2.8}$$

For a process with very persistent components, i.e. a low value of γ_1 and for a small value of k, the quantity $\gamma_1 b^{k-1}$ remains small, and the transition probability satisfies:

$$\gamma_k \approx \gamma_1 b^{k-1} \tag{2.9}$$

The parameter b measures the rate the transition probabilities grow for low frequencies. The rate of increase slows down as k gets closer to the final $\overline{k}$, and condition (2.9) ensures that the parameter γ_k to be lower than 1. In empirical applications the parameters k and b needs to be estimated. By having the transition probability k estimated, we can isolate γ_1, and thus calculate the probabilities recursively at all $\overline{k}$ volatility components using relationship (2.9). As it was mentioned the MSM uses only a few restrictions on the marginal distribution of the multipliers: $M > 0$ and $\mathbb{E}[M] = 1$. This allows flexible parametric or non-parametric specifications of the multipliers M. In the small number of available literature on multifractals, the multipliers M have been assumed to be either a Binomial or a Lognormal distributed. In this research we will employ the Binomial version for the distribution of the volatility components as there is no huge difference in performance, but Binomial case is more simple.

2.4.3 The Binomial Markov-Switching Multifractal (MSM) model

Based on Lauren Calvet (2004) the Binomial MSM (BMSM) is constructed by specifying the random variable M as a Binomial random variable with only two possible values, m_0 and m_1. For simplicity we assume that these two values have equal probability of occurring by specifying that $m_1 = 2 - m_0$. This guarantees a mean equal to one for all components of M. We may freely choose the number of frequency components we want to use in the MSM model. However the optimal number depends upon the number of observations and type of the considered data. Once the number of $\overline{k}$ volatility components has been chosen, the BMSM model with Gaussian error term is a specified Markov-Switching process with $2^{\overline{k}}$ states and the vector of parameters:

$$\phi = (m_0, \overline{\sigma}, b, \gamma_{\overline{k}})$$

where m_0 describes the distribution of the multipliers, $\overline{\sigma}$ is the unconditional standard deviation of returns, which defines the average level of volatility, b and $\gamma_{\overline{k}}$ define the set of probabilities to switch the value. To be more precise, $\overline{k}$ controls the frequency or persistence of the highest frequency component, and b determines the frequency of all the other frequency components relative to the highest one. by having specified the full parameter vector one may proceed to Maximum Likelihood estimation.

Maximum Likelihood Estimation

If one assumes that the multipliers M has a discrete distribution, then there exist a finite number of volatility states. This allows standard methods to provide the likelihood function of the MSM model in closed form.

Updating the State Vector

Under the above assumption, the Markov state vector M_t takes $d = 2^{\bar{k}}$ number of values $m_1, ..., m_d \in \mathbb{R}_+^{\bar{k}}$, and its dynamics are characterized by the transition matrix $A = (a_{i,j})_{1 \leq i,j \leq d}$ with components $a_{i,j} = \mathbb{P}(M_{t+1} = m^j | M_t = m^i)$, i.e. the probability of being in state j at time $t+1$ conditional on being in state i at time t. Conditional on the volatility state, the error term ϵ_t is assumed to have Gaussian density $f(r_t | M_t = m^i) = n[0; \sigma^2(m_i)]$, where $n(0; \sigma^2(m_i))$ denotes the density of normal distribution with 0 mean and variance equal to $\sigma^2(m_i)$. The researcher does not directly observe M_t but can compute the conditional probabilities:

$$\Pi_t^j = \mathbb{P}(M_t = m^j | r_1, ..., r_t)$$

i.e. the probability of being in state j depends on all available information. The row vector of these probabilities can be stacked as $\Pi_t = (\Pi_t^1, ..., \Pi_t^d) \in R_+^d$, and the conditional probability vector is then can be computed recursively. Using the Bayes rule, Π_t can be expressed as follows:

$$\Pi_t = \frac{\omega(r_t) * (\Pi_{t-1} A)}{[\omega(r_t) * (\Pi_{t-1} A)]\mathbf{1}'}$$

where $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^d$, $x * y$ stands for the Hadamard product $(x_1 y_1, ..., x_d y_d)$ for any $x, y \in \mathbb{R}^d$, and

$$\omega(r_t) = (n[\mu; \sigma^2(m_1)], ..., n[\mu; \sigma^2(m_d)])$$

In empirical applications, the initial vector Π_0 is chosen to be the ergodic distribution of the Markov process. Since the multipliers are mutually independent, the ergodic distribution is given by $\Pi_0^j = \Pi_{l=1}^{\bar{k}} \mathbb{P}(M = m_l^j)$ for all j, i.e. in the first iteration we assign equal probability to all the possible states, which is $\Pi_0 = 1/d$.

Closed-Form Likelihood

Lauren Calvet (2004) has shown that, based on the comments above, a closed-form log-likelihood function can be expressed as

$$lnL(r_1, ..., r_T; \phi) = \sum_{t=1}^{T} ln[\omega(r_t) \cdot (\Pi_{t-1} A)]$$

where $x \cdot y$ denotes the inner product $x_1 y_1, ..., x_d y_d$ for any $x, y \in R^d$. For a fixed $\overline{k}$, it is known that the ML estimator is consistent and asymptotically efficient as $T \to \infty$.

Even-though the development of a closed form likelihood algorithm was a large step forward for the improvement of MSM model, it has restrictions in the distributional assumptions of the multipliers by the sense that it only works for discrete distributions and is not applicable for continuous ones, e.g. a Lognormal distribution. ML estimation, in the Binomial case, also encounters bounds of computational performance for selecting the more than about $\overline{k} = 10$ volatility components due to the large vector of the possible states. For this reason we will use $\overline{k} = 10$ components in the empirical part.

However, an advantage of the ML procedure is that it is able to obtain optimal forecasts by updating of the conditional probabilities $\Pi_t^j = \mathbb{P}(M_t = m^j | r_1, ..., r_t)$ for the volatility states m^j.

Forecasting Volatility using the MSM model

In a very simple way one can extend the likelihood function algorithm of the MSM model to produce volatility forecasts by the any horizon. In order to obtain the optimal forecasts a vector of forward state probabilities needs to be calculated, i.e. a vector of elements $\mathbb{P}(M_{t+h} = m^j | r_1, ..., r_t)$, where h denotes the number of steps ahead. The forward vector can be derived using the transition matrix A and the conditional probability vector, such that

$$\hat{\Pi}_{t,h} = \Pi_t A^h$$

where A is the one period ahead transition matrix. Thus $\hat{\Pi}_{t,h}$ is the time t conditional forecast of the probability of being in each state at time $t + h$. Finally the dot-product is taken of this future vector and the vector of possible volatility levels in order to obtain the time $t + h$ volatility.

2.4.4 Heterogeneous Auto-Regressive (HAR) model

The HAR model as introduced in Corsi (2009) appeared to be popular as it has good forecasting accuracy, allows for economic interpretation and is easy to estimate. These are the reasons we choose this model to compare the precision level of volatility forecast with MSM model presented in the previous section. Also there are many versions and modifications of the HAR model, however we stick to the original model to focus on the actual volatility dynamics. For this reason we ignore other temporary effects, such as the leverage, which might be employed in a HAR framework as well. Then the HAR model is defined as

$$\ln RV_{t+1}^{(d)} = c + \beta^{(d)} \ln RV_t^{(d)} + \beta^{(w)} \ln RV_t^{(w)} + \beta^{(m)} \ln RV_t^{(m)} + \epsilon_{t+1} \tag{2.10}$$

where $\ln RV_t^{(w)} = \frac{1}{5}\sum_{i=1}^{5} \ln RV_{t-i+1}^{(d)}$ and $\ln RV_t^{(m)} = \frac{1}{22}\sum_{i=1}^{22} \ln RV_{t-i+1}^{(d)}$ are the weekly and monthly averages of daily log realized variances, and ϵ_t is an error term. When these log realized variances are known, the model can be consistently evaluated by the simple least squares method to calculate values for c, $\beta^{(d)}$, $\beta^{(w)}$, $\beta^{(m)}$.

It was noticed by Corsi (2009) that the HAR model can be rewritten to a constrained $AR(22)$ model:

$$\ln RV_{t+1}^{(d)} = \phi^{HAR} + \sum_{i=1}^{22} \phi_i^{HAR} \ln RV_{t-i+1}^{(d)} + \epsilon_{t+1} \tag{2.11}$$

where the restrictions as employed by (2.10) are required to be

$$\phi_i^{HAR} = \begin{cases} \beta^{(d)} + \frac{1}{5}\beta^{(w)} + \frac{1}{22}\beta^{(m)}, & \text{for } i = 1 \\ \frac{1}{5}\beta^{(w)} + \frac{1}{22}\beta^{(m)}, & \text{for } i = 2, ..., 5 \\ \frac{1}{22}\beta^{(m)}, & \text{for } i = 6, ..., 22 \end{cases} \tag{2.12}$$

Chapter 3

Empirical Results

This chapter introduce data sets used for research in section 2.1. Also it represents the reliability of multifractality detection methods based on Monte Carlo simulations in section 2.2. Section 2.3 contains a discussion on the results from detection methods of multifractality. Finally section 2.4 presents MSM and HAR models results and includes discussion on volatility forecasting.

3.1 Data Selection

For the purpose of applying the theories and methods described in the previous chapter, a dataset with a history from the beginning of 2017 has been selected. The dataset contains four major aggregated indices, which individually are composed of a number of minor indices. The selected indices are the Dow Jones Industrial Average (D&J), Australian Securities Exchange (ASX), Nikkei-225 (NI225) and NASDAQ-100 (NDX). These indices covers a wide range of industry sectors except the ASX index which is for security exchanges. The datasets were exported from finam.ru and the index prices are quoted in USD. Observations of 5 minutes frequency were obtained for each of the indices in the time period from January 1, 2017 to November 2, 2017. This gives a collection of 17437 synchronized 5 minutes observations for D&J index, 15714 - for ASX index, 12867 - for NI225 and 16615 for NDX index. The count of observations is inconsistent among these indices due to different working days and hours.

All the selected and derived data (returns, realized volatility) is depicted graphically in the appendix A.

3.2 Robustness of Multifractality Detection Methods

In this subsection we present the robustness of both selected methods to detect multifractality in the previous chapter. It is known that none of the mentioned methods have asymptotic theory. This leads to natural question about correctness of methods. In order to evaluate whether the multifractality detection methods are working as expected, we performed simple test by simulating a set of realizations in both mono-fractal and multifractal cases. Fractional Brownian motion (fBm) represents mono-fractal process. The fBm is a continuous-time Gaussian process $B_H(t)$ on $[0,T]$, which starts at zero, has expectation zero for all t in $[0,T]$, and has the following covariance function:

$$\mathbb{E}[B_H(t), B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where H is a real number in $(0,1)$, called the Hurst exponent associated with the fractional Brownian motion.

One hundred Monte Carlo simulations of Fractional Brownian motion with various Hurst exponents were generated to obtain mono-fractal processes. Table 3.1 depicts means and standard deviations of GHE estimates of Monte Carlo simulations. At the lower values of H the GHE method vary less than on higher values. Standard deviations for $H = 0.9$ fluctuates around 0.02, while at the lowest level of $H = 0.1$ it is observed to be from 0.007 to 0.009. Nevertheless, it is relatively small variations and we can state that GHE can detect mono-fractal process properly.

Table 3.2 depicts means and standard deviations of MF-DFA estimates of Monte Carlo simulations. The similar tendency is observed on MF-DFA method as well. Lower values of H varies less than higher Hurst exponents. Standard deviations for $H = 0.1$ fluctuates around 0.009, while at the higher level of $H = 0.9$ it is observed to be from 0.02 to 0.041. In addition to that standard deviations tend to be smaller at the positive order of moments.

TABLE 3.1: H estimations by GHE: mono-fractal case

Moment order	Measure \ H	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	Mean	0.100	0.199	0.300	0.400	0.498	0.599	0.690	0.776	0.862
	SD	0.009	0.011	0.012	0.015	0.014	0.016	0.019	0.022	0.021
2	Mean	0.100	0.202	0.300	0.398	0.499	0.595	0.690	0.781	0.858
	SD	0.007	0.009	0.011	0.014	0.020	0.018	0.017	0.021	0.021
3	Mean	0.010	0.198	0.298	0.397	0.498	0.595	0.692	0.777	0.859
	SD	0.008	0.010	0.013	0.014	0.015	0.017	0.020	0.019	0.020

TABLE 3.2: H estimations by MF-DFA: mono-fractal case

Moment order	Measure \ H	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
-3	Mean	0.124	0.215	0.316	0.410	0.513	0.619	0.714	0.824	0.916
	SD	0.009	0.015	0.010	0.029	0.028	0.035	0.035	0.030	0.041
0	Mean	0.114	0.206	0.306	0.402	0.504	0.607	0.670	0.809	0.901
	SD	0.009	0.015	0.020	0.030	0.028	0.033	0.034	0.034	0.041
3	Mean	0.105	0.198	0.297	0.395	0.494	0.595	0.686	0.795	0.885
	SD	0.008	0.010	0.013	0.014	0.015	0.017	0.012	0.019	0.020

In order to obtain robustness test on multi-fractal processes we choose to generate one hundred Monte Carlo simulations using MSM model, presented in the previous chapter, with various sets of parameters. Sets of parameters are showed in the 3.3 and 3.4 tables. Table 3.3 depicts means and standard deviations of GHE estimates of multi-fractal Monte Carlo simulations. It can be seen that standard deviations remains small in all cases regardless the set of parameters and order of moments. It fluctuates only from 0.002 to 0.012 and state that GHE method is stable to detect multifractality in the process.

Table 3.4 depicts means and standard deviations of MF-DFA estimates of multi-fractal Monte Carlo simulations. The very similar tendency is observed on MF-DFA method as well. It shows only small standard deviations disregarding the set of parameters and order of moments. In MF-DFA case it fluctuates only from 0.001 to 0.009 and state as well that MF-DFA method is robust to detect multifractality in the process. The brighter difference is that estimates of MF-DFA significantly differ from GHE estimates. This happens due to difference of methods and the fact that different H is observed for different Δt.

TABLE 3.3: H estimations by GHE: multi-fractal case

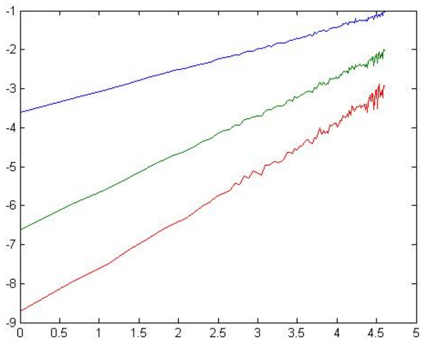
Moment order	Measure \ Parameters	$\overline{k}=10$ $b=2$ $m_0=1.5$ $\gamma_{\overline{k}}=0.9$ $\overline{\sigma}=0.1$	$\overline{k}=10$ $b=2$ $m_0=1.5$ $\gamma_{\overline{k}}=0.9$ $\overline{\sigma}=0.2$	$\overline{k}=15$ $b=5$ $m_0=1.5$ $\gamma_{\overline{k}}=0.8$ $\overline{\sigma}=0.1$	$\overline{k}=15$ $b=5$ $m_0=1.5$ $\gamma_{\overline{k}}=0.8$ $\overline{\sigma}=0.2$	$\overline{k}=15$ $b=10$ $m_0=1.5$ $\gamma_{\overline{k}}=0.5$ $\overline{\sigma}=0.1$
1	Mean	0.525	0.532	0.526	0.529	0.581
	SD	0.003	0.006	0.004	0.002	0.004
2	Mean	0.449	0.436	0.438	0.440	0.497
	SD	0.003	0.007	0.008	0.003	0.006
3	Mean	0.326	0.280	0.377	0.316	0.370
	SD	0.004	0.009	0.012	0.004	0.010

TABLE 3.4: H estimations by MF-DFA: multi-fractal case

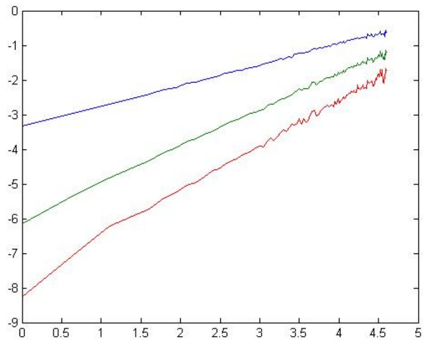
Moment order	Measure \ Parameters	$\overline{k}=10$ $b=2$ $m_0=1.5$ $\gamma_{\overline{k}}=0.9$ $\overline{\sigma}=0.1$	$\overline{k}=10$ $b=2$ $m_0=1.5$ $\gamma_{\overline{k}}=0.9$ $\overline{\sigma}=0.2$	$\overline{k}=15$ $b=5$ $m_0=1.5$ $\gamma_{\overline{k}}=0.8$ $\overline{\sigma}=0.1$	$\overline{k}=15$ $b=5$ $m_0=1.5$ $\gamma_{\overline{k}}=0.8$ $\overline{\sigma}=0.2$	$\overline{k}=15$ $b=10$ $m_0=1.5$ $\gamma_{\overline{k}}=0.5$ $\overline{\sigma}=0.1$
-3	Mean	0.466	0.576	0.660	0.838	0.783
	SD	0.004	0.004	0.007	0.006	0.004
0	Mean	0.371	0.411	0.420	0.506	0.526
	SD	0.002	0.001	0.001	0.001	0.002
3	Mean	0.213	0.149	0.166	0.161	0.182
	SD	0.002	0.002	0.005	0.007	0.009

3.3 Results of Multifractality Detection Methods

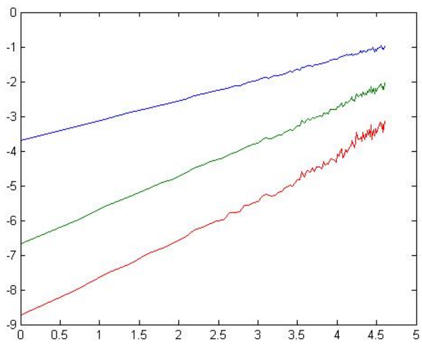
When we are sure that GHE and MF-DFA methods are robust enough to detect multi-fractality, we present the results received by using the real financial indices. Firstly, we estimated GHE statistics for $q = 1$, $q = 2$ and $q = 3$. The figure 3.1 depicts the results: logarithm values of GHE statistics lie on the vertical axis while horizontal axis marks the logarithm of Δt. Blue line stands for $q = 1$, green line - for $q = 2$ and red line - for $q = 3$. It can be seen that all four indices have similar behaviour of the GHE statistics. Set of the statistic's values for $q = 2$ and $q = 3$ have more similar slope while the slope for the blue line differs significantly. These results make a colculsion that GHE method found the trace of multifractality in all of the selected indices.



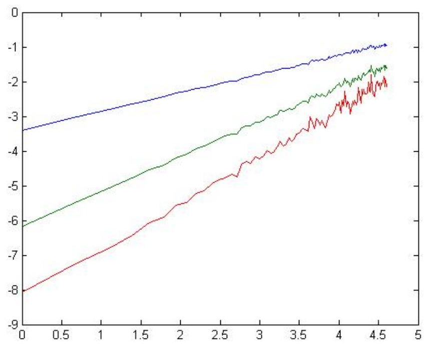
(A) D&J index



(B) ASX index



(C) NI225 index



(D) NDX index

FIGURE 3.1: GHE values for $q = 1$ (blue), $q = 2$ (green) and $q = 3$ (red)

Secondly, we estimated MF-DFA statistics for $q = -5$, $q = 0$, $q = 5$ and evaluated Hurst exponent. The figure 3.2 depicts the results of D&J index:

- the first graph shows the relations of MF-DFA statistics F_q and different segment sizes for different order of moments;
- the second graph depicts the relationship between Hurst exponent and the q order of moments;
- the third graph shows the dependancy of τ function and the q order of moments;
- the last graph indicates multifractal spectrum and its width.

It can be seen that statistics of MF-DFA and the slope of different moment order behaves similar to the GHE results. The second and the third graph illustrates that H and τ functions have non linear dependancy of q order of moments. According to theory, this indicates that D&J index has property of multi-fractality. In addition to that we calculated the multi-fractal spectrum described in the previous chapter. Typical width of multi-fractal spectrum for mono-fractal process fluctuates from 0.142 to 0.168. It was calculated by measuring multi-fractal width of the fBM simulations used for robustness test of multi-fractal detection methods in the previous section. In comparison, we observed 0.46 width of multi-fractal spectrum for D&J index and that also indicates the multi-fractal process.

The figure 3.3 depicts the results of ASX index, figure 3.4 - NI225 index, figure 3.5 - NDX index. Basically, all the figures represents very similar results to D&J index. All cases distinguish oneselfs by multifractality. Each figure illustrates that H and τ functions depends on q order of moments in a non linear way. Multi-fractal spectrum is wide enough to state that data is multi-fractal as well.

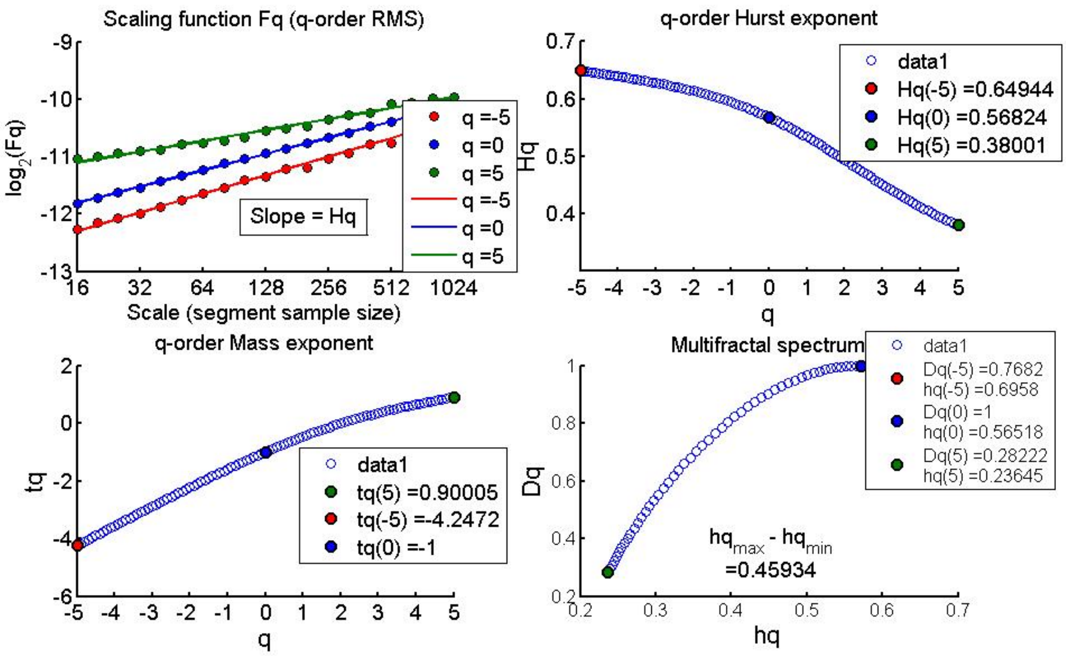


Figure 3.2: MF-DFA results for D&J

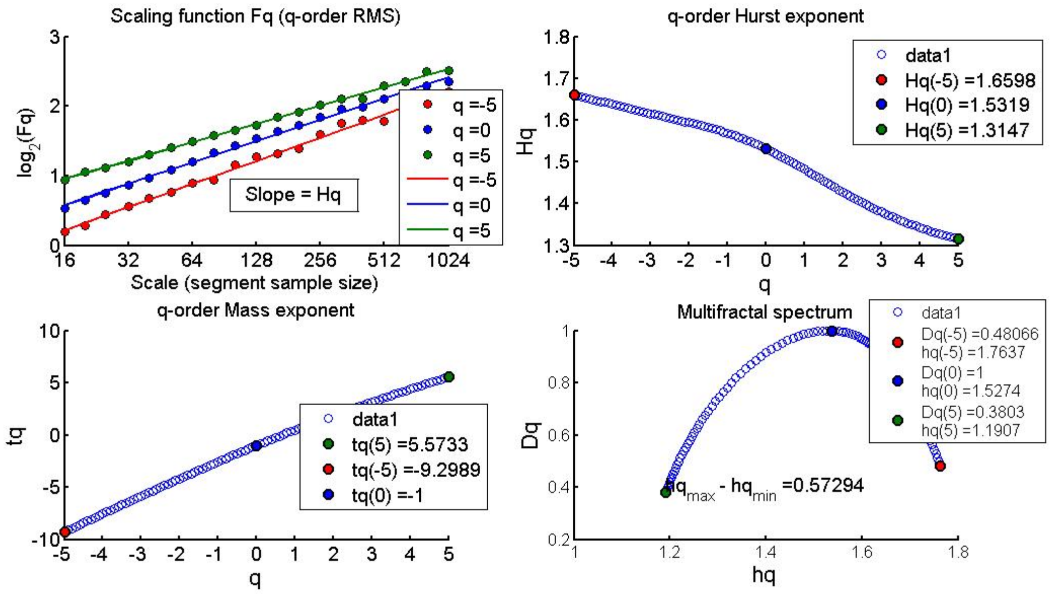


Figure 3.3: MF-DFA results for ASX

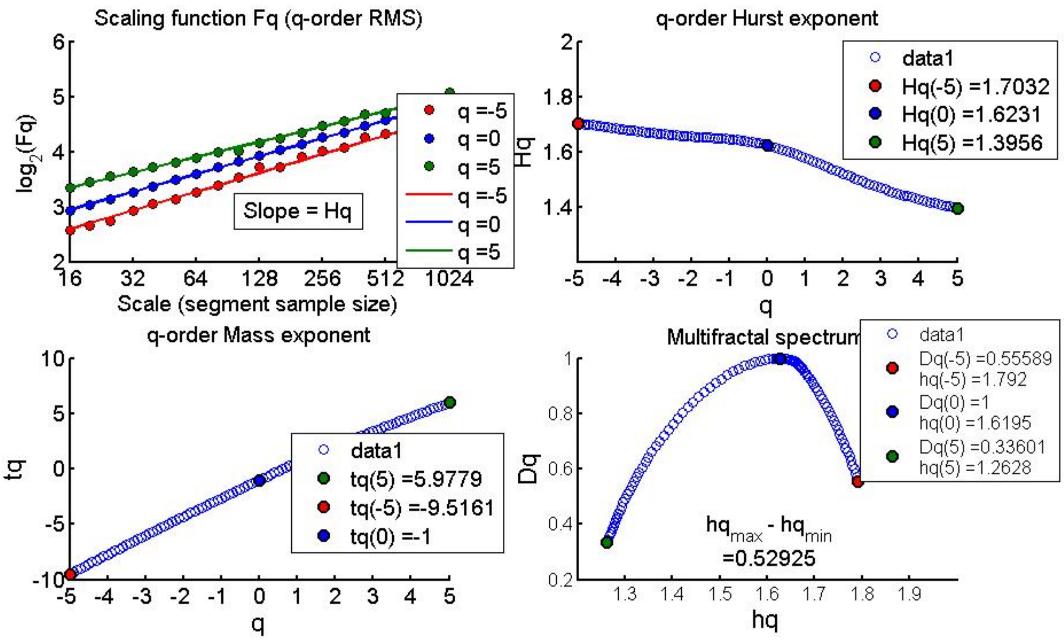


Figure 3.4: MF-DFA results for NI225

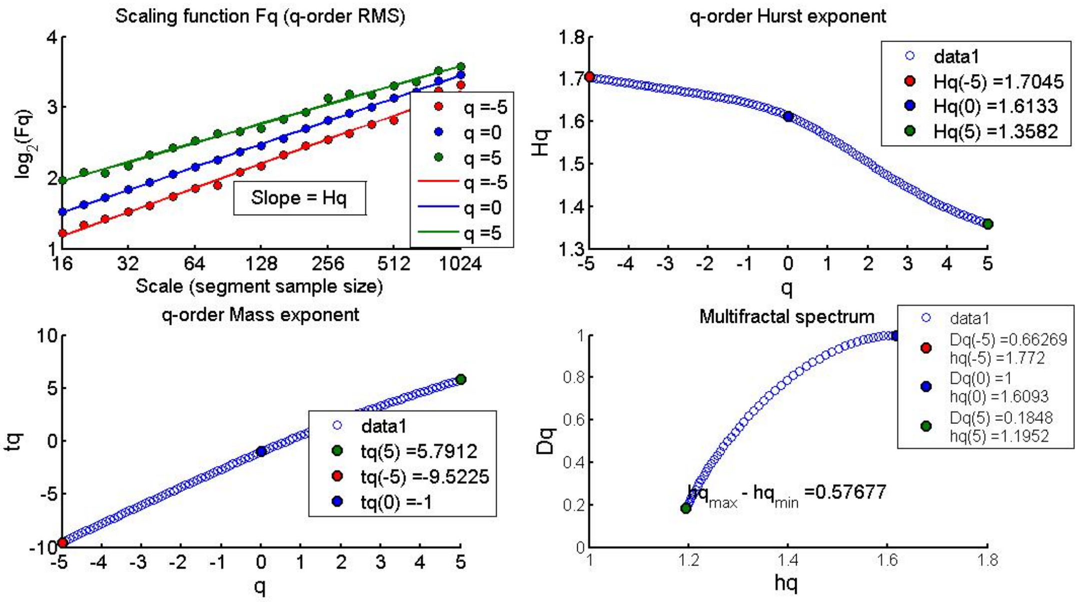


Figure 3.5: MF-DFA results for NDX

3.4 Estimated Models and Accuracy of Prediction

In this subsection we show the accuracy results of realized volatility forecasts by Markov switching multifractal model and compare it to forecasts of well-know Heterogeneous auto-regressive model. Models were fitted to realized volatility of each of selected index. Fitted models were used to predict the future values and compare it by two methods: in sample and out of sample. By using in sample method we are forecasting for an observations that was part of the data sample while forecasting for an observation that was not part of the data sample is defined by out of sample method. We used four different prediction horizons for comparison of accuracy: 1, 5, 10 and 20 steps ahead. Mean absolute error (MAE) was chosen as a measure of prediction goodness and let us to compare forecast errors between two models.

Table 3.5 represents the in sample MAE results retrieved from predictions by both models and real values of realized volatility . For D&J index it was clear that at lower horizons MSM performs better in terms of prediction. We can say the same about NI225 and NDX - for horizons up to 10 MSM predicts more accurate compared to HAR model. Meanwhile MSM model showed slightly worse results for one step ahead for ASX index, however it was better at 5 and 10 horizons as well. Clearly, the HAR model performed better at the highest horizon of 20 steps ahead for all selected indices.

Table 3.6 depicts out of sample MAE results obtained from difference of predictions by both models and real values of realized volatility. The better prediction of D&J index was observed for one step ahead only by MSM model while HAR model performed better at other horizons. Similar tendency remains on NI225 index, however MSM was better on $h = 2$ as well. The HAR model was outperformed fully by MSM on all horizons for ASX index, but the opposite results were obtain for NDX index where results are in favor of HAR model. All in all we can conlude that MSM model predict better at shorter horizons, while HAR model performs well at long horizons.

Table 3.5: $MAE * 10^7$: In Sample

Data set	Model \ Horizon	$h = 1$	$h = 5$	$h = 10$	$h = 20$
D&J	MSM	0.119	0.284	0.240	0.451
	HAR	0.440	0.307	0.324	0.318
ASX	MSM	0.494	0.707	0.926	1.165
	HAR	0.430	1.216	1.012	0.957
NI225	MSM	0.681	1.156	1.300	0.969
	HAR	1.477	1.149	1.452	1.396
NDX	MSM	1.226	1.372	1.277	1.502
	HAR	1.468	0.949	1.270	1.108

Table 3.6: $MAE * 10^7$: Out of Sample

Data set	Model \ Horizon	$h = 1$	$h = 5$	$h = 10$	$h = 20$
D&J	MSM	0.111	0.388	0.344	0.362
	HAR	0.202	0.278	0.281	0.258
ASX	MSM	0.241	0.508	0.538	0.489
	HAR	0.317	0.717	0.656	0.862
NI225	MSM	0.069	0.864	1.175	1.241
	HAR	1.326	1.329	1.071	0.929
NDX	MSM	0.768	0.915	1.150	1.257
	HAR	0.306	0.366	0.484	0.618

Chapter 4

Conclusions

In this thesis we have focused on multifractality detection of intra-day financial data and modeling realized volatility using discrete time model. More specifically, we have examined the properties of the multi-fractality and discrete time Markov Switching Multifractal model, and compared it to other model of realized volatility.

Firstly, we presented the main concepts of multi-fractality and some of its detection methods, then following this we introduced the main model of the paper and the HAR model we wanted to make comparison with MSM model.

In the empirical part of the paper, we introduced the selected data and measured how robust multi-fractality detection methods are as and used them to ascertain whether the intra-day financial indices have multifractal traces or not. Also we compared the prediction accuracy of models by mean absolute error in two ways: in-sample and out-of-sample.

The results of robustness of multi-fractal detection methods showed that both GHE and MF-DFA are stable and reliable to determine whether the proces is multifractal or not. Following this conclusion we could apply these methods on four real financial indices. Comparing to similar studies on daily returns data, both methods GHE and MF-DFA revealed that each of the four selected intra-day indices have the property of multi-fractal as well.

In the comparison of realized volatility forecasts with in-sample method by MSM and HAR models, we found slight evidence suggesting that the MSM model is competitive on the short horizons and is outperformed on long horizon by HAR model in the most cases. The similar results were observed by out-of-sample method as well.

Appendix A

Appendix

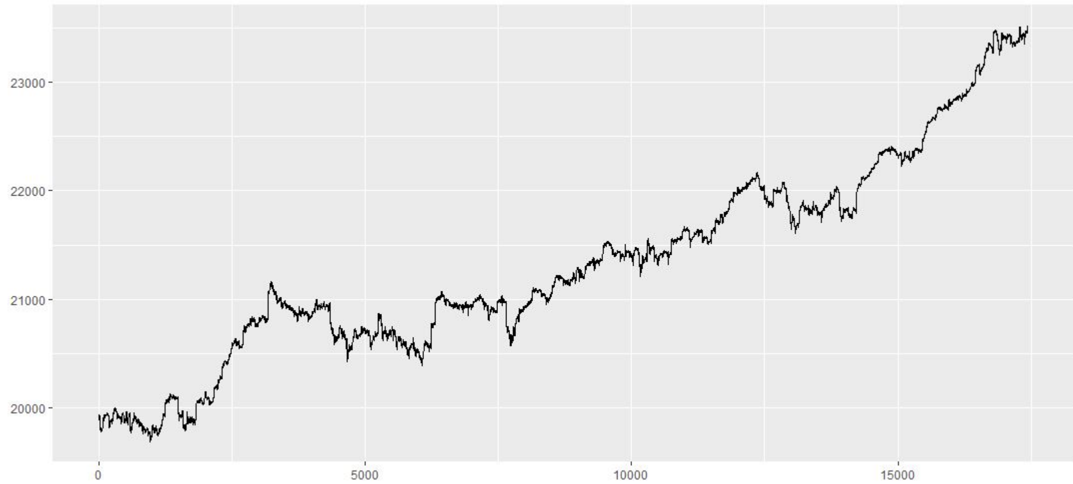


Figure A.1: D&J index 5-minutes prices from January 1, 2017 to November 2, 2017

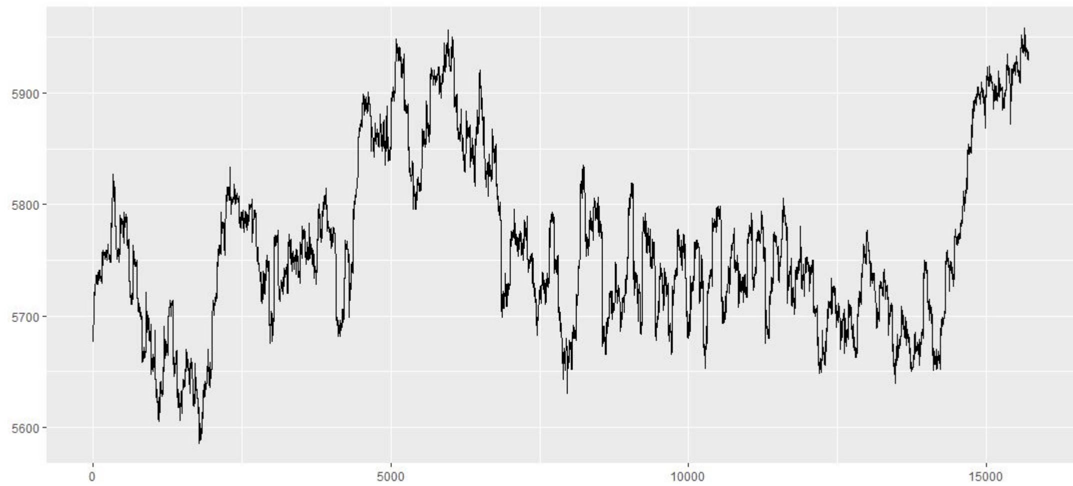


Figure A.2: ASX index 5-minutes prices from January 1, 2017 to November 2, 2017

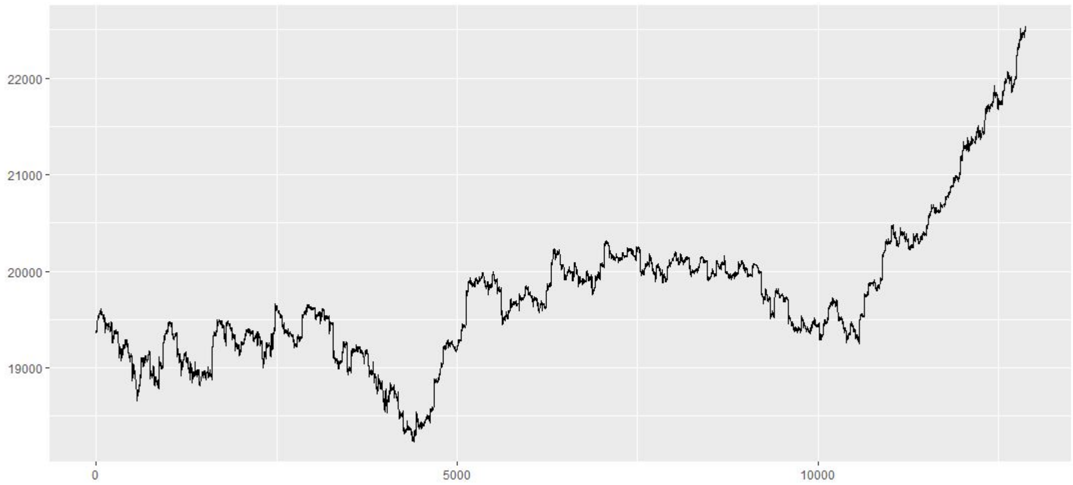


Figure A.3: NI225 index 5-minutes prices from January 1, 2017 to November 2, 2017

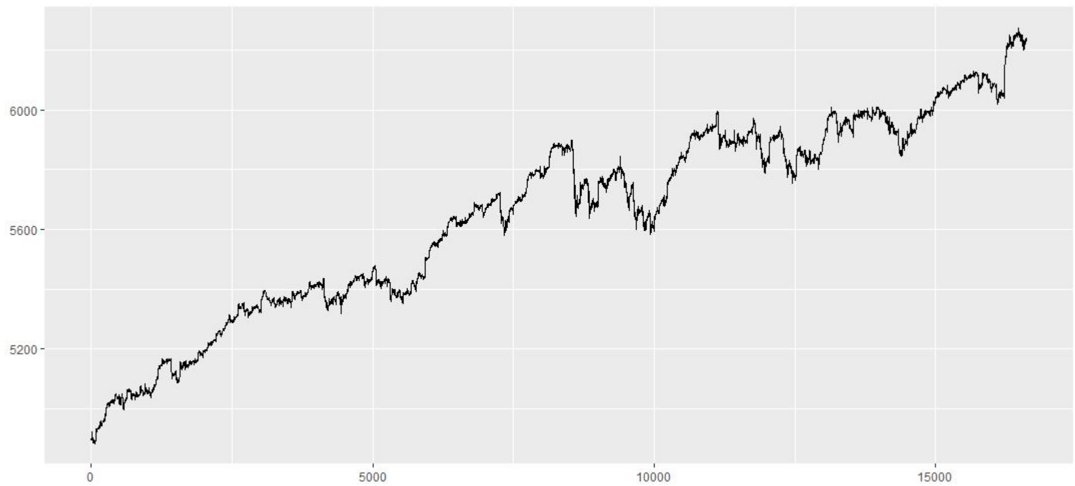


Figure A.4: NDX index 5-minutes prices from January 1, 2017 to November 2, 2017

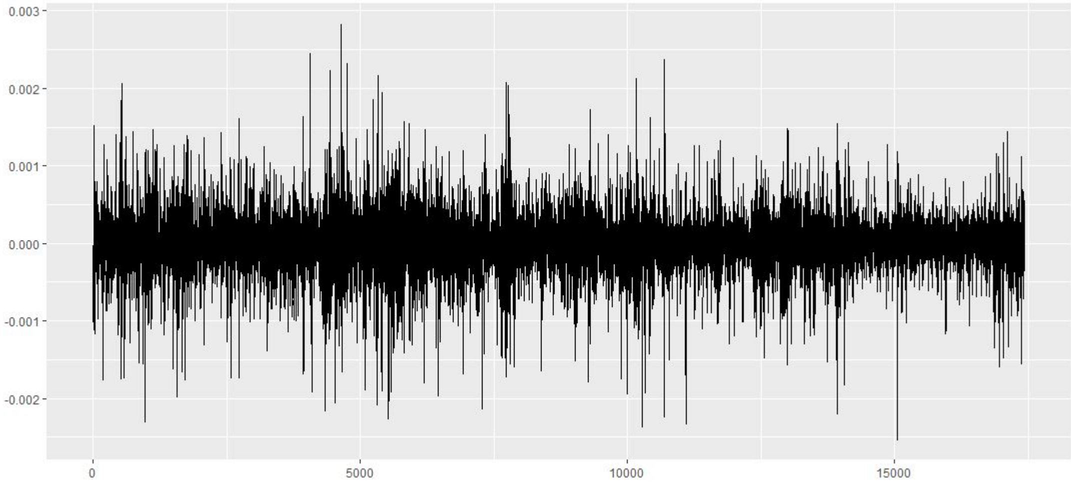


FIGURE A.5: D&J 5-minutes returns from January 1, 2017 to November 2, 2017

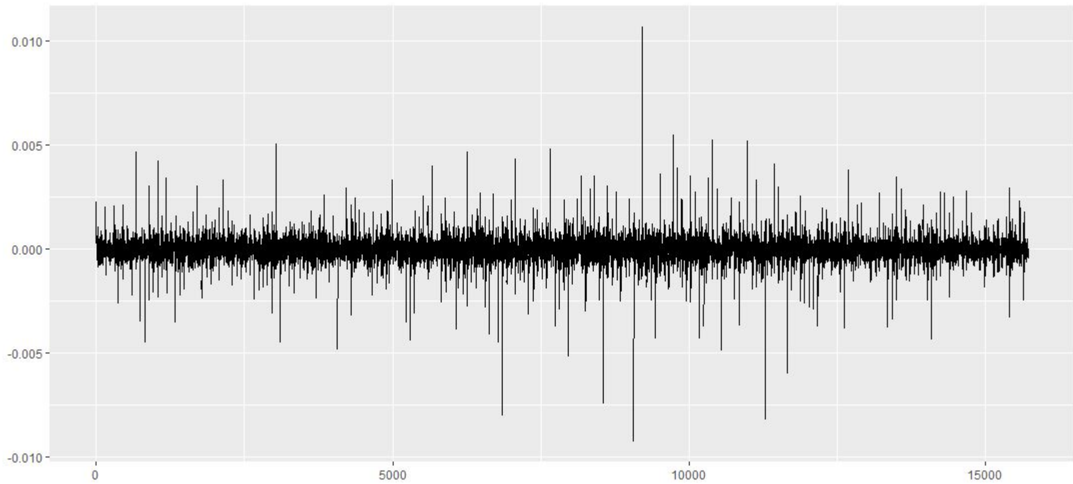


FIGURE A.6: ASX 5-minutes returns from January 1, 2017 to November 2, 2017

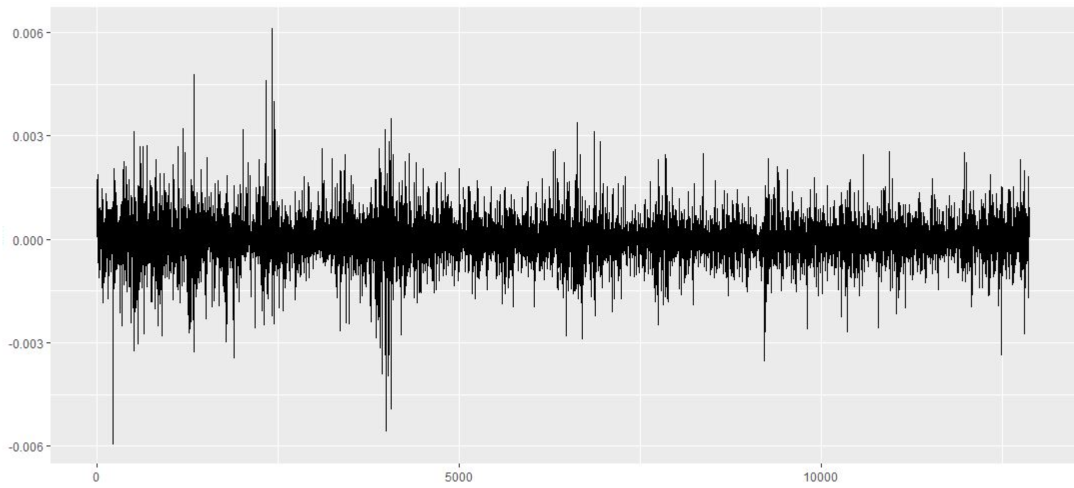


Figure A.7: NI225 5-minutes returns from January 1, 2017 to November 2, 2017

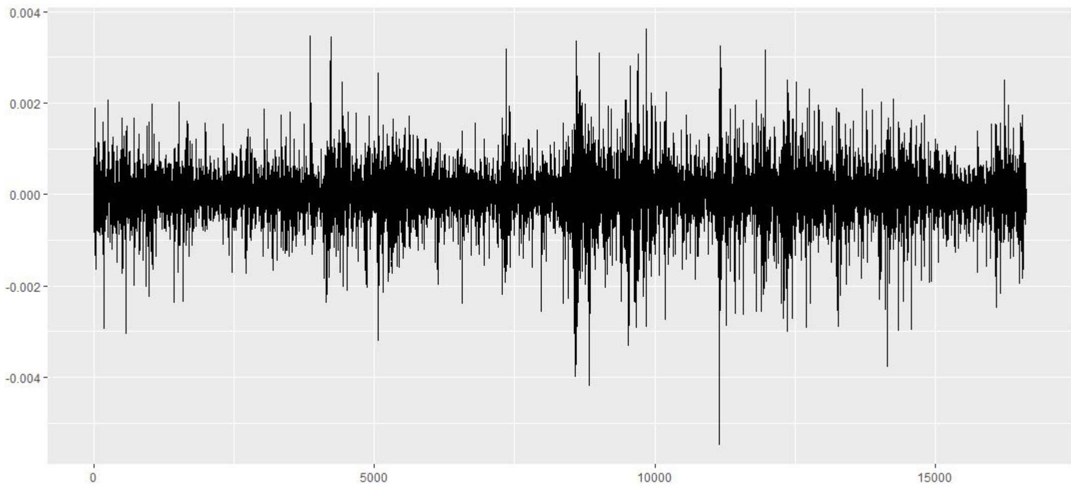


Figure A.8: NDX realized daily volatility from January 1, 2017 to November 2, 2017

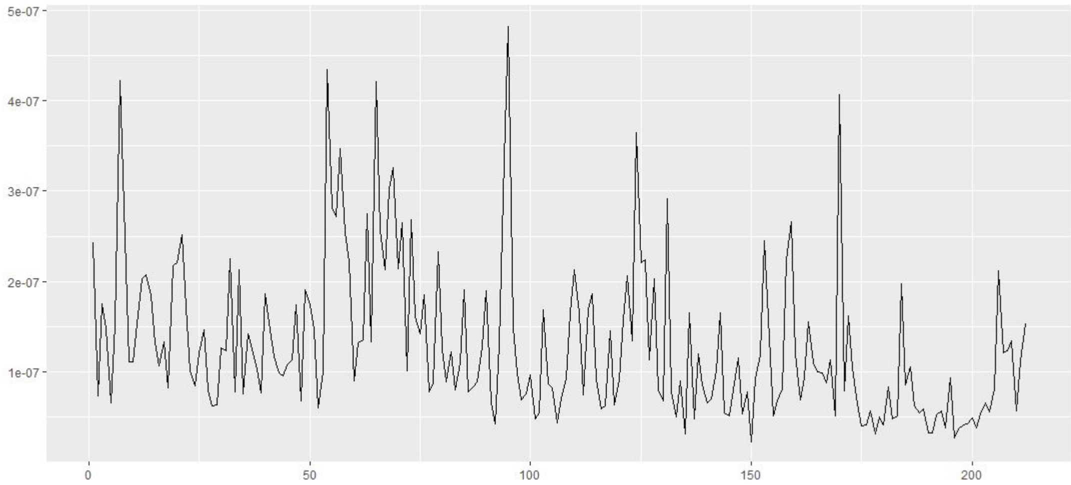


FIGURE A.9: D&J realized daily volatility from January 1, 2017 to November 2, 2017

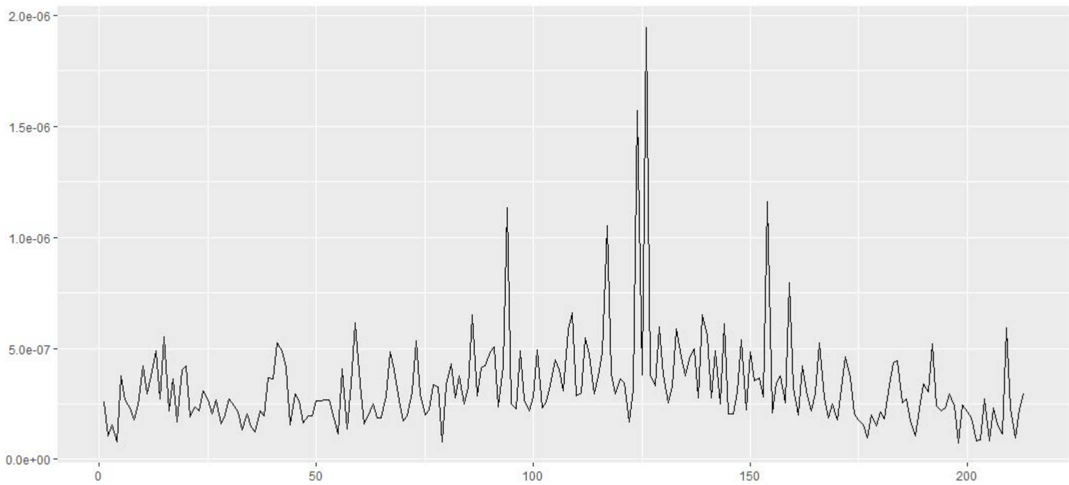


FIGURE A.10: ASX realized daily volatility from January 1, 2017 to November 2, 2017

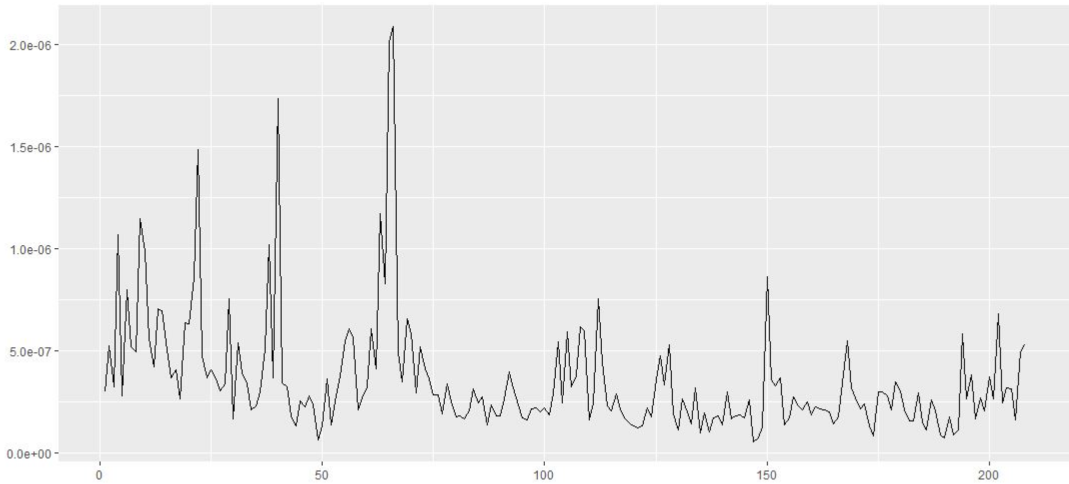


FIGURE A.11: NI225 realized daily volatility from January 1, 2017 to November 2, 2017

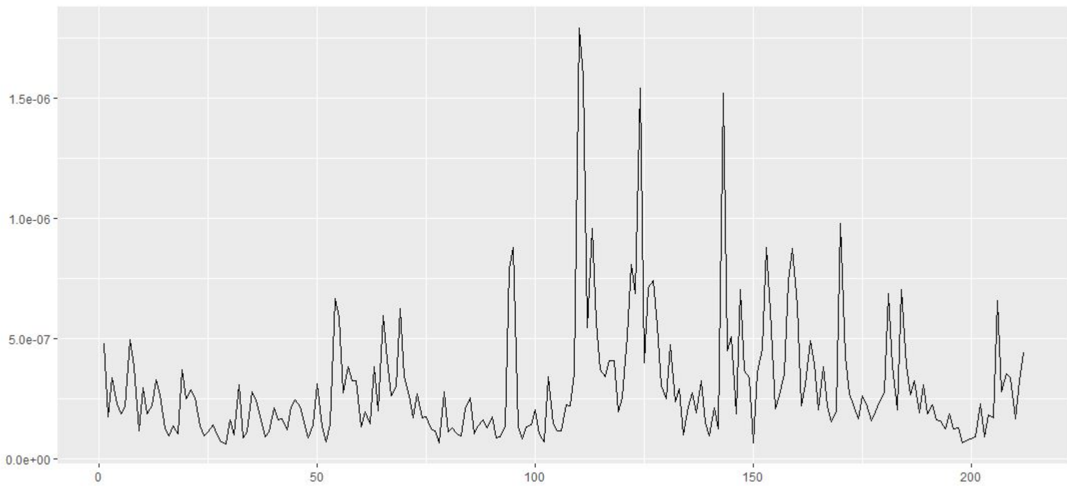


FIGURE A.12: NDX realized daily volatility from January 1, 2017 to November 2, 2017

Bibliography

Andersen (2006). “Volatility and correlation forecasting”. *Handbook of economic forecasting* 1, pp. 777–878.

Andersen, Torben and Bollerslev (2001). “The distribution of realized stock return volatility”. *Journal of financial economics* 61.1, pp. 43–76.

Black, Fischer and Myron Scholes (1973). “The pricing of options and corporate liabilities”. *Journal of political economy* 81.3, pp. 637–654.

Bollerslev, Tim (1986). “Generalized autoregressive conditional heteroskedasticity”. *Journal of econometrics* 31.3, pp. 307–327.

Calvet, Lauren (2004). “How to forecast long-run volatility: regime switching and the estimation of multifractal processes”. *Journal of Financial Econometrics* 2.1, pp. 49–83.

Calvet, Laurent and Fisher (2001). “Forecasting multifractal volatility”. *Journal of econometrics* 105.1, pp. 27–58.

Cont, Rama (2001). “Empirical properties of asset returns: stylized facts and statistical issues”. *Taylor & Francis.*

Cornelis, L. (2004). “Measuring the degree of financial market efficiency: An essay”. *SSRN, Social Science Research Network.*

Corsi, Fulvio (2009). “A simple approximate long-memory model of realized volatility”. *Journal of Financial Econometrics* 7.2, pp. 174–196.

Di Matteo, Tiziana (2007). “Multi-scaling in finance”. *Quantitative finance* 7.1, pp. 21–36.

Di Matteo, Tiziana, Tomaso Aste, and Michel Dacorogna (2005). "Long-term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development". *Journal of Banking & Finance* 29.4, pp. 827–851.

Engle, Robert F. (1982). "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation". *Econometrica: Journal of the Econometric Society*, pp. 987–1007.

Fama, Eugene F. (1965). "The behavior of stock-market prices". *The journal of Business* 38.1, pp. 34–105.

Kantelhardt, Jan (2008). "Fractal and multifractal time series". *Encyclopedia of Complexity and Systems Science*. Springer, pp. 3754–3779.

Kantelhardt, Jan and Zschiegner (2002). "Multifractal detrended fluctuation analysis of nonstationary time series". *Physica A: Statistical Mechanics and its Applications* 316.1, pp. 87–114.

Mandelbrot (1982). "The Fractal Geometry of Nature". *W.H. Freeman and Company, New York*.

Mandelbrot, Benoit and Fisher (1997). "A multifractal model of asset returns". *Cowles Foundation discussion paper no. 1164, Yale University*.

Mandelbrot, Benoit and Richard Hudson (2007). *The Misbehavior of Markets: A fractal view of financial turbulence*. New York.

Riedi, Rudolf H. (1999). *Multifractal processes*. Tech. rep. Rice University ,Department of Electrical and Computer Engineering,Houston,TX,77005.

Sharpe, William F. (1964). "Capital asset prices: A theory of market equilibrium under conditions of risk". *The journal of finance* 19.3, pp. 425–442.