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Singular solutions of the Stokes problems in the power cusp domains

DOCTORAL DISSERTATION

Life sciences,
mathematics N 001

VILNIUS 2019

This dissertation was written between 2012 and 2018 at Vilnius University.

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VILNIAUS UNIVERSITETAS

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Singuliarieji Stokso sistemų sprendiniai srityse su smailuma

DAKTARO DISERTACIJA

Gamtos mokslai,
matematika N 001

VILNIUS 2019

Disertacija rengta 2012–2018 metais Vilniaus Universitete.

Mokslinis vadovas:

prof. habil. dr. Konstantinas Pileckas (Vilniaus universitetas, gamtos mokslai, matematika, N 001)

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Chapter 1

Introduction

The Navier-Stokes equations, named after the Claude-Louis Navier (1785-1836) and George Gabriel Stokes (1819-1903), were derived in the early 1800's: Naviers original but inconsistent proof in 1822, later the equations were rediscovered by Cauchy in 1823, by Poisson in 1829, by Saint-Venant in 1837, and by Stokes in 1845 (see [5]). These equations are coupled partial differential equations used in fluid mechanics to describe the motion of an incompressible viscous fluid (for example water, oil, etc.). The Navier-Stokes equations illustrate the relation between the velocity, pressure and external force of a moving fluid (see [2]).

If we denote by $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, some domain and by $[0, T)$ a fixed time interval with $0 < T \leq \infty$, then the full nonlinear Navier-Stokes system in $\Omega \times [0, T)$ has the form (see, e.g. [46])

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, & x \in \Omega, \\ \operatorname{div}\mathbf{u} = 0, & x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{a}, & x \in \partial\Omega, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $\mathbf{u} = \mathbf{u}(x,t) = (u_1(x,t), ..., u_n(x,t))$ is the unknown velocity, $p = p(x,t)$ the unknown pressure, $\mathbf{f} = \mathbf{f}(x,t) = (f_1(x,t), ..., f_n(x,t))$ the given density of the exterior force, $\nu > 0$ the viscosity coefficient (which depends on physical properties of the fluid and is constant throughout the dissertation), $\mathbf{a} = \mathbf{a}(x,t) = (a_1(x,t), ..., a_n(x,t))$ the given boundary value and $\mathbf{u}_0(x)$ the given initial velocity with $t \in [0, T)$, $x = (x_1, ..., x_n) \in \Omega$.

The Navier-Stokes equations are derived from the basic principles of continuity of mass, momentum, and energy (see, e.g. [2]). The first equation $(1.1)_1$ describes the balance of forces. The second one $(1.1)_2$ means that the fluid is homogeneous and incompressible. The third and fourth equations $(1.1)_{3,4}$ give the boundary and initial conditions.

The Stokes equations are in fact the linearized Navier-Stokes equations, i.e. they can be described by the same system (1.1) without the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$:

$$\begin{cases} \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, & x \in \Omega, \\ \operatorname{div} \mathbf{u} &= 0, & x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{a}, & x \in \partial\Omega, \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The Navier-Stokes equations were (and still are) in the area of interest of many acknowledged mathematicians (see, e.g. [9], [18], [47]). Nevertheless the Navier-Stokes equations are still the source of many interesting problems, such as: *Navier-Stokes existence and smoothness problem* stated by the Clay Mathematics Institute on May 24, 2000, as one of the seven Millenium Prize problems, or the so-called *Flux Leray's problem* which was open for more than 80 years and was only recently solved in two dimensional and three dimensional axially-symmetric cases (see [15]) leaving the general three dimensional case still open. There are also many useful and interesting applications of the Navier-Stokes equations such as designing aerodynamically stable footballs, predicting the weather or simulating the fluids in computer animation.

Actuality and literature review

In order to solve the Navier-Stokes (1.1) or Stokes (1.2) equations one has to find the functions $\mathbf{u}$ and p. However, practice shows that these equations for most problems are too difficult to solve analytically. Therefore, it became usual to look for certain approximations or/and simplifications of the equations which could be solved[1]. Consequently, the asymptotic behaviour of the solutions to the Stokes and Navier-Stokes equations became a very important topic.

The asymptotic behaviour of the solutions to the Stokes and Navier–Stokes equations in singularly perturbed domains become of growing interest during the last fifty years. There is an extensive literature concerning these issues for various elliptic problems, see, e.g., [23], [25], [26], [30]–[37], [49], [12], [4], [3]. In particular, the steady Navier-Stokes equations are studied in a punctured domain $\Omega = \Omega_0 \setminus \{O\}$ with $O \in \Omega_0$ assuming that the point O is a sink or source of the fluid [14], [44], [45] (see also [16] for the review of these results). Although the steady Navier–Stokes equations in singulary

[1]Note, that nowadays these groups of simplified equations can be solved using the high speed computers. To do so, computational fluid dynamics uses different techniques: finite difference, finite volume, finite element, spectral methods.

perturbed domains are well studied, there are few papers studying the initial boundary value problem for the non-stationary Navier-Stokes equations in such domains (e.g., [38]–[40]). We can also mention the recent paper [17] where the Dirichlet problem for the nonstationary Stokes system is studied in a three-dimensional cone.

There are many applications of source singularities in fluid mechanical modelling. Point source-sink pairs are often of use as simple models for driving flow through a gap in a wall. The use of localized suction to control vortices around aerofoil sections is one of such problems. In oceanography, it is common to use point sources to model the influx of fluid from channels and holes. There are also applications of pulsed source-sink systems in the study of chaotic advection and many others.

Constructing the asymptotic representation of the solution to the Stokes problems near the cusp point O we use the ideas proposed in the paper [33] where the asymptotic behaviour of the solutions to stationary Stokes and Navier-Stokes problems was studied in unbounded domains with paraboloidal outlets to infinity. In turn the method used in [33] is a variant of the algorithm of constructing the asymptotics for solutions to elliptic equations in slender domains, see, e.g., [20], [21], [27], [22] for arbitrary elliptic problems, [28], [32] for the stationary Stokes and Navier–Stokes equations and [38]–[40] for the nonstationary and time-periodic Navier–Stokes equations.

It is mentioned in [33] that for solutions of the stationary Stokes and Navier–Stokes problems the asymptotic representation near a power cusp point on the boundary can be constructed just by the same arguments as for the case of a "paraboloidal outlet to infinity". The distinction between the cusp point and the "paraboloidal outlet" is that in the first case the asymptotic representation is constructed as x_n goes to 0, while in the second case the problem is considered as x_n goes to infinity. In both cases the same coordinate transformation is used. However, in [33] the explicit formulas were not presented.

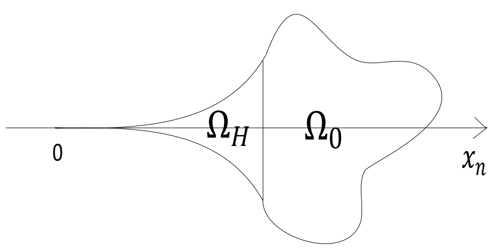

Figure 1.1: Domain Ω.

In the thesis we consider stationary, time-periodic and nonstationary

Stokes problems in domains $\Omega = \Omega_H \cup \Omega_0$ having a singular point O on the boundary (for more detailed description of the domain see Figure 1.1 and Chapter *Notation and auxiliary results*). We assume that there is a source or a sink of the fluid in the cusp point O. Therefore, the solutions are necessary singular. We prove the existence of singular solutions to stationary, time-periodic and nonstationary Stokes problems in the case when the boundary value $\mathbf{a}$ has nonzero fluxes.

Aims and problems

The main aim of the dissertation is the analysis of Stokes problems (stationary, time-periodic and nonstationary) in bounded domains having a peak-type singularity (power cusp singularity). More precisely, our objective is to prove the existence of singular solutions for Stokes problems in power cusp domains. In order to do that, we:

- construct the formal asymptotic expansion of the solutions to the Stokes problems (stationary, time-periodic and nonstationary) near the singular point of the boundary,
- prove the existence of singular solutions for the Stokes problems (stationary, time-periodic and nonstationary) in power cusp domains.

Methods

In the thesis we use methods of functional analysis, properties of Sobolev spaces, both ordinary and partial differential equations theory. We apply matched asymptotic expansion ideas and techniques. We construct a boundary-layer-in-time with the fast time depending on the space variable.

Novelty

All results obtained in the thesis are new. To our best knowledge, the formal asymptotical representation for singular solutions of stationary, time-periodic and nonstationary Stokes problems in a power cusp domain are presented for the first time. The existence of such solutions was not known. As far as we know, the construction of the boundary-layer-in-time with the fast time depending on the space variable is new. We do not know any papers considering fast time variable depending on spacial variables.

Structure of the dissertation and main results

The dissertation consists of five Chapters, Conclusions and Bibliography. The first Chapter is an introduction to the area of research. It contains history and actuality of the problem as well as required information concerning the dissemination of results presented in the thesis.

For the reader convenience in Chapter 2 we present the notation which we use in the dissertation. Some auxiliary results are also given.

In Chapter 3 we consider the stationary Stokes problem. We construct the formal asymptotic expansion of the solution near the singular point of the boundary. Then we prove that a solution to the given problem exist as a sum of the asymptotic expansion and the term with finite energy.

In Chapter 4 we consider the time-periodic Stokes problem. As before we construct the formal asymptotic expansion of the solution near the singular point of the boundary and then prove the existence of the solution which is represented as a sum of the asymptotic expansion and the term with finite energy.

In Chapter 5 we consider the initial boundary value problem or the Stokes system in a power cusp domain. We construct the formal asymptotic expansion of the solution near the singular point of the boundary. In this case the asymptotic expansion consists of two parts: outer and inner (boundary layer) asymptotics. We finish given chapter with the main result, i.e. we prove that a solution to the given problem exists as a sum of the asymptotic expansion and the term with finite energy.

Dissemination

The results of this thesis were presented at the following conferences

- *Asymptotic Problems, Elliptic and Parabolic Issues,* Vilnius, Lithuania, June 1 - 5, 2015.
- *Classical Problems and New Trends in Mathematical Fluid Mechanics,* Ferrara, Italy, September 29 - October 3, 2014.
- *Advances in Mathematical Fluid Mechanics,* Lisbon, Portugal, June 30 - July 5, 2014.
- *55th Conference of Lithuanian Mathematical Society,* Vilnius, Lithuania, June 26 - 27, 2014.

School

- *12th School "Mathematical Theory in Fluid Mechanics",* Kacov, Czech Republic, May 24 - 31, 2013.

Contributing talks were given at the seminars at the Department of Differential Equations and Computational Mathematics (VU) and at the Department of Mathematical Modelling (VGTU).

Publications

The results of this thesis are published in the following papers:

1. A. Eismontaite, K. Pileckas, On singular solutions of time-periodic and steady Stokes problems in a power cusp domain, *Applicable Analysis*, **97**(3) (2018), 415–437.

2. A. Eismontaite, K. Pileckas, On singular solutions of the initial boundary value problem for the Stokes system in a power cusp domain, *Applicable Analysis*, (2018), Advanced online publication: 20 Apr 2018, https://doi.org/10.1080/00036811.2018.1460815.

Acknowledgments

I am grateful to my husband for his encouragement and consistent support. I am highly thankful to my colleagues, especially Mindaugas Skujus and Kristina Kaulakytė, for long conversations, discussions and ideas exchange. I thank my Batchelor and Master studies advisor Professor Algirdas Ambrazevičius for his guidance while writing my first scientific works as well as encouraging me to continue with Doctoral studies. I would especially like to thank Professor Konstantinas Pileckas for his patient and subtile advising during Doctoral studies. I am very grateful for introducing me to so many interesting problems and many great mathematicians.

Chapter 2

Notation and auxiliary results

For the reader convenience we shall first introduce some notation. In the thesis vector-valued functions are denoted by bold letters. The vector valued function $\mathbf{u} = (u_1, ..., u_n)$ belongs to a Banach space V, if $u_i \in V$, $i = 1, \dots, n$, and $\|\mathbf{u}\|_V = \sum\limits_{i=1}^{n} \|u_i\|_V$.

We use C, C_j, $j = 1, 2, \dots$, to denote constants whose numerical values or whose dependence on parameters is unessential. Therefore, these constants may have different values in a single computation.

By $\lfloor x \rfloor$ we denote the integer part of the number x.

Let G be a bounded domain in $\mathbb{R}^n$. In the thesis, we use usual notations of functional spaces (e.g., [1]). By $L^p(G)$ and $W^{m,p}(G)$, $1 \le p < \infty$, we denote the usual Lebesgue and Sobolev spaces, respectively. The norms in $L^p(G)$ and $W^{m,p}$ are indicated by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, respectively, or, to be exact,

$$\|\mathbf{u}\|_{L^p(G)} = \left(\int\limits_{\Omega} |\mathbf{u}(x)|^p dt \right)^{1/p}$$

and

$$\|\mathbf{u}\|_{W^{m,p}(G)} = \sum_{|\alpha| \le m} \|D_x^\alpha \mathbf{u}\|_{L^p(G)},$$

where $D_x^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_n} x_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$. Space $W^{m-1/p,p}(\partial G)$ is the trace space of functions from $W^{m,p}(G)$ on ∂G with the norm

$$\|\mathbf{u}\|_{W^{m-1/p,p}(\partial G)} = \inf\{\|\hat{\mathbf{u}}\|_{W^{m,p}} : \hat{\mathbf{u}} = \mathbf{u} \text{ on } \partial G\}.$$

We denote by $C^\infty(G)$ the set of all infinitely differentiable functions defined on G and by $C_0^\infty(G)$ the subset of all functions from $C^\infty(G)$ with compact supports in G. By $\mathring{W}^{k,q}(G)$ we denote the completion of the $C_0^\infty(G)$ in the $\|\cdot\|_{W^{m,p}}$ norm. We shall write $\mathbf{u} \in W^{m,p}_{loc}(G)$ if $\mathbf{u} \in W^{m,p}(G')$ for any bounded subdomain G' such that $\overline{G'} \subset G$. The space $L^p(0,T;X)$ consists of all measurable functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{L^p(0,T;X)} = \left(\int_0^T \|\mathbf{u}(t)\|^p dt\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

A vector field $\mathbf{v}$ is called solenoidal (or divergence free) if

$$\nabla \cdot \mathbf{v} = \operatorname{div} \mathbf{u} = 0.$$

If functions u, $v \in W^{1,2}(G)$ then there holds the so-called integration by parts formula:

$$\int_G \frac{\partial u(x)}{\partial x_k} v(x)\, dx = -\int_G u(x) \frac{\partial v(x)}{\partial x_k}\, dx + \int_{\partial G} u(x) v(x) n_k(x)\, dS, \tag{2.1}$$

where (here and below) $\mathbf{n}$ is the unit outward (with respect to G) normal to ∂G, $n_k = \cos(\mathbf{n}, x_k)$ - cosine of an angle between normal $\mathbf{n}$ and x_k-axis.

Domain Ω

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain described as a union $\Omega = \Omega_H \cup \Omega_0$ (see Fig. 2.1), where Ω_H is the power cusp domain:

$$\Omega_H = \left\{x \in \mathbb{R}^n : |x'| < \varphi(x_n), x_n \in (0, H]\right\},$$

$\varphi(x_n) = \gamma_0 x_n^\lambda$, $\gamma_0 = \text{const}$, $\lambda > 1$, $x' = (x_1, ..., x_{n-1})$, $\partial\Omega_0$ is Lipschitz and the intersection $\overline{\Omega_H} \bigcap \overline{\Omega_0}$ is the plane $x_n = H$.

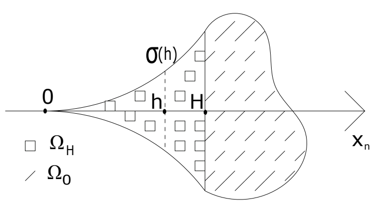

Figure 2.1: Domain Ω.

We denote by $\sigma(h)$ the cross-section of Ω_H by the plane $x_n = h$ (see Fig. 2.1):

$$\sigma(h) = \{x \in \Omega_H : x_n = h = \text{const}\}.$$

In the thesis we also use notation

$$\Omega_{H/2,H} = \left\{x \in \mathbb{R}^n : |x'| < \varphi(x_n),\, x_n \in [H/2, H]\right\} \cup \Omega_0. \tag{2.2}$$

In the thesis we will use the cut-off function ξ

$$\xi(t) = \begin{cases} 1, & t \le H/2, \\ 0, & t \ge H. \end{cases} \tag{2.3}$$

Poisson problem

Throughout the thesis the same Poisson problem appears

$$\begin{cases} -\nu\Delta'\varphi & = & 1, \quad y' \in \omega, \\ \varphi & = & 0, \quad y' \in \partial\omega, \end{cases} \tag{2.4}$$

where $y' = (y_1, ..., y_{n-1})$ and

$$\omega = \left\{y' \in \mathbb{R}^{n-1} : |y'| < \gamma_0\right\}.$$

The solution φ to (2.4) has the form

$$\varphi(y') = \frac{1}{2\nu(n-1)}\left(|y'|^2 - \gamma_0^2\right). \tag{2.5}$$

Obviously the function φ obeys the estimates

$$|\partial_i^j \varphi(y')| \le C_k, \quad i = 1, ..., n-1,\ j = 0, 1, ... \tag{2.6}$$

Moreover,

$$\int\limits_\omega \varphi(y')\, dy' = -\nu \int\limits_\omega |\nabla'\varphi(y')|^2 dy' \equiv \kappa_0 < 0 \tag{2.7}$$

and

$$\int\limits_\omega y' \cdot \nabla'\varphi(y')\, dy' = -(n-1)\int\limits_\omega \varphi(y')\, dy' = -(n-1)\kappa_0, \tag{2.8}$$

where $\kappa_0 = -\dfrac{\pi}{8\nu}\gamma_0^4$ for $n = 3$ and $\kappa_0 = -\dfrac{2}{3\nu}\gamma_0^3$ for $n = 2$.

Auxiliary results

Let G be an arbitrary bounded domain in $\mathbb{R}^n$.

The Dirichlet problem for the Laplace operator

Consider the Dirichlet problem for the Laplace operator

$$\begin{cases} -\Delta \mathbf{u} &= \mathbf{f}, \quad x \in G, \\ \mathbf{u} &= 0, \quad x \in \partial G, \end{cases} \tag{2.9}$$

Definition 2.1. *The weak solution of the problem* (2.9) *is a vector field* $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}(G)$ *satisfying the integral identity*

$$\int\limits_G \nabla \mathbf{u}(x) \cdot \nabla \boldsymbol{\eta}(x)\, dx = \int\limits_G \mathbf{f}(x) \cdot \boldsymbol{\eta}(x)\, dx,$$

for every function $\boldsymbol{\eta} \in \overset{\circ}{W}{}^{1,2}(G)$.

Here and in all dissertation by $\nabla\mathbf{u}\cdot\nabla\mathbf{v}$ we mean $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum\limits_{j=1}^{n}\sum\limits_{k=1}^{n} \frac{\partial u_j}{\partial x_k}\frac{\partial v_j}{\partial x_k}$.

There holds the following theorem (e.g. [8]).

Theorem 2.1. *Let* $\mathbf{f} \in L^2(G)$. *Then the problem* (2.9) *admits a unique weak solution* $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}(G)$ *and there holds the estimate*

$$\|\mathbf{u}\|_{W^{1,2}(G)} \le c\|\mathbf{f}\|_{L^2(G)}. \tag{2.10}$$

Moreover, if ∂G *is* C^∞ *and* $\mathbf{f} \in C^\infty(G)$, *then solution* $\mathbf{u} \in C^\infty(G)$.

The initial boundary value problem for the heat equation

Consider the initial boundary value problem for the heat equation

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} &= \mathbf{f}, \qquad x \in G, \\ \mathbf{u} &= 0, \qquad x \in \partial G, \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \quad x \in G. \end{cases} \tag{2.11}$$

Definition 2.2. *The weak solution of the problem* (2.11) *is a vector field* $\mathbf{u} \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G))$ *with* $\mathbf{u}_t \in L^2(0,T;L^2(G))$ *satisfying the initial condition* $u(x,0) = u_0(x)$ *and the integral identity*

$$\int\limits_0^t\int\limits_G \mathbf{u}_\tau(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\, dx d\tau + \int\limits_0^t\int\limits_G \nabla\mathbf{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\, dx d\tau$$
$$= \int\limits_0^t\int\limits_G \mathbf{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\, dx d\tau,$$

for every function $\boldsymbol{\eta} \in L^2(0,T;C_0^\infty(G))$.

There holds the following theorem (e.g. [8]).

Theorem 2.2. *Let* $\mathbf{f} \in L^2(0,T;L^2(G))$, $\mathbf{u}_0 \in \overset{\circ}{W}{}^{1,2}(G)$. *Then the problem* (2.11) *admits a unique weak solution* $\mathbf{u} \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G))$ *with* $\mathbf{u}_t \in L^2(0,T;L^2(G))$ *and there holds the estimate*

$$\max_{t\in[0,T)} \|\mathbf{u}(\cdot,t)\|^2_{W^{1,2}(G)} + \|\mathbf{u}\|^2_{L^2(0,T;W^{1,2}(G))} + \|\mathbf{u}_t\|^2_{L^2(0,T;L^2(G))} \leq c\|\mathbf{f}\|_{L^2(0,T;L^2(G))}. \tag{2.12}$$

Moreover, if ∂G *is* C^∞, $\mathbf{f} \in C^\infty(0,T;C^\infty(G))$ *and* $\mathbf{u}_0 \in C^\infty(G)$, *then solution* $\mathbf{u} \in C^\infty(0,T;C^\infty(G))$.

The divergence problem

Consider the divergence problem

$$\begin{cases} \operatorname{div}\mathbf{u} = g, & x \in G, \\ \mathbf{u} = 0, & x \in \partial G. \end{cases} \tag{2.13}$$

From integration by parts formula (2.1) we get

$$\int_G \sum_{k=1}^n \frac{\partial u_k(x)}{\partial x_k}\,dx = \int_{\partial G} \sum_{k=1}^n u_k(x) n_k(x)\,dS,$$

i.e. there holds Stokes formula

$$\int_G \operatorname{div}\mathbf{u}(x)\,dx = \int_{\partial G} \mathbf{u}(x)\cdot\mathbf{n}(x)\,dS.$$

Hence, if $\mathbf{u}(x)$ is a solution of the problem (2.13), we necessarily have

$$\int_G g(x)\,dx = \int_G \operatorname{div}\mathbf{u}(x)\,dx = \int_{\partial G} \mathbf{u}(x)\cdot\mathbf{n}(x)\,dS = 0.$$

Thus, the condition

$$\int_G g(x)\,dx = 0 \tag{2.14}$$

is the necessary for the solvability of the problem (2.13). There holds the following theorem (see [19] for the proof).

Theorem 2.3. *Let* $g \in L^2(G)$ *and there holds the necessary solvability condition* (2.14). *Then the problem* (2.13) *has at least one solution* $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}(G)$ *satisfying the following estimate*

$$\|\nabla\mathbf{u}\|_{L^2(G)} \leq c\|g\|_{L^2(G)}. \tag{2.15}$$

The steady Stokes problem with the homogeneous divergence equation

Consider in G the steady Stokes problem with the homogeneous divergence equation:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \quad x \in G, \\ \operatorname{div}\mathbf{u} &= 0, \quad x \in G, \\ \mathbf{u} &= 0, \quad x \in \partial G. \end{cases} \tag{2.16}$$

Definition 2.3. *The weak solution of the problem* (2.16) *is a solenoidal vector field* $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}(G)$ *satisfying the integral identity*

$$\int\limits_G \nabla\mathbf{u}(x) \cdot \nabla\boldsymbol{\eta}(x)\, dx = \int\limits_G \mathbf{f}(x) \cdot \boldsymbol{\eta}(x)\, dx,$$

for every solenoidal $\boldsymbol{\eta} \in \overset{\circ}{W}{}^{1,2}(G)$.

Solvability of (2.16) follows from well known results (the unique solvability of the Stokes problem is proved for arbitrary domains G, see, e.g., [18]).

Theorem 2.4. *Let* $\mathbf{f} \in L^2(G)$. *Then the problem* (2.16) *admits a unique weak solution* $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}(G)$ *and there holds the estimate*

$$\|\mathbf{u}\|^2_{W^{1,2}(G)} \le c\|\mathbf{f}\|^2_{L^2(G)}. \tag{2.17}$$

Moreover, there exists a corresponding pressure function $p \in L^2(G)$ *such that* $\int\limits_G p(x)\, dx = 0$ *and the following estimate holds*

$$\|p\|^2_{L^2(G)} \le c\|\mathbf{f}\|^2_{L^2(G)}.$$

If ∂G *is* C^∞, $\mathbf{f} \in C^\infty(G)$, *then solution* $\mathbf{u}$, $p \in C^\infty(G)$.

The steady Stokes problem with the nonhomogeneous divergence equation

Consider in G the steady Stokes problem with the nonhomogeneous divergence equation:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \quad x \in G, \\ \operatorname{div}\mathbf{u} &= g, \quad x \in G, \\ \mathbf{u} &= 0, \quad x \in \partial G. \end{cases} \tag{2.18}$$

Definition 2.4. *We look for a weak solution of the problem* (2.18) *in the form* $\mathbf{u} = \mathbf{V}+\mathbf{v}$, *where a function* $\mathbf{V}$ *is the solution of the divergence problem* $(2.18)_{2,3}$ *and* $\mathbf{v} \in \mathring{W}^{1,2}(G)$ *is a solenoidal vector field satisfying the integral identity*

$$\nu \int\limits_G \nabla \mathbf{v} \cdot \nabla \boldsymbol{\eta}\, dx = \int\limits_G \mathbf{f} \cdot \boldsymbol{\eta} dx - \nu \int\limits_G \nabla \mathbf{V} \cdot \nabla \boldsymbol{\eta}\, dx$$

for every solenoidal $\boldsymbol{\eta} \in \mathring{W}^{1,2}(G)$.

From Theorems 2.3 and 2.4 follows

Theorem 2.5. *Let* $\mathbf{f} \in L^2(G)$ *and* $g \in L^2(G)$ *satisfies the necessary solvability condition* (2.14). *Then the problem* (2.18) *admits a unique weak solution* $\mathbf{u} = \mathbf{V} + \mathbf{v}$ *and there holds the estimate*

$$\|\mathbf{u}\|^2_{W^{1,2}(G)} \leq c\left(\|\mathbf{f}\|^2_{L^2(G)} + \|g\|^2_{L^2(G)}\right). \tag{2.19}$$

Moreover, there exists a corresponding pressure function $p \in L^2(G)$ *such that* $\int\limits_G p(x)\, dx = 0$ *and the following estimate holds*

$$\|p\|^2_{L^2(G)} \leq c\left(\|\mathbf{f}\|^2_{L^2(G)} + \|g\|^2_{L^2(G)}\right).$$

If ∂G *is* C^∞, $\mathbf{f}$, $g \in C^\infty(G)$, *then solution* $\mathbf{u}$, $p \in C^\infty(G)$.

The time-periodic Stokes problem

Consider in G the time-periodic Stokes equations:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &=& \mathbf{f}, & x \in G, \\ \operatorname{div}\mathbf{u} &=& 0, & x \in G, \\ \mathbf{u} &=& 0, & x \in \partial G, \\ \mathbf{u}(x,0) &=& \mathbf{u}(x,2\pi), & x \in G, \end{cases} \tag{2.20}$$

Definition 2.5. *The weak solution of the problem* (2.20) *is a time-periodic solenoidal vector field* $\mathbf{u} \in L^2(0,2\pi;\mathring{W}^{1,2}(G))$ *with* $\mathbf{u}_t \in L^2(0,2\pi;L^2(G))$ *satisfying the integral identity*

$$\int\limits_0^{2\pi}\int\limits_G \mathbf{u}_t(x,t)\cdot\boldsymbol{\eta}(x,t)\, dxdt + \nu\int\limits_0^{2\pi}\int\limits_G \nabla\mathbf{u}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)\, dxdt$$

$$= \int\limits_0^{2\pi}\int\limits_G \mathbf{f}(x,t)\cdot\boldsymbol{\eta}(x,t)\, dxdt,$$

for every time-periodic solenoidal function $\boldsymbol{\eta} \in L^2(0,2\pi;\mathring{W}^{1,2}(G))$.

Theorem 2.6. *Let $\mathbf{f} \in L^2(0, 2\pi; L^2(G))$ be a time-periodic function. Then the problem* (2.20) *admits a unique time-periodic weak solution* $\mathbf{u} \in L^2(0, 2\pi; \mathring{W}^{1,2}(G))$ *with* $\mathbf{u}_t \in L^2(0, 2\pi; L^2(G))$ *and there holds the estimate*

$$\begin{aligned} \sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot, t)\|^2_{W^{1,2}(G)} + \|\mathbf{u}\|^2_{L^2(0,2\pi;W^{1,2}(G))} + \|\mathbf{u}_t\|^2_{L^2(0,2\pi;L^2(G))} \\ \leq c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}. \end{aligned} \tag{2.21}$$

Proof. This theorem is proved by standard methods, for example, following the ideas from [10] (see also [11], [48], [24]). However, for the reader convenience, we present the sketch of the proof. The solution $\mathbf{u}$ can be found as the limit of Galerkin approximations $\mathbf{u}_N(x, t) = \sum\limits_{J=1}^{N} \gamma_{kN}(t)\mathbf{e}_k(x)$, where $\mathbf{e}_k \in \mathring{W}^{1,2}(G) \bigcap W^{2,2}_{loc}(G)$ are the eigenfunctions of the Stokes problem (the existence of the eigenfunctions $\mathbf{e}_k$ is known for arbitrary bounded domains, e.g., [18]) and $\gamma_{kN}(t)$ are found as periodic solutions of the following system of ordinary differential equations:

$$\begin{aligned} &\int_G \mathbf{u}_{Nt}(x, t) \cdot \mathbf{e}_l(x)\, dx + \nu \int_G \nabla\mathbf{u}_N(x, t) \cdot \nabla\mathbf{e}_l(x)\, dx \\ &= \int_G \mathbf{f}(x, t) \cdot \mathbf{e}_l(x)\, dx,\ l = 1, 2, \dots, N,\ t \in [0, 2\pi]. \end{aligned} \tag{2.22}$$

Multiplying (2.22) by $\gamma_{lN}(t)$ and summing over l we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_N\|^2_{L^2(G)} + \nu\|\nabla\mathbf{u}_N\|^2_{L^2(G)} = \int_G \mathbf{f} \cdot \mathbf{u}_N\, dx. \tag{2.23}$$

Using the Poincaré and Cauchy–Schwarz inequalities from (2.23) follows

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_N\|_{L^2(G)} + c_0\|\mathbf{u}_N\|_{L^2(G)} \leq \|\mathbf{f}\|_{L^2(G)}. \tag{2.24}$$

Multiplying (2.24) by $e^{c_0 t}$ and integrating by t gives after simple calculations

$$\|\mathbf{u}_N(\cdot, 2\pi)\|_{L^2(G)} \leq e^{-2c_0\pi}\|\mathbf{u}_N(\cdot, 0)\|_{L^2(G)} + \frac{1}{\sqrt{2c_0}}\|\mathbf{f}\|_{L^2(0,2\pi;L^2(G))}.$$

Hence, the map $M : \mathbf{u}_N(\cdot, 0) \mapsto \mathbf{u}_N(\cdot, 2\pi)$ is continuous and maps the ball of $L^2(\Omega)$ of radius $\sqrt{2}\|\mathbf{f}\|_{L^2(0,2\pi;L^2(G))}/\sqrt{c_0}$ into itself, and, therefore, there exists a smooth 2π-periodic solution to the Galerkin approximations (2.22) (e.g., [11]).

Integrating (2.23) by t over $[0, 2\pi]$ and using the periodicity property of $\mathbf{u}_N$ we also derive the estimate

$$\|\nabla\mathbf{u}_N\|^2_{L^2(0,2\pi;L^2(G))} \leq c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}. \tag{2.25}$$

Analogously, multiplying (2.22) by $\gamma'_{lN}(t)$ and summing over l yield

$$\|\mathbf{u}_{Nt}\|^2_{L^2(G)} + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{u}_N\|^2_{L^2(G)} = \int\limits_G \mathbf{f}\cdot \mathbf{u}_{Nt}\, dx, \tag{2.26}$$

and, therefore,

$$\|\mathbf{u}_{Nt}\|^2_{L^2(0,2\pi;L^2(G))} \le c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}. \tag{2.27}$$

Let us now multiply (2.22) by $t\,\gamma_{lN}(t)$:

$$\frac{1}{2}\frac{d}{dt}\Big(t\|\mathbf{u}_N\|^2_{L^2(G)}\Big) - \|\mathbf{u}_N\|^2_{L^2(G)} + \nu\, t\|\nabla \mathbf{u}_N\|^2_{L^2(G)} = t\int\limits_G \mathbf{f}\cdot \widehat{\mathbf{U}}_N\, dx.$$

Integrating by t and applying the Poincaré and Cauchy–Schwarz inequalities and (2.25) we derive

$$\begin{aligned}&\|\mathbf{u}_N(\cdot,2\pi)\|^2_{L^2(G)} \le \|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))} + \frac{\pi+1}{\pi}\|\mathbf{u}_N\|^2_{L^2(0,2\pi;L^2(G))}\\ &\le \|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))} + c\|\nabla\mathbf{u}_N\|^2_{L^2(0,2\pi;L^2(G))} \le c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}.\end{aligned}$$

By the periodicity property of $\mathbf{u}_N$, we also get

$$\|\mathbf{u}_N(\cdot,0)\|^2_{L^2(G)} \le c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}. \tag{2.28}$$

Multiplying (2.22) by $t\,\gamma'_{lN}(t)$, using (2.25), (2.27) and arguing in the same way yield

$$\begin{aligned}&\|\nabla\mathbf{u}_N(\cdot,0)\|^2_{L^2(G)} = \|\nabla\mathbf{u}_N(\cdot,2\pi)\|^2_{L^2(G)} \le \frac{1}{2\pi}\Big(\|\nabla\mathbf{u}_N\|^2_{L^2(0,2\pi;L^2(G))}\\ &+\|\mathbf{u}_{Nt}\|^2_{L^2(0,2\pi;L^2(G))} + \|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}\Big) \le c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}.\end{aligned} \tag{2.29}$$

Consequently, integrating (2.23) and (2.26) by t from 0 to $\tau\in(0,2\pi]$ and using (2.28), (2.29), we show

$$\begin{aligned}\sup_{\tau\in[0,2\pi]}\|\mathbf{u}_N(\cdot,\tau)\|^2_{W^{1,2}(G)} + \|\mathbf{u}_N\|^2_{L^2(0,2\pi;W^{1,2}(G))} + \|\mathbf{u}_{Nt}\|^2_{L^2(0,2\pi;L^2(G))}&\\ \le c\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(G))}.&\end{aligned} \tag{2.30}$$

Since the linear span of the eigenfunction $\{\mathbf{e}_k\}$ of the Stokes problem is dense in the space $H(G)=\{\mathbf{v}\in \mathring{W}^{1,2}(G) : \mathrm{div}\mathbf{v}=0\}$ and in view of (2.30), we can pass to the limit as $N\to\infty$ and to prove by standard arguments that the limit function $\mathbf{u}$ is a weak solution to problem (5.77). For $\mathbf{u}$ holds the estimate (2.33). The uniqueness of the solution is obvious. $\square$

The inverse problem for the heat equation

Now consider in G the following (inverse) problem for the heat equation

$$\begin{cases} u_t(x,t) - \nu\Delta u(x,t) = q(t), \quad (x,t) \in G \times (0,T), \\ u(x,t)|_{\partial G} = 0, \quad u(x,0) = u_0(x), \\ \int\limits_G u(x,t)\,dx = F(t). \end{cases} \tag{2.31}$$

Here $T \in (0,+\infty]$, u_0 and F are given functions, while u, q have to be found, i.e., we have to solve the heat equation with an unknown right-hand side q which has to be found so that u satisfies the additional flux condition $(2.31)_3$.

Definition 2.6. *By a weak solution of the problem* (2.31) *we understand a pair* $(u,q) \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G)) \times L^2(0,T)$ *with* $\partial_t u \in L^2(0,T;L^2(G))$ *satisfying the integral identity*

$$\int\limits_0^t \int\limits_G \partial_\tau u(x,\tau)\eta(x,\tau)\,dxd\tau + \nu \int\limits_0^t \int\limits_G \nabla u(x,\tau)\cdot\nabla\eta(x,\tau)\,dxd\tau$$
$$= \int\limits_0^t q(\tau) \int\limits_G \eta(x,\tau)\,dxd\tau, \quad \textit{for every } \eta \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G)), \quad \forall t \in (0,T],$$

the initial condition $u(x,0) = u_0(x)$ *and the flux condition*

$$\int\limits_G u(x,t)\,dx = F(t).$$

The theorem below is proved in [42] (see also [43]).

Theorem 2.7. *Let* $u_0 \in \overset{\circ}{W}{}^{1,2}(G)$, $F \in W^{1,2}(0,\infty)$ *and let the following compatibility condition*

$$F(0) = \int\limits_G u_0(x)\,dx \tag{2.32}$$

holds. Then there exists a unique weak solution (u,q) *of the problem* (2.31) *such that*

$$\begin{aligned} \sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{W^{1,2}(G)} + \|u\|_{L^2(0,\infty;W^{1,2}(G))} + \|\partial_t u\|_{L^2(0,\infty;L^2(G))} \\ +\|q\|_{L^2(0,\infty)} \le c\left(\|F\|_{W^{1,2}(0,\infty)} + \|u_0\|_{W^{1,2}(G)}\right). \end{aligned} \tag{2.33}$$

Let $W^{1,2}_\mu(0,\infty)$ be the space of functions with the finite norm

$$\|F\|_{W^{1,2}_\mu(0,\infty)} = \left(\int\limits_0^\infty \exp(2\mu t)\Big(|F(t)|^2 + |F'(t)|^2\Big)\,dt\right)^{1/2}.$$

If $\mu > 0$, the elements of $W_\mu^{1,2}(0,\infty)$ vanish exponentially as $t \to +\infty$.

The behavior of the solution of the problem (2.31) as $t \to \infty$ is described by the following theorem (see [41], [43]).

Theorem 2.8. *Let* $u_0 \in \overset{\circ}{W}{}^{1,2}(G)$, $F \in W_\mu^{1,2}(0,\infty)$ *with* $\mu > 0$ *and there holds the compatibility condition* (2.32)*. Then the solution* $(u,\, q)$ *of the problem* (2.31) *satisfies the estimate*

$$\begin{aligned} &\max_{t\in[0,\infty)} \left[\exp(\nu_* t)\left(\int_G |u(x,t)|^2\,dx + \nu\int_G |\nabla u(x,t)|^2\,dx\right)\right] \\ &+ \int_0^\infty \exp(\nu_* t)\int_G |\partial_t u(x,t)|^2\ dx\,dt + \int_0^\infty \exp(\nu_* t)|q(t)|^2\,dt \\ &\qquad\qquad \le c\left(\|F\|^2_{W_\mu^{1,2}(0,\infty)} + \|u_0\|^2_{W^{1,2}(G)}\right), \end{aligned} \tag{2.34}$$

where $\nu_* = \min\{\lambda_1, 1, 2\mu\}$*. Here* λ_1 *is the first eigenvalue of the Dirichlet problem for the Laplace equation*

$$\begin{cases} -\nu\Delta u(x) = \lambda u(x), \quad x \in G, \\ u(x)|_{\partial G} = 0. \end{cases}$$

The non-steady Stokes problem with homogeneous divergence equation and boundary condition

Let us now consider the non-steady Stokes problem

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} + \sum\limits_{i=1}^n \frac{\partial \mathbf{f}_i}{\partial x_i}, \quad (x,t) \in G\times(0,T), \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}|_{\partial G} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x). \end{cases} \tag{2.35}$$

Definition 2.7. *The weak solution of the problem* (2.35) *is a solenoidal vector field* $\mathbf{u} \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G))$ *with* $\mathbf{u}_t \in L^2(0,T;L^2(G))$ *satisfying the initial condition* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ *and the integral identity*

$$\begin{aligned} &\int_0^t\int_G \mathbf{u}_\tau(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau + \nu\int_0^t\int_G \nabla\mathbf{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\,dxd\tau \\ &= \int_0^t\int_G \mathbf{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau + \sum_{i=1}^n\int_0^t\int_G \mathbf{f}_i(x,\tau)\cdot\frac{\partial\boldsymbol{\eta}}{\partial x_i}(x,\tau)\,dxd\tau, \end{aligned}$$

for every $\boldsymbol{\eta} \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G))$, $\ \forall t \in (0,T]$.

There holds the following theorem (e.g., [18]).

Theorem 2.9. *Let* $\mathbf{f} \in L^2(0,\infty;L^2(G))$, $\mathbf{f}_i, \mathbf{f}_{i,t} \in L^2(0,\infty;L^2(G)), i = 1,\dots, n$, $\mathbf{u}_0 \in \overset{\circ}{W}{}^{1,2}(G)$, $\operatorname{div}\mathbf{u}_0 = 0$. *Then the problem* (2.35) *admits a unique weak solution* $\mathbf{u}$ *such that*

$$\max_{t\in[0,\infty)} \|\mathbf{u}(\cdot,t)\|^2_{W^{1,2}(G)} + \|\mathbf{u}\|^2_{L^2(0,\infty;W^{1,2}(G))} + \|\mathbf{u}_t\|^2_{L^2(0,\infty;L^2(G))}$$
$$\le c\Big(\|\mathbf{u}_0\|^2_{W^{1,2}(G)} + \|\mathbf{f}\|^2_{L^2(0,\infty;L^2(G))} \tag{2.36}$$
$$+\sum_{i=1}^{n}\big(\|\mathbf{f}_i\|^2_{L^2(0,\infty;L^2(G))} + \|\mathbf{f}_{i,t}\|^2_{L^2(0,\infty;L^2(G))}\big)\Big).$$

Moreover, there exists a number $\nu_0 > 0$ *such that if* $\mathbf{f} \in L^2_\mu(0,\infty;L^2(G))$ *with* $\mu \in (0,\nu_0)$, *then*

$$\max_{t\in[0,\infty)}\left[\exp(\mu t)\left(\int_G |\mathbf{u}(x,t)|^2\,dx + \nu\int_G |\nabla\mathbf{u}(x,t)|^2\,dx\right)\right]$$
$$+\int_0^\infty \exp(\mu t)\int_G |\mathbf{u}_t(x,t)|^2\,dx\,dt \le \Big(\|\mathbf{u}_0\|^2_{W^{1,2}(G)} \tag{2.37}$$
$$+\|\mathbf{f}\|^2_{L^2_\mu(0,\infty;L^2(G))} + \sum_{i=1}^{n}\big(\|\mathbf{f}_i\|^2_{L^2_\mu(0,\infty;L^2(G))} + \|\mathbf{f}_{i,t}\|^2_{L^2_\mu(0,\infty;L^2(G))}\big)\Big).$$

The non-steady Stokes problem with the nonhomogeneous divergence equation and boundary condition

Finally, consider the non-steady Stokes problem with nonhomogeneous divergence equation and boundary condition

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad (x,t) \in G\times(0,T), \\ \operatorname{div}\mathbf{u} = d, \\ \mathbf{u}|_{\partial G} = \mathbf{a}, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \end{cases} \tag{2.38}$$

and assume that the compatibility conditions

$$\operatorname{div}\mathbf{u}_0(x) = d(x,0), \quad \mathbf{u}_0(x)|_{\partial G} = \mathbf{a}(x,0), \tag{2.39}$$

$$\int_G d(x,t)\,dx = \int_{\partial G} \mathbf{a}(x,t)\cdot\mathbf{n}(x)\,ds \quad \forall t \in (0,T) \tag{2.40}$$

hold.

We look for the solution of the problem (2.38) in the form $\mathbf{u} = \mathbf{V} + \mathbf{v}$, where a function $\mathbf{V}$ the solution to the following problem

$$\begin{cases} \operatorname{div}\mathbf{V} = d, \\ \mathbf{V}|_{\partial G} = \mathbf{a}, \end{cases} \tag{2.41}$$

while $\mathbf{v}$ is the solution to the following problem

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} - \mathbf{V}_t + \nu\Delta\mathbf{V}, \quad (x,t) \in G \times (0,T), \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}|_{\partial G} = 0, \quad \mathbf{v}(x,0) = \mathbf{u}_0(x) - \mathbf{V}(x,0). \end{cases} \tag{2.42}$$

Then, from the conditions (2.39) we have that

$$\operatorname{div}(\mathbf{u}_0(x) - \mathbf{V}(x,0)) = 0, \quad (\mathbf{u}_0(x) - \mathbf{V}(x,0))|_{\partial G} = 0.$$

Consider problem (2.41). First we consider the linear extension operator $E : W^{1/2,2}(\partial G) \to W^{1,2}(G)$ given by the formula $E\mathbf{a} = \mathbf{w}$, where $\mathbf{w}|_{\partial G} = \mathbf{a}$. The operator E is bounded:

$$\|E\mathbf{a}\|^2_{W^{1,2}(G)} = \|\mathbf{w}\|^2_{W^{1,2}(G)} \le C\|\mathbf{a}\|^2_{W^{1/2,2}(\partial G)}, \tag{2.43}$$

see, e.g. [1].

If $\mathbf{a} = \mathbf{a}(x,t)$ depends on t and $\mathbf{a}_t \in W^{1/2,2}(\partial G)$, then due to the fact that the operator E is linear, we have $E\mathbf{a}_t = \mathbf{w}_t$ and

$$\|E\mathbf{a}_t\|^2_{W^{1,2}(G)} \le C\|\mathbf{a}_t\|^2_{W^{1/2,2}(\partial G)}. \tag{2.44}$$

If $\mathbf{a} \in L^2(0,T;W^{1/2,2}(\partial G))$ and $\mathbf{a}_t \in L^2(0,T;W^{1/2,2}(\partial G))$, then integrating (2.43), (2.44) by t we get

$$\|\mathbf{w}\|^2_{L^2(0,T;W^{1,2}(G))} \le C\|\mathbf{a}\|^2_{L^2(0,T;W^{1/2,2}(\partial G))}, \tag{2.45}$$

$$\|\mathbf{w}_t\|^2_{L^2(0,T;W^{1,2}(G))} \le C\|\mathbf{a}_t\|^2_{L^2(0,T;W^{1/2,2}(\partial G))}. \tag{2.46}$$

We look for the solution $\mathbf{V}$ to the problem (2.41) in the form $\mathbf{V} = \mathbf{w} + \mathbf{z}$, where the vector function $\mathbf{z}$ is the solution of the problem

$$\begin{cases} \operatorname{div}\mathbf{z} = d - \operatorname{div}\mathbf{w} := h, \\ \mathbf{z}|_{\partial G} = 0. \end{cases} \tag{2.47}$$

Note, that the variable t in the equations (2.47) plays the role of a parameter. Thus, we can differentiate with respect to time variable t. Then, from (2.40) it follows that $\int_G h\,dx = 0$ and $\int_G h_t\,dx = 0$ for all t. Therefore, the problem (2.47) has a solution $\mathbf{z}$ satisfying the estimates

$$\begin{aligned} &\|\nabla\mathbf{z}\|_{L^2(0,T;W^{1,2}(G))} \le C\|h\|_{L^2(0,T;L^2(G))} \\ &\le C\left(\|d\|_{L^2(0,T;L^2(G))} + \|\mathbf{a}\|_{L^2(0,T;W^{1/2,2}(\partial G))}\right), \end{aligned} \tag{2.48}$$

$$\begin{aligned} &\|\nabla\mathbf{z}_t\|_{L^2(0,T;W^{1,2}(G))} \le C\|h_t\|_{L^2(0,T;W^{1/2,2}(\partial G))} \\ &\le C\left(\|d_t\|_{L^2(0,T;L^2(G))} + \|\mathbf{a}_t\|_{L^2(0,T;W^{1/2,2}(\partial G))}\right), \end{aligned} \tag{2.49}$$

(see [19] or the corresponding subsection about divergence problem in the present chapter).

Note, that if d and $\mathbf{a}$ exponentially vanish as $t \to \infty$, then also the vector function $\mathbf{V} = \mathbf{w} + \mathbf{z}$ exponentially vanishes.

Now consider problem (2.42). Function $\mathbf{v}$ is a solenoidal vector field $\mathbf{v} \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G))$ with $\mathbf{v}_t \in L^2(0,T;L^2(G))$ satisfying the initial condition $\mathbf{v}(x,0) = \mathbf{u}_0(x) - \mathbf{V}(x,0)$ and the integral identity

$$\begin{aligned}
&\int_0^t\int_G \mathbf{v}_t(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau + \nu\int_0^t\int_G \nabla\mathbf{v}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\,dxd\tau \\
&= \int_0^t\int_G (\mathbf{f}(x,\tau) - \mathbf{V}_\tau(x,\tau))\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau - \nu\int_0^t\int_G \nabla\mathbf{V}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\,dxd\tau,
\end{aligned}$$

for every $\boldsymbol{\eta} \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(G))$, $\forall t \in (0,T]$.

From Theorem 2.9 and estimates (2.45), (2.46), (2.48), (2.49) follows

Theorem 2.10. *Let* $\mathbf{f} \in L^2(0,\infty;L^2(G))$, d, $d_t \in L^2(0,T;L^2(G))$, $\mathbf{a}$, $\mathbf{a}_t \in L^2(0,T;W^{1/2,2}(\partial G))$, $\mathbf{u}_0 \in W^{1,2}(G)$. *Moreover, suppose that* $\operatorname{div}\mathbf{u}_0 = d(x,0)$, $\mathbf{a}(x,0) = \mathbf{u}_0(x)|_{\partial G}$ *and the compatibility condition* (2.40) *holds. Then the problem* (2.38) *admits a unique weak solution* $\mathbf{u} = \mathbf{V} + \mathbf{v}$ *such that*

$$\begin{aligned}
&\max_{t\in[0,\infty)} \|\mathbf{u}(\cdot,t)\|^2_{W^{1,2}(G)} + \|\mathbf{u}\|^2_{L^2(0,\infty;W^{1,2}(G))} + \|\mathbf{u}_t\|^2_{L^2(0,\infty;L^2(G))} \\
&\le c\Big(\|\mathbf{f}\|^2_{L^2(0,\infty;L^2(G))} + \|d\|^2_{L^2(0,\infty;L^2(G))} + \|d_t\|^2_{L^2(0,\infty;L^2(G))} \\
&+ \|\mathbf{a}\|^2_{L^2(0,T;W^{1/2,2}(\partial G))} + \|\mathbf{a}_t\|^2_{L^2(0,T;W^{1/2,2}(\partial G))} + \|\mathbf{u}_0\|^2_{W^{1,2}(G)}\Big).
\end{aligned} \tag{2.50}$$

Moreover, there exists a number $\nu_0 > 0$ *such that if* $\mathbf{f} \in L^2_\mu(0,\infty;L^2(G))$, $\mu \in (0,\nu_0)$, *then*

$$\begin{aligned}
&\max_{t\in[0,\infty)}\left[\exp(\mu t)\left(\int_G |\mathbf{u}(x,t)|^2\,dx + \nu\int_G |\nabla\mathbf{u}(x,t)|^2\,dx\right)\right] \\
&\qquad + \int_0^\infty \exp(\mu t)\int_G |\mathbf{u}_t(x,t)|^2\;dx\,dt \\
&\le c\Big(\|\mathbf{f}\|^2_{L^2_\mu(0,\infty;L^2(G))} + \|d\|^2_{L^2_\mu(0,\infty;L^2(G))} + \|d_t\|^2_{L^2_\mu(0,\infty;L^2(G))} \\
&+ \|\mathbf{a}\|^2_{L^2_\mu(0,T;W^{1/2,2}(\partial G))} + \|\mathbf{a}_t\|^2_{L^2_\mu(0,T;W^{1/2,2}(\partial G))} + \|\mathbf{u}_0\|^2_{W^{1,2}(G)}\Big).
\end{aligned} \tag{2.51}$$

Chapter 3

The steady Stokes problem

Let us consider the boundary value problem for the steady Stokes system in the domain Ω (see Fig. 1.1 or Fig. 2.1)

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}+\nabla p &=& \mathbf{f}, & x\in\Omega,\\ \operatorname{div}\mathbf{u} &=& 0, & x\in\Omega,\\ \mathbf{u} &=& \mathbf{a}, & x\in\partial\Omega. \end{array}\right. \tag{3.1}$$

Here $\mathbf{u}$ stands for the velocity field, p stands for the pressure, $\nu > 0$ is the constant kinematic viscosity. We assume that $\mathbf{f} \in L^2(\Omega)$, $\operatorname{supp}\mathbf{a} \subset \partial\Omega \cap \partial\Omega_0$, $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and that

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = -F \neq 0.$$

Thus we have to look for a solution $(\mathbf{u}, p)$ of problem (3.1) having a sink or a source of the constant intensity F in the cusp point. Such solution will satisfy the following flux condition

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS = F. \tag{3.2}$$

Here $\mathbf{n}$ is the unit normal to $\sigma(h)$.

The results of this chapter are presented in [6].

3.1 Formal asymptotic decomposition

In this section we discuss the construction near a cusp point of the asymptotics of solutions to the stationary Stokes problem.

3.1.1 The leading-order term of the asymptotic decomposition

Consider first the homogeneous problem (3.1) with zero boundary condition in the domain Ω_H. Remind that $\mathbf{u}|_{\partial\Omega_H\cap\partial\Omega} = 0$. We change the variables as follows

$$(x', x_n) \longrightarrow \left(\frac{x'}{x_n^\lambda}, x_n\right) := (y', y_n)$$

and rewrite the problem (3.1) in the following form

$$\left\{\begin{array}{rcll} -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p &=& 0, & y \in \Pi, \\ -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p &=& 0, & y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}u_n &=& 0, & y \in \Pi, \\ \mathbf{u} &=& 0, & y \in \partial\Pi, \end{array}\right. \tag{3.3}$$

where $\Pi = \{y \in \mathbb{R}^n : |y'| < \gamma_0,\, y_n \in (0, H)\}$ and by $\partial\Pi$ we understand the side of the cylinder Π ommiting the top and bottom bases,

$$\mathbf{u}' = (u_1, ..., u_{n-1}), \quad \nabla' = (\partial_1, ..., \partial_{n-1}), \quad \partial_k = \frac{\partial}{\partial y_k}, \quad k = 1, ..., n,$$

$$\mathrm{div}'\,\mathbf{u}' = \nabla' \cdot \mathbf{u}', \quad \Delta' = \nabla' \cdot \nabla', \quad \mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla'.$$

We look for the approximate solution of (3.3) in the form

$$\begin{aligned} &\mathbf{U}_0(y', y_n) = \left(\mathbf{U}_0'(y', y_n), U_{n,0}(y', y_n)\right), \\ &P_0(y', y_n) = q_0(y_n) + Q_0(y', y_n) \end{aligned} \tag{3.4}$$

with

$$U_{n,0}(y', y_n) = y_n^{2\lambda}\partial_n q_0(y_n)\varphi(y'). \tag{3.5}$$

Substituting the approximate solution (3.4) into the equations (3.3) and collecting the "most singular" terms as $y_n \to 0$ we get

$$\left\{\begin{array}{rcll} -\nu y_n^{-2\lambda}\Delta'\mathbf{U}_0' + y_n^{-\lambda}\nabla' Q_0 &=& 0, & y' \in \omega, \\ \mathrm{div}'\mathbf{U}_0' &=& G_0(y', y_n), & y' \in \omega, \\ \mathbf{U}_0' &=& 0, & y' \in \partial\omega, \end{array}\right. \tag{3.6}$$

and

$$\left\{\begin{array}{rcll} -\nu\Delta'\varphi &=& 1, & y' \in \omega, \\ \varphi &=& 0, & y' \in \partial\omega, \end{array}\right. \tag{3.7}$$

where

$$\omega = \left\{y' \in \mathbb{R}^{n-1} : |y'| < \gamma_0\right\},$$

$$G_0(y', y_n) = -y_n^{\lambda}\mathfrak{D}(y_n^{2\lambda}\partial_n q_0(y_n)\varphi(y')).$$

Poisson problem (3.7) has an exact solution which is described in Chapter 2 (see Subsection *Poisson problem*, Eq. (2.5)).

A solution $(\mathbf{U}'_0, y_n^{\lambda}Q_0)$ exists if and only if the compatibility condition

$$\int_{\omega} G_0\, dy' = 0$$

is satisfied (see [18] or Chapter 2). Using relations (2.7) and (2.8) we can rewrite the compatibility condition in the following form

$$-y_n^{\lambda}\partial_n\left[y_n^{2\lambda}\partial_n q_0(y_n)\right] + \lambda(n-1)y_n^{3\lambda-1}\partial_n q_0(y_n) = 0.$$

Thus function q_0 satisfies the second order ODE which is equivalent to

$$\partial_n\left[y_n^{\lambda(n+1)}\partial_n q_0(y_n)\right] = 0.$$

Thus,

$$q_0(y_n) = C_1 y_n^{1-\lambda(n+1)} + C_2. \tag{3.8}$$

Since the pressure is defined up to an additive constant and in the expression of $U_{n,0}$ only the derivative $\partial_n q_0(y_n)$ appears (see (3.4), (3.5)), we can set $C_2 = 0$.

From (3.5), taking into account (3.8), we get

$$U_{n,0}(y', y_n) = C_1[1-\lambda(n+1)]y_n^{-\lambda(n-1)}\varphi(y') \tag{3.9}$$

and the function G_0 takes the form

$$G_0(y', y_n) = \lambda C_1(1-\lambda(n+1))y_n^{-\lambda(n-2)-1}A(y', \nabla')\varphi(y'), \tag{3.10}$$

where the operator A is given by

$$A(y', \nabla') = n - 1 + y' \cdot \nabla'. \tag{3.11}$$

Comparing the power exponents of y_n in (3.6), (3.10) we conclude that functions $\mathbf{U}'_0(y', y_n)$, $Q_0(y', y_n)$ can be taken in the form

$$\begin{aligned} \mathbf{U}'_0(y', y_n) &= y_n^{-\lambda(n-2)-1}\boldsymbol{\mathcal{U}}'_0(y'), \\ Q_0(y', y_n) &= y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y'), \end{aligned} \tag{3.12}$$

and we can rewrite (3.6) as the following Stokes problem

$$\left\{ \begin{array}{rcll} -\nu\Delta'\boldsymbol{\mathcal{U}}'_0 + \nabla'\mathcal{Q}_0 &=& 0, & y' \in \omega, \\ \mathrm{div}'\boldsymbol{\mathcal{U}}'_0 &=& \mathcal{G}_0(y'), & y' \in \omega, \\ \boldsymbol{\mathcal{U}}'_0 &=& 0, & y' \in \partial\omega, \end{array} \right. \tag{3.13}$$

with

$$\mathcal{G}_0(y') = \lambda C_1(1-\lambda(n+1))A(y',\nabla')\varphi(y').$$

Moreover, after simple computations we get

$$\int_\omega U_{n,0}(y',y_n)\,dy' = \kappa_0 C_1(1-\lambda(n+1))y_n^{-\lambda(n-1)}$$

or, coming back to variables x,

$$\int_{\sigma(t)} \mathbf{u}_0\cdot\mathbf{n}\,dx' = \kappa_0 C_1(1-\lambda(n+1)),$$

where κ_0 is constant described by (2.7). Thus, taking

$$C_1 = F\left[\kappa_0(1-\lambda(n+1))\right]^{-1}, \tag{3.14}$$

we satisfy the flux condition (3.2).

Note, that the necessary solvability condition for the problem (3.13)

$$\int_\omega \mathcal{G}_0(y')\,dy' = 0$$

is satisfied due to the construction. Using Theorem 2.5 we can formulate the following lemma concerning the solvability of the problem (3.13).

Lemma 3.1. *The problem* (3.13) *admits a unique weak solution* $\mathfrak{U}_0' \in \overset{\circ}{W}{}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathfrak{U}_0'\|^2_{W^{1,2}(\omega)} \le c\|\mathcal{G}_0\|^2_{L^2(\omega)}. \tag{3.15}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_0 \in L^2(\omega)$ *such that* $\int\limits_\omega \mathcal{P}_0(y')\,dy' = 0$ *and the following estimate holds*

$$\|\mathcal{P}_0\|^2_{L^2(\omega)} \le c\|\mathcal{G}_0\|^2_{L^2(\omega)}.$$

Since $\mathcal{G}_0 \in C^\infty(\omega)$, *the solution* $\mathfrak{U}_0'$, $\mathcal{P}_0 \in C^\infty(\omega)$, *i.e. is infinitely differentiable up to the boundary.*

Finally, the main approximation term is represented as follows

$$\begin{aligned}
\mathbf{U}_0'(y',y_n) &= y_n^{-\lambda(n-2)-1}\mathfrak{U}_0'(y'),\\
U_{n,0}(y',y_n) &= \frac{F}{\kappa_0}y_n^{-\lambda(n-1)}\varphi(y'),\\
P_0(y',y_n) &= \frac{F}{\kappa_0(1-\lambda(n+1))}y_n^{-\lambda(n+1)+1} + y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y').
\end{aligned} \tag{3.16}$$

Discrepancies

The discrepancies $\mathbf{H}_0'(y', y_n)$, $H_{n,0}(y', y_n)$ left by functions $\mathbf{U}_0$, P_0 in the equations $(3.3)_1$, $(3.3)_2$ can be written in the form

$$\begin{aligned}
\mathbf{H}_0'(y', y_n) = &\ \nu \mathfrak{D}^2 \mathbf{U}_0'(y', y_n) = \nu \mathfrak{D}^2 \left(y_n^{-\lambda(n-1)-1} \boldsymbol{\mathfrak{U}}_0'(y') \right) \\
= &\ y_n^{-\lambda(n-2)-3} \boldsymbol{\mathfrak{F}}_0'(y'), \\
H_{n,0}(y', y_n) = &\ \nu \mathfrak{D}^2 U_{n,0}(y', y_n) - \mathfrak{D} \mathfrak{Q}_0(y', y_n) \\
= &\ \nu \mathfrak{D}^2 \left(y_n^{-\lambda(n-1)} \mathfrak{U}_{n,0}(y') \right) - \mathfrak{D} \left(y_n^{-\lambda(n-1)-1} \mathfrak{Q}_0(y') \right) \\
= &\ y_n^{-\lambda(n-1)-2} \mathfrak{F}_{n,0}(y').
\end{aligned} \tag{3.17}$$

Estimates of the leading-order term

Note, that in a bounded domain ω with the smooth boundary $\partial\omega$ and smooth data solutions of the Stokes problem (3.13) are infinitely differentiable up to the boundary and obey the estimates

$$|\partial_i^j \boldsymbol{\mathfrak{U}}_0'(y')| + |\partial_i^j \mathfrak{Q}_0(y')| \leq C_j |F|, \quad i = 1, ..., n-1, \ j = 0, 1, ... \tag{3.18}$$

(see (3.13) and (3.14)). Therefore, for $\boldsymbol{U}_0$, $\mathfrak{Q}_0$ we obtain

$$|\partial_i^j \partial_n^l \boldsymbol{\mathfrak{U}}_0'(y', y_n)| \leq C|F| y_n^{-(n-2)\lambda-1-l}, \ j, l = 0, 1, ..., \tag{3.19}$$

$$|\partial_i^j \partial_n^l \mathfrak{U}_{n,0}(y', y_n)| \leq C|F| y_n^{-(n-1)\lambda-l}, \ j, l = 0, 1, ..., \tag{3.20}$$

$$|\partial_i^j \partial_n^l \mathfrak{Q}_0(y', y_n)| \leq C|F| y_n^{-(n+1)\lambda+1-l}, \ j, l = 0, 1, ... \tag{3.21}$$

Let us come back to the variables x and define

$$\mathbf{u}^0(x) = \mathbf{U}_0(x'/x_n^\lambda, x_n), \quad p^0(x) = P_0(x'/x_n^\lambda, x_n),$$

where

$$P_0(x'/x_n^\lambda, x_n) = \frac{F}{\kappa_0(1 - \lambda(n+1))} x_n^{-(n+1)\lambda+1} + x_n^{-(n-1)\lambda-1} \mathfrak{Q}_0(x'/x_n^\lambda).$$

By construction

$$\text{div } \mathbf{u}^0(x) = 0 \ \text{ in } \ \Omega_H, \quad \mathbf{u}^0(x) = 0 \ \text{ on } \ \partial\Omega_H \cap \partial\Omega,$$

and

$$\int\limits_{\sigma(h)} \mathbf{u}^0 \cdot \mathbf{n} \, dx' = F.$$

Functions $\mathbf{u}^0$, p^0 satisfy the Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u}^0 + \nabla p^0 &= \mathbf{H}^0, \quad x \in \Omega_H, \\ \operatorname{div}\mathbf{u}^0 &= 0, \quad x \in \Omega_H, \\ \mathbf{u}^0 &= 0, \quad x \in \partial\Omega_H \cap \partial\Omega. \end{cases} \tag{3.22}$$

where the right-hand side $\mathbf{H}^0(x) = \mathbf{H}_0(x'/x_n^\lambda, x_n)$ is defined by formula (3.17). Moreover, the following estimates hold

$$\begin{aligned} |D_x^\alpha \mathbf{H}^{0\prime}(x)| &\leq C|F|x_n^{-(n-2)\lambda-1-2-(\alpha_1+...+\alpha_{n-1})\lambda-\alpha_n} \\ &= C|F|x_n^{-(n-2+|\alpha|)\lambda-3+\alpha_n(\lambda-1)}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} |D_x^\alpha H_n^0(x)| &\leq C|F|x_n^{-(n-1)\lambda-2-(\alpha_1+...+\alpha_{n-1})\lambda-\alpha_n} \\ &= C|F|x_n^{-(n-1+|\alpha|)\lambda-2+\alpha_n(\lambda-1)}, \end{aligned} \tag{3.24}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$.

Finally, by construction there hold the following estimates for the functions $\mathbf{u}^0$, p^0:

$$|D_x^\alpha \mathbf{u}^{0\prime}(x)| \leq C|F|x_n^{-(n-2)\lambda-1-(\alpha_1+...+\alpha_{n-1})\lambda-\alpha_n}, \quad |\alpha| \geq 0, \tag{3.25}$$

$$|D_x^\alpha u_n^0(x)| \leq C|F|x_n^{-(n-1)\lambda-(\alpha_1+...+\alpha_{n-1})\lambda-\alpha_n}, \quad |\alpha| \geq 0, \tag{3.26}$$

$$|D_x^\alpha p^0(x)| \leq C|F|x_n^{-(n+1)\lambda+1-(\alpha_1+...+\alpha_{n-1})\lambda-\alpha_n}, \quad |\alpha| \geq 0, \tag{3.27}$$

3.1.2 Higher-order terms of the asymptotic decomposition

In this section we will describe the formal procedure of constructing the complete asymptotic series. In order to do this, first of all, we compensate discrepancy terms in (3.17) and show that each new discrepancy appearing after compensation has the similar form and is "smaller" than that compensated. In fact our final goal is to construct the sufficient number of asymptotical terms which guarantees that the discrepancy is in the $L^2(\Omega)$ space.

System (3.3) with the right-hand sides having special form

Lets consider equations (3.3) with the right-hand sides having the special form

$$\begin{cases} -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p &= y_n^{-\lambda(n-2)-3}\boldsymbol{\mathfrak{F}}_0'(y'), \quad y \in \Pi, \\ -\nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p &= y_n^{-\lambda(n-1)-2}\mathfrak{F}_{n,0}(y'), \quad y \in \Pi, \\ y_n^{-\lambda}\operatorname{div}'\mathbf{u}' + \mathfrak{D}u_n &= 0, \quad y \in \Pi, \\ \mathbf{u} &= 0, \quad y \in \partial\Pi, \end{cases} \tag{3.28}$$

where functions $\boldsymbol{\mathcal{F}}'_0$, $\mathcal{F}_{n,0}$ are described in (3.17). We look for the solution $(\mathbf{U}_1, P_1)$ of (3.28) in the form

$$\begin{aligned}
\mathbf{U}'_1(y', y_n) &= y_n^{-\lambda(n-4)-3}\boldsymbol{\mathcal{U}}'_1(y'),\\
U_{n,1}(y', y_n) &= y_n^{-\lambda(n-3)-2}\mathcal{U}_{n,1}(y'),\\
P_1(y', y_n) &= C_2 y_n^{-\lambda(n-1)-1} + y_n^{-\lambda(n-3)-3}\mathcal{Q}_1(y'),
\end{aligned} \tag{3.29}$$

with

$$\mathcal{U}_{n,1}(y') = C_2(-\lambda(n-1)-1)\varphi(y') + \mathcal{U}^*_{n,1}(y'), \tag{3.30}$$

where φ is the solution to (3.7), the function $\mathcal{U}^*_{n,1}$ satisfies the equations

$$\left\{\begin{array}{rcll} -\nu\Delta'\mathcal{U}^*_{n,1} &=& \mathcal{F}_{n,0}, & y' \in \omega,\\ \mathcal{U}^*_{n,1} &=& 0, & y' \in \partial\omega, \end{array}\right. \tag{3.31}$$

and $(\boldsymbol{\mathcal{U}}'_1, \mathcal{Q}_1)$ is the solution to

$$\left\{\begin{array}{rcll} -\nu\Delta'\boldsymbol{\mathcal{U}}'_1 + \nabla'\mathcal{Q}_1 &=& \boldsymbol{\mathcal{F}}'_0, & y' \in \omega,\\ \mathrm{div}'\boldsymbol{\mathcal{U}}'_1 &=& [\lambda A(y', \nabla') - 2(\lambda-1)]\mathcal{U}_{n,1}, & y' \in \omega,\\ \boldsymbol{\mathcal{U}}'_1 &=& 0, & y' \in \partial\omega. \end{array}\right. \tag{3.32}$$

The constant C_2 has to be determined from the solvability condition for the problem (3.32)

$$\int\limits_\omega \big[\lambda A(y', \nabla') - 2(\lambda-1)\big]\,\mathcal{U}_{n,1}(y')\,dy' = 0.$$

Indeed, we can rewrite the solvability condition as follows (see (3.30))

$$\kappa_0 C_2 2[\lambda(n-2)-1][-\lambda(n-1)-1] = -\int\limits_\omega [\lambda A(y', \nabla') - 2(\lambda-1)]\mathcal{U}^*_{n,1}(y')\,dy'.$$

Since neither $\lambda(n-2)-1 \neq 0$, nor $-\lambda(n-1)-1 \neq 0$ (remind that $\lambda > 1$), we can uniquely find constant C_2 from the previous equation.

Compensation of the discrepancies

The discrepancies $\mathbf{H}'_1(y', y_n)$, $H_{n,1}(y', y_n)$ left by functions $\mathbf{U}_1$, P_1 in the equations $(3.28)_1$, $(3.28)_2$ can be written in the form

$$\begin{aligned}
\mathbf{H}'_1(y', y_n) &= \nu\mathfrak{D}^2\mathbf{U}'_1(y', y_n) = \nu\mathfrak{D}^2\left(y_n^{-\lambda(n-4)-3}\boldsymbol{\mathcal{U}}'_1(y')\right)\\
&= y_n^{-\lambda(n-4)-5}\boldsymbol{\mathcal{F}}'_1(y'),\\
H_{n,1}(y', y_n) &= \nu\mathfrak{D}^2 U_{n,1}(y', y_n) - \mathfrak{D}\mathcal{Q}_1(y', y_n)\\
&= \nu\mathfrak{D}^2\left(y_n^{-\lambda(n-3)-2}\mathcal{U}_{n,1}(y')\right) - \mathfrak{D}\left(y_n^{-\lambda(n-3)-3}\mathcal{Q}_1(y')\right)\\
&= y_n^{-\lambda(n-3)-4}\mathcal{F}_{n,1}(y').
\end{aligned} \tag{3.33}$$

Next we want to compensate the discrepancies $\mathbf{H}'_1$, $H_{n,1}$. It is easy to see that the discrepancies $\mathbf{H}_1$ has the similar form as $\mathbf{H}_0$ except for the decay exponent μ, which has changed by the following rule

$$\mu \xrightarrow{\mathbf{H}_0 \to \mathbf{H}_1} \mu + 2(\lambda - 1), \tag{3.34}$$

i.e.

$$\left(y_n^{\mu+\lambda-1} \boldsymbol{\mathcal{F}}'_0, y_n^{\mu} \mathcal{F}_{n,0} \right) \xrightarrow{\mathbf{H}_0 \to \mathbf{H}_1} \left(y_n^{\mu+3(\lambda-1)} \boldsymbol{\mathcal{F}}'_1, y_n^{\mu+2(\lambda-1)} \mathcal{F}_{n,1} \right).$$

Since $\lambda > 1$, the rule (3.34) provides that after constructing a finite number of terms $\mathbf{U}_k, P_k$, $k = 0, 1, ..., J^* - 1$, we will reach our goal, i.e.

$$(\mathbf{H}'_{J^*-1}, H_{n,J^*-1}) \in L^2(\Omega), \tag{3.35}$$

where number J^* is described in Section 3.2.

Therefore, repeating the above described procedure we obtain the approximate solution $(\mathbf{U}^{[J]}, P^{[J]})$ of the problem (3.3) in the form of series in powers of y_n:

$$\begin{aligned}
\mathbf{U}'^{[J]}(y', y_n) &= \sum_{k=0}^{J} y_n^{-(n-2k-2)\lambda-(1+2k)} \boldsymbol{\mathcal{U}}'_k(y'), \\
U_n^{[J]}(y', y_n) &= \sum_{k=0}^{J} y_n^{-(n-2k-1)\lambda-2k} \mathcal{U}_{n,k}(y'), \\
P^{[J]}(y', y_n) &= \sum_{k=0}^{J} \left(C_{k+1} y_n^{-(n+1-2k)\lambda+1-2k} + y_n^{-(n-2k-1)\lambda-(1+2k)} \mathcal{Q}_k(y') \right),
\end{aligned} \tag{3.36}$$

where

$$\begin{cases}
-\nu \Delta' \boldsymbol{\mathcal{U}}'_k + \nabla' \mathcal{Q}_k = \boldsymbol{\mathcal{F}}'_{k-1}, & y' \in \omega, \\
\mathrm{div}' \boldsymbol{\mathcal{U}}'_k = -[\lambda A(y', \nabla') - 2k(\lambda - 1)] \mathcal{U}_{n,k}, & y' \in \omega, \\
\boldsymbol{\mathcal{U}}'_k = 0, & y' \in \partial\omega,
\end{cases} \tag{3.37}$$

$$\mathcal{U}_{n,k}(y') = C_{k+1} \left[-(n+1-2k)\lambda + 1 - 2k \right] \varphi(y') + \mathcal{U}^*_{n,k}(y'),$$

the function φ is the solution to (3.7), the functions $\mathcal{U}^*_{n,k}$, $k = 1, 2, ...$, satisfy the equations

$$\begin{cases}
-\nu \Delta' \mathcal{U}^*_{n,k} = \mathcal{F}_{n,k-1}, & y' \in \omega, \\
\mathcal{U}^*_{n,k} = 0, & y' \in \partial\omega,
\end{cases} \tag{3.38}$$

with

$$\boldsymbol{F}'_k(y', y_n) = \nu \mathfrak{D}^2 \mathbf{U}'_k(y', y_n) = y_n^{-(n-2(k+1))\lambda-1-2(k+1)} \boldsymbol{\mathcal{F}}'_k(y'),$$

$$F_{n,k}(y', y_n) = \nu \mathfrak{D}^2 U_{n,k}(y', y_n) - \mathcal{Q}_k(y', y_n) = y_n^{-(n-1-2k)\lambda-2(k+1)} \mathcal{F}_{n,k}(y');$$

$$\partial q_0(y_n) = C_1(1-\lambda(n+1))y_n^{-\lambda(n+1)},\ C_1 = F/\left[\kappa_0(1-\lambda(n+1))\right].$$

The constants C_{k+1}, $k = 1, 2, ...$, are determined from the solvability condition for the problem (3.37):

$$\int_\omega [\lambda A(y', \nabla') - 2k(\lambda - 1)]\mathcal{U}_{n,k}(y')\, dy' = 0, \tag{3.39}$$

i.e.

$$C_{k+1} = -\frac{1}{\kappa_0[1-2k-(n+1-2k)\lambda]} \int_\omega \mathcal{U}^*_{n,k}(y')dy'. \tag{3.40}$$

If $1 - 2k - (n+1-2k)\lambda \neq 0$, the constants C_{k+1}, $k = 1, 2, ...$, are uniquely determined from (3.40). However, if $1 - 2k - (n+1-2k)\lambda = 0$, i.e.,

$$k = \frac{\lambda(n+1)-1}{2(\lambda-1)},$$

then C_{k+1} cannot be found from (3.40). Let us denote this particular $k := \bar{k}$ (if such $\bar{k} \in \mathbb{N}$ exists; for example, this is not the case when $\lambda = 2$). In this case we look for $(\mathbf{U}_{\bar{k}}, P_{\bar{k}})$ in the form

$$\begin{aligned} \mathbf{U}_{\bar{k}}(y', y_n) &= \mathbf{U}_k(y', y_n), \\ P_{\bar{k}}(y', y_n) &= C_{\bar{k}+1}\ln\, y_n + y_n^{-(n-(2\bar{k}+1))\lambda-(1+2\bar{k})}\mathcal{Q}_{\bar{k}}(y'). \end{aligned} \tag{3.41}$$

For $\mathcal{U}_{n,\bar{k}}(y') = C_{\bar{k}+1}\varphi(y') + \mathcal{U}^*_{n,\bar{k}}(y')$ and $(\boldsymbol{\mathcal{U}}'_{\bar{k}}(y'), \mathcal{Q}_{\bar{k}}(y'))$ we get the same equations (3.37), (3.38); the solvability condition for the problem (3.37) is changed into

$$C_{\bar{k}+1}2\bar{k}(\lambda-1)\kappa_0 = \int_\omega [\lambda A(y', \nabla') - 2\bar{k}(\lambda-1)]\mathcal{U}^*_{n,\bar{k}}(y')dy'.$$

Using Theorem 2.1 we can formulate the following lemma concerning the solvability of the problem (3.38).

Lemma 3.2. *The problem* (3.38) *admits a unique weak solution* $\mathcal{U}^*_{n,k} \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathcal{U}^*_{n,k}\|_{W^{1,2}(\omega)} \leq c\|\mathcal{F}_{n,k-1}\|_{L^2(\omega)}. \tag{3.42}$$

Moreover, since by construction $\mathcal{F}_{n,k-1}$ *is infinitely smooth, the solution* $\mathcal{U}^*_{n,k}$ *is infinitely smooth up to the boundary.*

Note, that the necessary solvability condition (3.39) of the problem (3.37) holds due to the construction. Using Theorem 2.5 we can formulate the following lemma concerning solvability of the problem (3.37).

Lemma 3.3. *The problem* (3.37) *admits a unique weak solution* $\boldsymbol{\mathfrak{U}}'_k \in \overset{\circ}{W}{}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\boldsymbol{\mathfrak{U}}'_k\|^2_{W^{1,2}(\omega)} \leq c\left(\|\boldsymbol{\mathfrak{F}}'_{k-1}\|^2_{L^2(\omega)} + \|\mathcal{U}^*_{n,k}\|^2_{L^2(\omega)}\right). \tag{3.43}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_k \in L^2(\omega)$ *such that* $\int\limits_\omega \mathcal{P}_k(y')\,dy' = 0$ *and the following estimate holds*

$$\|\mathcal{P}_k\|^2_{L^2(\omega)} \leq c\left(\|\boldsymbol{\mathfrak{F}}'_{k-1}\|^2_{L^2(\omega)} + \|\mathcal{U}^*_{n,k}\|^2_{L^2(\omega)}\right).$$

Since $\boldsymbol{\mathfrak{F}}'_{k-1}$, $\mathcal{U}^*_{n,k} \in C^\infty(\omega)$, *then solution* $\boldsymbol{\mathfrak{U}}'_k$, $\mathcal{P}_k \in C^\infty(\omega)$, *i.e. is infinitely differentiable up to the boundary.*

Finally, we get

$$\begin{aligned}
\mathbf{U}'^{[J]}(y', y_n) &= \sum_{k=0}^{J} y_n^{-(n-2k-2)\lambda-(1+2k)}\boldsymbol{\mathfrak{U}}'_k(y'),\\
U_n^{[J]}(y', y_n) &= \sum_{k=0}^{J} y_n^{-(n-2k-1)\lambda-2k}\mathcal{U}_{n,k}(y'),\\
P^{[J]}(y', y_n) &= \sum_{k=0}^{J}\Big[C_{k+1}y_n^{-(n+1-2k)\lambda+1-2k}[1+\\
+\delta_k^{\bar{k}}(y_n^{(n+1-2k)\lambda-1+2k}&\ln y_n - 1)] + y_n^{-(n-2k-1)\lambda-(1+2k)}\mathcal{Q}_k(y')\Big],
\end{aligned} \tag{3.44}$$

where $J \in \mathbb{N}$.

Discrepancies

The discrepancies $\mathbf{H}'_J(y', y_n)$, $H_{n,J}(y', y_n)$ left after $J+1$ steps are

$$\begin{aligned}
\mathbf{H}'_J(y', y_n) &= y_n^{-(n-2(J+1))\lambda-1-2(J+1)}\boldsymbol{\mathfrak{F}}'_J(y'),\\
H_{n,J}(y', y_n) &= y_n^{-(n-1-2J)\lambda-2(J+1)}\mathcal{F}_{n,J}(y').
\end{aligned} \tag{3.45}$$

Estimates of the higher-order terms

Let us come back to the variables x and define

$$\mathbf{u}^{[J]}(x) = \mathbf{U}^{[J]}(x'/x_n^\lambda, x_n),$$

$$p^{[J]}(x) = P^{[J]}(x'/x_n^\lambda, x_n).$$

By the construction

$$\operatorname{div}\,\mathbf{u}^{[J]}(x) = 0 \ \text{ in } \ \Omega_H, \quad \mathbf{u}^{[J]}(x) = 0 \ \text{ on } \ \partial\Omega_H \cap \partial\Omega,$$

and

$$\int\limits_{\sigma(h)} \mathbf{u}^{[J]} \cdot \mathbf{n}\, dx' = F.$$

Functions $\mathbf{u}^{[J]}$, $p^{[J]}$ satisfy the Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u}^{[J]} + \nabla p^{[J]} &= \mathbf{H}^{J-1}, \quad x \in \Omega_H, \\ \operatorname{div} \mathbf{u}^{[J]} &= 0, \qquad\quad x \in \Omega_H, \\ \mathbf{u}^{[J]} &= 0, \qquad\quad x \in \partial\Omega_H \cap \partial\Omega. \end{cases} \tag{3.46}$$

where the right-hand side $\mathbf{H}^{J-1}(x) = \mathbf{H}'_J(x'/x_n^\lambda, x_n)$ is described by formula (3.45) and the following estimates hold

$$\begin{aligned} |D_x^\alpha \mathbf{H}^{J-1\prime}(x)| &\le C|F| x_n^{-(n-2J)\lambda-1-2J-(\alpha_1+\ldots+\alpha_{n-1})\lambda-\alpha_n} \\ &= C|F| x_n^{-(n-2J+|\alpha|)\lambda-1-2J+\alpha_n(\lambda-1)}, \end{aligned} \tag{3.47}$$

$$\begin{aligned} |D_x^\alpha H_n^{J-1}(x)| &\le C|F| x_n^{-(n+1-2J)\lambda-2J-(\alpha_1+\ldots+\alpha_{n-1})\lambda-\alpha_n} \\ &= C|F| x_n^{-(n+1-2J+|\alpha|)\lambda-2J+\alpha_n(\lambda-1)}, \end{aligned} \tag{3.48}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$.

Finally, there hold the following estimates for the functions $\mathbf{u}^{[J]}$, $p^{[J]}$:

$$|D_x^\alpha \mathbf{u}'^{[J]}(x)| \le C|F| x_n^{-(n-2)\lambda-1-(\alpha_1+\ldots+\alpha_{n-1})\lambda-\alpha_n}, \tag{3.49}$$

$$|D_x^\alpha u_n^{[J]}(x)| \le C|F| x_n^{-(n-1)\lambda-(\alpha_1+\ldots+\alpha_{n-1})\lambda-\alpha_n}, \tag{3.50}$$

$$|D_x^\alpha p^{[J]}(x)| \le C|F| x_n^{-(n+1)\lambda+1-(\alpha_1+\ldots+\alpha_{n-1})\lambda-\alpha_n}, \tag{3.51}$$

where $|\alpha| \ge 0$.

3.2 Existence of the solution

Let $\xi \in C^\infty[0,\infty)$ be a nonnegative cut-off function described by (2.3). We look for the solution $(\mathbf{u}, p)$ of the problem (3.1), (3.2) in the form

$$\mathbf{u}(x', x_n) = \xi(x_n)\mathbf{U}^{[J^*-1]}(x'/x_n^\lambda, x_n) + \mathbf{V}(x', x_n) + \widehat{\mathbf{U}}(x', x_n), \tag{3.52}$$

$$p(x', x_n) = \xi(x_n)P^{[J^*-1]}(x'/x_n^\lambda, x_n) + \widehat{P}(x', x_n), \tag{3.53}$$

where $(\mathbf{U}^{[J^*-1]}, P^{[J^*-1]})$ is given by the formula (3.44). We put $\mathbf{V} = 0$ for $x_n \le H/2$ and for $x_n \ge H$, while for $x_n \in (H/2, H)$ the function $\mathbf{V} \in W^{1,2}(\Omega_{H/2,H})$ is the solution to the problem

$$\begin{cases} \operatorname{div}\mathbf{V} &= -\xi' U_n^{[J^*-1]}, \quad x \in \Omega_{H/2,H}, \\ \mathbf{V} &= \mathbf{a}, \qquad\qquad\quad x \in \partial\Omega_{H/2,H}, \end{cases} \tag{3.54}$$

and we extend boundary value $\mathbf{a}$ by zero on $x_n = H/2$ (and denote by the same letter $\mathbf{a}$).

Finally, $(\widehat{\mathbf{U}}, \widehat{P})$ is the weak solution of the problem

$$\begin{cases} -\nu\Delta\widehat{\mathbf{U}} + \nabla\widehat{P} &= \mathbf{f}^*, \quad x \in \Omega, \\ \operatorname{div}\widehat{\mathbf{U}} &= 0, \quad x \in \Omega, \\ \widehat{\mathbf{U}} &= 0, \quad x \in \partial\Omega, \end{cases} \tag{3.55}$$

where

$$\mathbf{f}^* = \mathbf{f} + \xi\mathbf{H}_{J^*-1} + \nu\Delta\mathbf{V} + \nu(2\nabla\xi\cdot\nabla\mathbf{U}^{[J^*-1]} + \Delta\xi\mathbf{U}^{[J^*-1]}) - \nabla\xi\cdot P^{[J^*-1]},$$

the discrepancy $\mathbf{H}_{J^*-1}(x'/x_n^\lambda, x_n)$ is described by the formula (3.45). Moreover, from (3.45) we compute that the inclusion $\mathbf{H}_{J^*-1} \in L^2(\Omega)$ holds (see (3.35)) if

$$J^* \geq \min\left\{J \in \mathbb{N} :\ J > \frac{1}{4}\left[\frac{n+2}{\lambda-1} + n + 3\right]\right\}.$$

Consider the divergence problem (3.54). Since

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx = -\int\limits_{\sigma(H/2)} U_n^{[J^*-1]}\,dS = -F,$$

the necessary compatibility condition

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx = \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a}\cdot\mathbf{n}\,dS \tag{3.56}$$

is satisfied and the problem (3.54) is solvable. Indeed, there holds the following lemma:

Lemma 3.4. *Let* $\mathbf{a} \in W^{1/2,2}(\partial\Omega_{H/2,H})$, $U^{[J^*-1]} \in L^2(\Omega_{H/2,H})$ *be given functions,* $\xi \in C^\infty[0,\infty)$ *be a nonnegative cut-off function described by* (2.3). *Then there exists at least one solution* $V \in W^{1,2}(\Omega_{H/2,H})$ *of the problem* (3.54) *and the following estimate holds*

$$\|\mathbf{V}\|_{W^{1,2}(\Omega_{H/2,H})} \leq C\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega_{H/2,H})}. \tag{3.57}$$

Proof. Since $\mathbf{a} \in W^{1/2,2}(\partial\Omega_{H/2,H})$ and (3.56) holds, there exists an extension $\mathbf{w} \in W^{1,2}(\Omega_{H/2,H})$ such that

$$\mathbf{w}|_{\partial\Omega_{H/2,H}} = \mathbf{a}, \tag{3.58}$$

and

$$\|\mathbf{w}\|_{W^{1,2}(\Omega_{H/2,H})} \leq C\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega_{H/2,H})}, \tag{3.59}$$

see, e.g., [1]. We look for the solution V of the problem (3.54) in the form $\mathbf{V} = \mathbf{w} + \mathbf{v}$, where the function $\mathbf{v}$ is a solution to

$$\begin{cases} \mathrm{div}\mathbf{v} &= -\xi' U_n^{[J^*-1]} - \mathrm{div}\mathbf{w} := h, \quad x \in \Omega_{H/2,H}, \\ \mathbf{v} &= 0, \qquad\qquad\qquad\qquad\quad x \in \partial\Omega_{H/2,H}. \end{cases} \tag{3.60}$$

From (3.56) and (3.58) it follows that

$$\int\limits_{\Omega_{H/2,H}} h\,dx = -\int\limits_{\Omega_{H/2,H}} \xi' U^{[J^*-1]}\,dx - \int\limits_{\Omega_{H/2,H}} \mathrm{div}\mathbf{w}\,dx$$
$$= -\int\limits_{\Omega_{H/2,H}} \xi' U^{[J^*-1]}\,dx - \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a}\cdot\mathbf{n}\,dS = 0.$$

Therefore, the problem (3.60) has a solution $\mathbf{v}$ satisfying the estimate

$$\|\nabla\mathbf{v}\|_{L^2(\Omega_{H/2,H})} \le C\|h\|_{L^2(\Omega_{H/2,H})}, \tag{3.61}$$

(for details see the corresponding subsection in Chapter 2 or [19]). Finally, collecting estimates (3.50), (3.59), (3.61) we get (3.57) (note that the flux F is the integral of the normal component of the boundary value $\mathbf{a}(x)$ and, therefore, can be estimated by the L^2 norm of the function $\mathbf{a}$).$\square$

Remark 3.1. *Note that* $\mathrm{supp}\,\mathbf{a} \subset \partial\Omega_H \cap \partial\Omega$ *and* $\mathbf{a} = 0$ *on* $x_n = H/2$ *while* $\mathrm{supp}\,h = -\xi' U_n^{[J^*-1]} - div\mathbf{w} \subset \Omega_H \cap \Omega$*. Therefore, the function* $\mathbf{V}$ *can be extended by zero into* $\Omega_{H/2}$*.*

From (3.57), (3.47)-(3.51) and the fact that the flux F can be estimated by the L^2 norm of the function $\mathbf{a}$ we conclude that the following estimate for the right-hand side $\mathbf{f}^*$ of the problem (3.55)

$$\|\mathbf{f}^*\|^2_{L^2(\Omega)} \le C\left(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega_{H/2,H})}\right) \tag{3.62}$$

holds.

Definition 3.1. *A weak solution of the problem* (3.55) *is a solenoidal vector field* $\widehat{\mathbf{U}} \in \mathring{W}^{1,2}(\Omega)$ *satisfying the integral identity*

$$\nu\int\limits_{\Omega} \nabla\widehat{\mathbf{U}}\cdot\nabla\boldsymbol{\eta}\,dx =$$

$$\int\limits_{\Omega} \Big[\mathbf{f}+\xi\mathbf{H}_{J^*-1}+\nu\Delta\mathbf{V}+\nu(2\nabla\xi\cdot\nabla\mathbf{U}^{[J^*-1]}+\Delta\xi\mathbf{U}^{[J^*-1]})-\nabla\xi\cdot P^{[J^*-1]}\Big]\cdot\boldsymbol{\eta}dx$$

for every solenoidal $\boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega)$*.*

Since the right-hand side of the problem (3.55) belongs to $L^2(\Omega)$, solvability of it follows from well known results (the unique solvability of the Stokes problem is proved for arbitrary domains Ω, see, e.g., [18] or the corresponding subsection in Chapter 2). Therefore, the following lemma holds

Lemma 3.5. *Let* $\mathbf{f}^* \in L^2(\Omega)$. *Then the problem* (3.55) *admits a unique weak solution* $\widehat{\mathbf{U}} \in \overset{\circ}{W}{}^{1,2}(\Omega)$ *and there holds the estimate*

$$\|\widehat{\mathbf{U}}\|^2_{W^{1,2}(\Omega)} \le C\|\mathbf{f}^*\|^2_{L^2(\Omega)} \le C\left(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega_{H/2,H})}\right). \tag{3.63}$$

3.2.1 Existence theorem

Let $\mathbf{f} \in L^2(\Omega)$, $\mathbf{a} \in W^{1/2,2}(\partial\Omega \cap \partial\Omega_0)$.

Definition 3.2. *By a weak solution of problem* (3.1) *we understand a solenoidal vector field* $\mathbf{u} \in W^{1,2}_{loc}(\Omega)$ *satisfying the boundary condition* $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$ *and the integral identity*

$$\nu \int\limits_{\Omega} \nabla\mathbf{u} \cdot \nabla\boldsymbol{\eta}\, dx = \int\limits_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta}\, dx, \quad \textit{for every } \boldsymbol{\eta} \in C_0^\infty(\Omega),\ \ \mathrm{div}\boldsymbol{\eta} = 0.$$

Theorem 3.1. *Let* $\mathbf{f} \in L^2(\Omega)$, $\mathbf{a} \in W^{1/2,2}(\partial\Omega_{H/2,H})$ *be given functions,* $\mathrm{supp}\,\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_{H/2,H} \cap \partial\Omega$. *Then the problem* (3.1) *admits at least one weak solution* $\mathbf{u} \in W^{1,2}_{loc}(\Omega) \cap W^{1,2}(\Omega_{H/2,H})$, *which can be represented as a sum* (3.52), *where* $\mathbf{U}^{[J^*-1]}$ *is the constructed in Section* 3.1 *asymptotic expansion* (*see formulas* (3.36), (3.44)), $\mathbf{V}$ *is a solution to the problem* (3.54), *and* $\widehat{\mathbf{U}}$ *is a weak solution to the problem* (3.55). *Moreover, the following estimate*

$$\|\mathbf{u} - \xi\mathbf{U}^{[J^*-1]}\|_{W^{1,2}(\Omega)} \le C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega_{H/2,H})}\right) \tag{3.64}$$

holds.

Proof. The difference $\mathbf{u} - (\xi\mathbf{U}^{[J^*-1]} + \mathbf{V}) = \widehat{\mathbf{U}}$ is a weak solution of the problem (3.55). Therefore estimate (3.64) follows from (3.63), (3.57). □

Chapter 4

The time-periodic Stokes problem

In this chapter we consider the time-periodic Stokes problems in domains Ω (see Fig. 1.1 or Fig. 2.1) having a power cusp (peak) type singular point on the boundary

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, & x \in \Omega, \\ \operatorname{div}\mathbf{u} &= 0, & x \in \Omega, \\ \mathbf{u} &= \mathbf{a}, & x \in \partial\Omega, \\ \mathbf{u}(x,0) &= \mathbf{u}(x,2\pi), & x \in \Omega. \end{cases} \tag{4.1}$$

As before $\mathbf{u}$ stands for the velocity field, p stands for the pressure, $\nu > 0$ is the constant kinematic viscosity.

We assume that the external force $\mathbf{f}(x,t)$ and the boundary value $\mathbf{a}(x,t)$ are time-periodic functions with the period 2π, i.e. $\mathbf{f}(x,0) = \mathbf{f}(x,2\pi)$, $\mathbf{a}(x,0) = \mathbf{a}(x,2\pi)$, and that $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega$. We consider the case where the flux $-F(t)$ of $\mathbf{a}(x,t)$ is nonzero:

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = -F(t). \tag{4.2}$$

Because of (4.2) the solution $\mathbf{u}$ of (4.1) has to satisfy the condition

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS = F(t), \tag{4.3}$$

so that the necessary compatibility condition

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS + \int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = 0$$

holds.

The results of this chapter are presented in [6].

Remark 4.1. Time period 2π is not essential and other period may be chosen.

4.1 Formal asymptotic decomposition

In this section we construct the formal asymptotic decomposition of the solution near the singular point.

4.1.1 The leading-order term of the asymptotic decomposition

Let us consider the solution $\mathbf{u}$ of the problem (4.1) in a neighborhood of the cuspidal point, i.e. in the domain Ω_H. Remind that $\mathbf{u}|_{\partial\Omega_H\cap\partial\Omega} = 0$. Changing the variables

$$(x', x_n, t) \longrightarrow \left(\frac{x'}{x_n^\lambda}, x_n, t\right) := (y', y_n, t)$$

we rewrite the problem (4.1) in the following form:

$$\begin{cases} \mathbf{u}'_t - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p &= 0, & y \in \Pi, \\ u_{n,t} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p &= 0, & y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}u_n &= 0, & y \in \Pi, \\ \mathbf{u} &= 0, & y \in \partial\Pi, \\ \mathbf{u}(y', y_n, 0) &= \mathbf{u}(y', y_n, 2\pi), & y \in \Pi, \end{cases} \tag{4.4}$$

where $\Pi = \{y \in \mathbb{R}^n : |y'| < \gamma_0,\ y_n \in (0, H)\}$. For the reader convenience we remind that

$$\mathbf{u}' = (u_1, ..., u_{n-1}), \quad \nabla' = (\partial_1, ..., \partial_{n-1}), \quad \partial_k = \frac{\partial}{\partial y_k}, \quad k = 1, ..., n,$$

$$\mathrm{div}'\,\mathbf{u}' = \nabla' \cdot \mathbf{u}', \quad \Delta' = \nabla' \cdot \nabla', \quad \mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla'.$$

We look for an approximate solution $(\mathbf{U}_0, P_0)$ of (4.4) in the form

$$\begin{aligned} \mathbf{U}_0(y', y_n, t) &= (\mathbf{U}'_0(y', y_n, t), U_{n,0}(y', y_n, t)) \\ P_0(y', y_n, t) &= q_0(y_n)g_0(t) + Q_0(y', y_n, t), \end{aligned} \tag{4.5}$$

with

$$U_{n,0}(y', y_n, t) = y_n^{2\lambda}\partial_n q_0(y_n)\Phi(y', t). \tag{4.6}$$

Substituting $(\mathbf{U}_0, P_0)$ into the equations (4.4) and collecting the terms with highest singularity at the cusp point O, we get the following systems of equations:

$$\left\{\begin{array}{rcll} -\nu y_n^{-2\lambda}\Delta'\mathbf{U}_0' + y_n^{-\lambda}\nabla' Q_0 & = & 0, & y' \in \omega, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{U}_0' & = & -\mathfrak{D}U_{n,0}, & y' \in \omega, \\ \mathbf{U}_0' & = & 0, & y' \in \partial\omega, \end{array}\right.$$

and

$$\left\{\begin{array}{rcll} -\nu\partial_n q_0(y_n)\Delta'\Phi(y',t) + g_0(t)\partial_n q_0(y_n) & = & 0, & y' \in \omega, \\ \Phi(y',t) & = & 0, & y' \in \partial\omega, \end{array}\right.$$

where $\omega = \{y' \in \mathbb{R}^{n-1} : |y'| < \gamma_0\}$. After the obvious simplifications these problems can be rewritten as

$$\left\{\begin{array}{rcll} -\nu\Delta'\mathbf{U}_0' + \nabla' y_n^{\lambda} Q_0 & = & 0, & y' \in \omega, \\ \mathrm{div}'\mathbf{U}_0' & = & G_0(y', y_n, t), & y' \in \omega, \\ \mathbf{U}_0' & = & 0, & y' \in \partial\omega, \end{array}\right. \tag{4.7}$$

$$\left\{\begin{array}{rcll} \nu\Delta'\Phi(y',t) & = & g_0(t), & y' \in \omega, \\ \Phi(y',t) & = & 0, & y' \in \partial\omega, \end{array}\right. \tag{4.8}$$

where

$$G_0(y', y_n, t) = -y_n^{\lambda}\mathfrak{D}\big(y_n^{2\lambda}\partial_n q_0(y_n)\Phi(y',t)\big).$$

The solution $\Phi(y',t)$ to (4.8) has the form

$$\Phi(y',t) = g_0(t)\varphi(y'), \tag{4.9}$$

where φ is the solution to the Poisson equation

$$\left\{\begin{array}{rcll} \nu\Delta'\varphi(y') & = & 1, & y' \in \omega, \\ \varphi(y') & = & 0, & y' \in \partial\omega, \end{array}\right. \tag{4.10}$$

and is described in Chapter 2 (see Subsection *Poisson problem*).

The solution $(\mathbf{U}_0', y_n^{\lambda} Q_0)$ of the problem (4.7) exists if and only if the following compatibility condition

$$\int\limits_{\omega} G_0(y', y_n, t)\, dy' = 0$$

holds. Using relations (2.7) and (2.8) we can rewrite the compatibility condition as follows

$$-y_n^{\lambda}\partial_n\left[y_n^{2\lambda}\partial_n q_0(y_n)\right]g_0(t)-\lambda(n-1)y_n^{3\lambda-1}\partial_n q_0(y_n)g_0(t)=0.$$

Thus, the function q_0 is a solution to the second order ODE:

$$\partial_n\left[y_n^{\lambda(n+1)}\partial_n q_0(y_n)\right]=0,$$

i.e.,

$$q_0(y_n)=C_1 y_n^{1-\lambda(n+1)}+C_2. \tag{4.11}$$

Since the pressure is defined up to an additive function depending only on t and in the expression of $U_{n,0}$ only the derivative $\partial_n q_0(y_n)$ appears (see (4.5), (4.6)), we can set $C_2=0$. Moreover, with no loss of generality, we can take $C_1=1$ (C_1 can be included into the definition of $g_0(t)$ (see (4.17)). From (4.6), taking into account (4.9) and (4.11), we derive

$$U_{n,0}(y',y_n,t)=[1-\lambda(n+1)]y_n^{-\lambda(n-1)}g_0(t)\varphi(y'). \tag{4.12}$$

Now the function G_0 can be expressed as

$$G_0(y',y_n,t)=\lambda(1-\lambda(n+1))y_n^{-\lambda(n-2)-1}g_0(t)A(y',\nabla')\varphi(y'), \tag{4.13}$$

where the operator A is given by

$$A(y',\nabla')=n-1+y'\cdot\nabla'. \tag{4.14}$$

Comparing the power exponents of y_n in (4.7) and (4.13) we conclude that functions $\mathbf{U}_0'(y',y_n,t)$, $Q_0(y',y_n,t)$ have to be taken in the forms

$$\begin{aligned}\mathbf{U}_0'(y',y_n,t)&=y_n^{-\lambda(n-2)-1}\boldsymbol{\mathcal{U}}_0'(y',t),\\ Q_0(y',y_n,t)&=y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y',t),\end{aligned} \tag{4.15}$$

and we rewrite (4.7) as follows

$$\begin{cases} -\nu\Delta'\boldsymbol{\mathcal{U}}_0'+\nabla'\mathcal{Q}_0 = 0, & y'\in\omega,\\ \mathrm{div}'\boldsymbol{\mathcal{U}}_0' = \mathcal{G}_0(y',t), & y'\in\omega,\\ \boldsymbol{\mathcal{U}}_0' = 0, & y'\in\partial\omega,\end{cases} \tag{4.16}$$

with

$$\mathcal{G}_0(y',t)=\lambda(1-\lambda(n+1))g_0(t)A(y',\nabla')\varphi(y').$$

Moreover, after simple computations we see that

$$\int_{\omega} U_{n,0}(y', y_n, t)\, dy' = \kappa_0(1 - \lambda(n+1))g_0(t)y_n^{-\lambda(n-1)}$$

or, coming back to the variables (x, t),

$$\int_{\sigma(h)} \mathbf{u}_0(x, t) \cdot \mathbf{n}(x)\, dx' = \kappa_0(1 - \lambda(n+1))g_0(t).$$

Thus, taking

$$g_0(t) = F(t) \left[\kappa_0(1 - \lambda(n+1))\right]^{-1}, \tag{4.17}$$

we will satisfy the flux condition (4.3).

Note, that the necessary solvability condition of the problem (4.16)

$$\int_{\omega} \mathcal{G}_0(y', t)\, dy' = 0$$

is satisfied due to the construction. The time variable t is only a parameter in the problem (4.16). Thus, using Theorem 2.5 we can formulate the following lemma concerning the solvability of the problem.

Lemma 4.1. *The problem* (4.16) *admits a unique weak solution* $\mathfrak{U}'_0 \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathfrak{U}'_0\|^2_{W^{1,2}(\omega)} \le c\|\mathcal{G}_0\|^2_{L^2(\omega)}. \tag{4.18}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_0 \in L^2(\omega)$ *such that* $\int_{\omega} \mathcal{P}_0(y')\, dy' = 0$ *and the following estimate holds*

$$\|\mathcal{P}_0\|^2_{L^2(\omega)} \le c\|\mathcal{G}_0\|^2_{L^2(\omega)}.$$

Since by construction the data is smooth, solution $\mathfrak{U}'_0$*,* $\mathcal{P}_0$ *is also smooth, i.e. is infinitely differentiable up to the boundary.*

Remark 4.2. Since t is a parameter in the problem (4.16) we can differentiate the equation with respect to time variable t and get the analogous results for the time derivatives of the solution.

Discrepancies

Functions $(\mathbf{U}_0, Q_0)$ leave in the equations $(4.4)_1$, $(4.4)_2$ the discrepancies $\mathbf{H}'_0(y', y_n, t)$, $H_{n,0}(y', y_n, t)$:

$$
\begin{aligned}
\mathbf{H}'_0(y',y_n,t) = &\ \ \nu\mathfrak{D}^2\mathbf{U}'_0(y',y_n,t)-\mathbf{U}'_{0,t}(y',y_n,t)\\
= &\ \ y_n^{-\lambda(n-2)-3}\widehat{\boldsymbol{\mathfrak{F}}}'_0(y',t)+y_n^{-\lambda(n-2)-1}\widetilde{\boldsymbol{\mathfrak{F}}}'_0(y',t),\\
H_{n,0}(y',y_n,t) = &\ \ \nu\mathfrak{D}^2U_{n,0}(y',y_n,t)-U_{n,0,t}(y',y_n,t)-\\
&\ \ \mathfrak{D}\left(y_n^{-\lambda(n-1)-1}\mathfrak{Q}_0(y',t)\right)\\
= &\ \ y_n^{-\lambda(n-1)-2}\widehat{\mathfrak{F}}_{n,0}(y',t)+y_n^{-\lambda(n-1)}\widetilde{\mathfrak{F}}_{n,0}(y',t).
\end{aligned}
\tag{4.19}
$$

The discrepancies in (4.19) split into two parts: terms with $\widetilde{\mathfrak{F}}$ and terms with $\widehat{\mathfrak{F}}$, where $\widetilde{\mathfrak{F}}$ denotes the discrepancies appearing due to $\mathbf{U}_{0,t}$, and $\widehat{\mathfrak{F}}$ - the rest of the discrepancies. To go further, we have to compensate the most singular terms in the expressions of the discrepancies, that is

$$
(y_n^{-\lambda(n-2)-3}\widehat{\boldsymbol{\mathfrak{F}}}'_0(y',t),\, y_n^{-\lambda(n-1)-2}\widehat{\mathfrak{F}}_{n,0}(y',t))=(\widehat{\mathbf{F}}'_0(y',y_n,t),\widehat{F}_{n,0}(y',y_n,t)).
$$

Estimates of the leading-order term

Since time variable t is only a parameter in the problem (4.16) we can write down similar estimates to its solutions as in the stationary Stokes case (see Section 3.1.1)

$$
|\partial_i^j\mathbf{\mathfrak{U}}'_0(y',t)|+|\partial_i^j\mathfrak{Q}_0(y',t)|\leq C_j|F(t)|,\quad i=1,...,n-1,\ j=0,1,... \tag{4.20}
$$

(see (4.16) and (4.17)). We also get

$$
|\partial_i^j\partial_n^l\mathbf{\mathfrak{U}}'_0(y',y_n,t)|\leq C|F(t)|y_n^{-(n-2)\lambda-1-l},\ j,l=0,1,..., \tag{4.21}
$$

$$
|\partial_i^j\partial_n^l\mathfrak{U}_{n,0}(y',y_n,t)|\leq C|F(t)|y_n^{-(n-1)\lambda-l},\ j,l=0,1,..., \tag{4.22}
$$

$$
|\partial_i^j\partial_n^l\mathfrak{Q}_0(y',y_n,t)|\leq C|F(t)|y_n^{-(n+1)\lambda+1-l},\ j,l=0,1,... \tag{4.23}
$$

Let us come back to the variables x, t and define

$$
\mathbf{u}^0(x,t)=\mathbf{U}_0(x'/x_n^\lambda,x_n,t),\quad p^0(x,t)=P_0(x'/x_n^\lambda,x_n,t).
$$

By construction

$$
\text{div }\mathbf{u}^0(x,t)=0\ \text{ in }\ \Omega_H,\quad \mathbf{u}^0(x,t)=0\ \text{ on }\ \partial\Omega_H\cap\partial\Omega,
$$

$$
\mathbf{u}^0(x,0)=\mathbf{u}^0(x,2\pi)\ \text{ in }\ \Omega_H,\quad \int\limits_{\sigma(h)}\mathbf{u}^0\cdot\mathbf{n}\,dx'=F(t).
$$

Functions $\mathbf{u}^0$, p^0 satisfy the Stokes equations

$$
\begin{cases}
-\nu\Delta\mathbf{u}^0 + \nabla p^0 &= \mathbf{H}^0, \quad x \in \Omega_H, \\
\operatorname{div}\mathbf{u}^0 &= 0, \quad x \in \Omega_H, \\
\mathbf{u}^0 &= 0, \quad x \in \partial\Omega_H \cap \partial\Omega,
\end{cases} \tag{4.24}
$$

where the right-hand side $\mathbf{H}^0(x,t) = \mathbf{H}_0(x'/x_n^\lambda, x_n, t)$ is defined by formula (4.19). Moreover, there hold the following estimates

$$
|D_x^\alpha D_t^\beta \mathbf{H}^{0\prime}(x,t)| \leq C\left(|F_t^\beta(t)| + |F_t^{\beta+1}(t)|\right) \cdot x_n^{-(n-2+|\alpha|)\lambda-3+\alpha_n(\lambda-1)}, \tag{4.25}
$$

$$
|D_x^\alpha D_t^\beta H_n^0(x,t)| \leq C\left(|F_t^\beta(t)| + |F_t^{\beta+1}(t)|\right) \cdot x_n^{-(n-1+|\alpha|)\lambda-2+\alpha_n(\lambda-1)}, \tag{4.26}
$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\beta \in \mathbb{N}$.

By construction there hold the following estimates for the functions $\mathbf{u}^0$, p^0

$$
|D_x^\alpha D_t^\beta \mathbf{u}^{0\prime}(x,t)| \leq C|F_t^\beta(t)| x_n^{-(n-2+|\alpha|)\lambda-1+\alpha_n(\lambda-1)}, \tag{4.27}
$$

$$
|D_x^\alpha D_t^\beta u_n^0(x,t)| \leq C|F_t^\beta(t)| x_n^{-(n-1+|\alpha|)\lambda+\alpha_n(\lambda-1)}, \tag{4.28}
$$

$$
|D_x^\alpha D_t^\beta p^0(x,t)| \leq C|F_t^\beta(t)| x_n^{-(n+1+|\alpha|)\lambda+1+\alpha_n(\lambda-1)}, \tag{4.29}
$$

with $|\alpha| \geq 0$, $\beta \in \mathbb{N}$.

4.1.2 Higher-order terms of the asymptotic decomposition

Our next step is to find functions $(\mathbf{U}_1, P_1)$ which satisfy equations $(4.4)_1$-$(4.4)_4$ with the right-hand sides $\widehat{\mathbf{F}}'_0$, $\widehat{F}_{n,0}$. Functions $(\mathbf{U}_1, P_1)$ leave some new discrepancies $\mathbf{H}'_1$, $H_{n,1}$ in the equations (4.4), etc. We shall keep constructing the functions $(\mathbf{U}_k, P_k)$, $k \in \mathbb{N}$, until the discrepancies belong to the space $L^2(0, 2\pi; L^2(\Omega))$.

Problem (4.4) with the right-hand sides having the special form

Let us consider the problem (4.4) with the right-hand sides having the special form

$$
\begin{cases}
\mathbf{u}'_t - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p = y_n^{-\lambda(n-2)-3}\widehat{\boldsymbol{\mathfrak{F}}}'_0(y',t), & y' \in \Pi, \\
u_{n,t} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}^2)u_n + \mathfrak{D}p = y_n^{-\lambda(n-1)-2}\widehat{\mathfrak{F}}_{n,0}(y',t), & y' \in \Pi, \\
y_n^{-\lambda}\operatorname{div}'\mathbf{u}' + \mathfrak{D}u_n = 0, & y' \in \Pi, \\
\mathbf{u} = 0, & y' \in \partial\Pi, \\
\mathbf{u}(y', y_n, 0) = \mathbf{u}(y', y_n, 2\pi), & y' \in \Pi,
\end{cases} \tag{4.30}
$$

where $\widehat{\boldsymbol{\mathcal{F}}}'_0$, $\widehat{\mathcal{F}}_{n,0}$ are described in (4.19). We look for the approximate solution of (4.30) in the form

$$\begin{aligned} &P_1(y', y_n, t) = g_1(t) y_n^{-1-\lambda(n-1)} + y_n^{-3-\lambda(n-3)} \mathcal{Q}_1(y', t), \\ &\mathbf{U}'_1(y', y_n, t) = y_n^{-3-\lambda(n-4)} \boldsymbol{\mathcal{U}}'_1(y', t), \\ &U_{n,1}(y', y_n, t) = y_n^{-2-\lambda(n-3)} \mathcal{U}_{n,1}(y', t), \end{aligned} \tag{4.31}$$

with

$$\mathcal{U}_{n,1}(y', t) = g_1(t)(-1 - \lambda(n-1))\varphi(y') + \mathcal{U}^*_{n,1}(y', t), \tag{4.32}$$

where the function φ is the solution of the problem (4.10), $\mathcal{U}^*_{n,1}$ is the solution of the boundary value problem

$$\begin{cases} -\nu \Delta' \mathcal{U}^*_{n,1} &= \widehat{\mathcal{F}}_{n,0}, \quad y' \in \omega, \\ \mathcal{U}^*_{n,1} &= 0, \qquad y' \in \partial\omega, \end{cases} \tag{4.33}$$

and $(\boldsymbol{\mathcal{U}}'_1, \mathcal{Q}_1)$ is the solution to

$$\begin{cases} -\nu \Delta' \boldsymbol{\mathcal{U}}'_1 + \nabla' \mathcal{Q}_1 &= \widehat{\boldsymbol{\mathcal{F}}}'_0, & y' \in \omega, \\ \operatorname{div}' \boldsymbol{\mathcal{U}}'_1 &= [\lambda A(y', \nabla') - 2(\lambda - 1)] \mathcal{U}_{n,1}, & y' \in \omega, \\ \boldsymbol{\mathcal{U}}'_1 &= 0, & y' \in \partial\omega. \end{cases} \tag{4.34}$$

The function g_1 is uniquely determined by the solvability condition for the problem (4.34):

$$\int\limits_\omega \big[\lambda A(y', \nabla') - 2(\lambda - 1)\big] \, \mathcal{U}_{n,1}(y', t) \, dy' = 0,$$

i.e.,

$$g_1(t) = \frac{1}{\kappa_0 \left[1 + \lambda(n-1)\right]} \int\limits_\omega \mathcal{U}^*_{n,1}(y', t) \, dy'.$$

Note, that all functions in the problems (4.33), (4.34) depend on time variable t as a parameter. Thus, using Theorems 2.1 and 2.5 we can formulate the following lemmas concerning the solvability of these problems.

Lemma 4.2. *The problem* (4.33) *admits a unique weak solution* $\mathcal{U}^*_{n,1} \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathcal{U}^*_{n,1}\|_{W^{1,2}(\omega)} \le c \|\widehat{\mathcal{F}}_{n,0}\|_{L^2(\omega)}. \tag{4.35}$$

Moreover, since by construction $\widehat{\mathcal{F}}_{n,0}$ *is infinitely smooth, the solution* $\mathcal{U}^*_{n,1}$ *is infinitely smooth up to the boundary.*

Lemma 4.3. *The problem* (4.34) *admits a unique weak solution* $\mathfrak{U}'_1 \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathfrak{U}'_1\|^2_{W^{1,2}(\omega)} \le c\left(\|\widehat{\mathfrak{F}}'_0\|^2_{L^2(\omega)} + \|\mathcal{U}^*_{n,0}\|^2_{L^2(\omega)}\right). \tag{4.36}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_1 \in L^2(\omega)$ *such that* $\int\limits_\omega \mathcal{P}_1(y')\,dy' = 0$ *and*

$$\|\mathcal{P}_1\|^2_{L^2(\omega)} \le c\left(\|\widehat{\mathfrak{F}}'_0\|^2_{L^2(\omega)} + \|\mathcal{U}^*_{n,0}\|^2_{L^2(\omega)}\right).$$

Since by construction all the data is smooth, the solution $\mathfrak{U}'_1$, $\mathcal{P}_1$ *is also smooth, i.e. is infinitely differentiable up to the boundary..*

Remark 4.3. Since t is a parameter in the problems (4.33), (4.34) we can differentiate the equations with respect to time variable t and get the analogous results for the time derivatives of the solution.

Compensation of the discrepancies

The discrepancies $\mathbf{H}'_1$, $H_{n,1}$ can be written in the form

$$\begin{aligned}
\mathbf{H}'_1(y', y_n, t) = {} & \nu\mathfrak{D}^2\mathbf{U}'_1(y', y_n, t) - \mathbf{U}'_{1,t}(y', y_n, t) \\
& + y_n^{-\lambda(n-2)-1}\widetilde{\mathfrak{F}}'_0(y', t) = y_n^{-\lambda(n-4)-5}\widehat{\mathfrak{F}}'_1(y', t) \\
& + y_n^{-\lambda(n-2)-1}\widetilde{\mathfrak{F}}'_0(y', t) + y_n^{-\lambda(n-4)-3}\widetilde{\mathfrak{F}}'_1(y', t), \\
H_{n,1}(y', y_n, t) = {} & \nu\mathfrak{D}^2 U_{n,0}(y', y_n, t) - U_{n,0,t}(y', y_n, t) \\
& - \mathfrak{D}\left(y_n^{-\lambda(n-3)-3}\mathcal{Q}_1(y', t)\right) + y_n^{-\lambda(n-1)}\widetilde{\mathcal{F}}_{n,0}(y', t) \\
= {} & y_n^{-\lambda(n-3)-4}\widehat{\mathcal{F}}_{n,1}(y', t) + y_n^{-\lambda(n-1)}\widetilde{\mathcal{F}}_{n,0}(y', t) \\
& + y_n^{-\lambda(n-3)-2}\widetilde{\mathcal{F}}_{n,1}(y', t).
\end{aligned} \tag{4.37}$$

Since $\lambda > 1$, $H_{n,1}$ does not belong to the space $L^2(0, 2\pi; L^2(\Omega))$ and we have to construct more terms $(\mathbf{U}_k, P_k)$. First we have to compensate the most singular term in $H_{n,1}$. We distinguish three cases: for $\lambda \in (1, 2)$ the most singular term is $y_n^{-\lambda(n-3)-4}\widehat{\mathcal{F}}_{n,1}(y', t)$, for $\lambda > 2$ the most singular term is $y_n^{-\lambda(n-1)}\widetilde{\mathcal{F}}_{n,0}(y', t)$, while for $\lambda = 2$ these terms have the same power exponents of y_n.

To explain heuristically the algorithm of the asymptotics construction, consider the equation $(4.4)_2$. In order to compensate the term $y_n^\mu F_{n,k-1}$, we construct a function $y_n^{\mu+2\lambda}\mathcal{U}_{n,k}$ which produces a new term in the discrepancy ${}_n^{\mu+2(\lambda-1)}F_{n,k}$ in the right-hand side, i.e. after every step the power exponents of y_n (describing the singularity) are changing by the following rule:

$$\mu \longrightarrow \mu + 2(\lambda - 1). \tag{4.38}$$

While constructing $\mathbf{U}_1, P_1$ we have compensated the term $(\widehat{\mathbf{F}}'_0, \widehat{F}_{n,0})$. Now we have to distinguish the above three cases. If $\lambda > 2$, we compensate $(\widetilde{\mathbf{F}}'_0, \widetilde{F}_{n,0})$, and if it's still not enough, we continue with $(\widehat{\mathbf{F}}'_1, \widehat{F}_{n,1})$ and so on, i.e., terms $\hat{F}$ and $\tilde{F}$ alternate as it is shown below

$$\widehat{\mathbf{F}}_0 \Rightarrow \widetilde{\mathbf{F}}_0 \Rightarrow \widehat{\mathbf{F}}_1 \Rightarrow \widetilde{\mathbf{F}}_1 \Rightarrow ... \Rightarrow \widehat{\mathbf{F}}_k \Rightarrow \widetilde{\mathbf{F}}_{k+1} \Rightarrow ...$$

If $\lambda = 2$, we compensate the sum $\widetilde{\mathbf{F}}_0 + \widehat{\mathbf{F}}_1$ (the terms $\widetilde{\mathbf{F}}_0$ and $\widehat{\mathbf{F}}_1$ have the same power exponents of y_n) and continue doing so till we reach the satisfactory discrepancy from $L^2(0, 2\pi; L^2(\Omega))$, i.e.,

$$\widetilde{\mathbf{F}}_0 + \widehat{\mathbf{F}}_1 \Rightarrow \widetilde{\mathbf{F}}_1 + \widehat{\mathbf{F}}_2 \Rightarrow ... \Rightarrow \widetilde{\mathbf{F}}_k + \widehat{\mathbf{F}}_{k+1} \Rightarrow ...$$

If $\lambda < 2$, then up to the number $\hat{I}$ the terms with $(\widehat{\boldsymbol{F}}'_k, \widehat{F}_{n,k})$, $k = 0, 2, ..., \widehat{I}$, are "more singular" than $(\widetilde{\boldsymbol{F}}'_k, \widetilde{F}_{n,k})$. When we reach this number there are two possibilities: either we compensate the sum $\widetilde{\mathbf{F}} + \widehat{\mathbf{F}}$ (these terms have the same power exponents of y_n if $\dfrac{1}{\lambda - 1} \in \mathbb{N}$, i.e. $\lambda = \dfrac{N+1}{N}$, $N = 1, 2, ...$) or terms $\hat{F}$ and $\tilde{F}$ are alternating (if $\lambda \neq \dfrac{N+1}{N}$), i.e.,

$$\widehat{\mathbf{F}}_0 \Rightarrow ... \Rightarrow \widehat{\mathbf{F}}_{\hat{I}} \Rightarrow \begin{cases} \widehat{\mathbf{F}}_{\hat{I}+1} + \widetilde{\mathbf{F}}_0 \Rightarrow ... \Rightarrow \widehat{\mathbf{F}}_{\hat{I}+1+k} + \widetilde{\mathbf{F}}_k \Rightarrow ..., \\ \widetilde{\mathbf{F}}_0 \Rightarrow \widehat{\mathbf{F}}_{\hat{I}+1} \Rightarrow ... \Rightarrow \widetilde{\mathbf{F}}_k \Rightarrow \widehat{\mathbf{F}}_{\hat{I}+1+k} \Rightarrow ... \end{cases}$$

The number $\hat{I}$ is given by

$$\widehat{I} = \min\left\{ I \in \mathbb{N} : I \geq \frac{1}{\lambda - 1} - 1 \right\} = \begin{cases} \frac{1}{\lambda-1} - 1, & \lambda = \frac{N+1}{N}, \\ \left\lfloor \frac{1}{\lambda-1} \right\rfloor, & \lambda \neq \frac{N+1}{N}. \end{cases} \tag{4.39}$$

Case $\lambda = \frac{N+1}{N}$

The right-hand sides which we have to compensate on the step $k+1$ have the form

$$\mathbf{F}'_k(y', y_n, t) = y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)} \mathfrak{F}'_k(y', t),$$
$$F_{n,k}(y', y_n, t) = y_n^{-\lambda(n-1)-2+2k(\lambda-1)} \mathfrak{F}_{n,k}(y', t),$$

where

$$\mathfrak{F}'_k(y', t) = \begin{cases} \widehat{\mathfrak{F}}'_k(y', t), & k \leq \widehat{I}, \\ \widehat{\mathfrak{F}}'_k(y', t) + \widetilde{\mathfrak{F}}'_{k-\widehat{I}-1}(y', t), & k > \widehat{I}, \end{cases}$$

and

$$\mathcal{F}_{n,k} = \begin{cases} \widehat{\mathcal{F}}_{n,k}(y',t), & k \le \widehat{I}, \\ \widehat{\mathcal{F}}_{n,k}(y',t) + \widetilde{\mathcal{F}}_{n,k-\widehat{I}-1}(y',t), & k > \widehat{I}, \end{cases}$$

$k = 0, 1, 2, \ldots$. Therefore, we can look for the approximate solution $(\mathbf{U}^{[J]}, P^{[J]})$ to the nonhomogeneous problem $(4.30)_1$-$(4.30)_4$ in the form of series in powers of y_n:

$$\begin{aligned} &\mathbf{U}'^{[J]}(y,t) = y_n^{-\lambda(n-2)-1}\boldsymbol{\mathcal{U}}_0'(y',t) + \sum_{k=1}^{J} y_n^{-\lambda(n-2-2k)-2k-1}\boldsymbol{\mathcal{U}}_k'(y',t), \\ &U_n^{[J]}(y,t) = \kappa_0^{-1} y_n^{-\lambda(n-1)} F(t)\varphi(y') + \sum_{k=1}^{J} y_n^{-\lambda(n-1-2k)-2k}\mathcal{U}_{n,k}(y',t), \\ &P^{[J]}(y,t) = [\kappa_0(1-\lambda(n+1))]^{-1}F(t) y_n^{1-\lambda(n+1)} + y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y',t) \\ &\quad + \sum_{k=1}^{J}\left[y_n^{-\lambda(n+1-2k)+1-2k} g_k(t) + y_n^{-\lambda(n-1-2k)-1-2k}\mathcal{Q}_k(y',t)\right], \end{aligned} \tag{4.40}$$

where $J \in \mathbb{N}$, $(\boldsymbol{\mathcal{U}}_k', \mathcal{Q}_k)$, $k = 1, 2, \ldots$, are the solutions of the problems

$$\begin{cases} -\nu\Delta'\boldsymbol{\mathcal{U}}_k' + \nabla'\mathcal{Q}_k &= \boldsymbol{\mathcal{F}}_{k-1}', & y' \in \omega, \\ \operatorname{div}'\boldsymbol{\mathcal{U}}_k' &= \left[\lambda A(y',\nabla') - 2k(\lambda-1)\right]\mathcal{U}_{n,k}, & y' \in \omega, \\ \boldsymbol{\mathcal{U}}_k' &= 0, & y' \in \partial\omega, \end{cases} \tag{4.41}$$

$$\mathcal{U}_{n,k}(y',t) = g_k(t)(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathcal{U}_{n,k}^*(y',t),$$

the function φ is the solution to the problem (4.10), $\mathcal{U}_{n,k}^*$ satisfy the equations

$$\begin{cases} -\nu\Delta'\mathcal{U}_{n,k}^* &= \mathcal{F}_{n,k-1}, & y' \in \omega, \\ \mathcal{U}_{n,k}^* &= 0, & y' \in \omega, \end{cases} \tag{4.42}$$

and the functions g_k, $k = 1, 2, \ldots$, are determined from the solvability condition for the problem (4.41):

$$\int_\omega \left[\lambda A(y',\nabla') - 2k(\lambda-1)\right]\mathcal{U}_{n,k}(y',t)\,dy' = 0, \tag{4.43}$$

i.e.,

$$\begin{aligned} &2k g_k(t)\kappa_0(\lambda-1)[-\lambda(n+1-2k)+1-2k] \\ &= \int_\omega \left[\lambda A(y',\nabla') - 2k(\lambda-1)\right]\mathcal{U}_{n,k}^*(y',t)\,dy'. \end{aligned} \tag{4.44}$$

Notice that the functions g_k can be uniquely determined from the equality

(4.44) if there holds the relation

$$-\lambda(n+1-2k)+1-2k \neq 0. \tag{4.45}$$

If (4.45) is not valid, we have

$$k=\frac{\lambda(n+1)-1}{2(\lambda-1)}. \tag{4.46}$$

Since $k \in \mathbb{N}$, it is easy to verify that (4.46) is true only for $N=2j$, $j=1,2,...$ and $n=3$ (then $k=2+\frac{3N}{2}$). Let us denote this particular k by $\bar{k}=2+\frac{3N}{2}$. In this case we look for $(\mathbf{U}_{\bar{k}}, P_{\bar{k}})$ in the form

$$\begin{aligned}
&\mathbf{U}_{\bar{k}}(y', y_n, t)=\mathbf{U}_k(y', y_n, t),\\
&P_{\bar{k}}(y', y_n, t)=g_{\bar{k}}(t)\ln\, y_n+y_n^{\lambda(2\bar{k}-n+1)-1-2\bar{k}}\mathcal{Q}_{\bar{k}}(y', t).
\end{aligned} \tag{4.47}$$

For $\mathcal{U}_{n,\bar{k}}(y', t)=g_{\bar{k}}(t)\varphi(y')+\mathcal{U}^*_{n,\bar{k}}(y', t)$ and $(\boldsymbol{\mathcal{U}}'_{\bar{k}}, \mathcal{Q}_{\bar{k}})$ we get the same equations (4.10), (4.41), (4.42); the solvability condition for the problem (4.41) is changed into

$$-g_{\bar{k}}(t)[\lambda(n+1)-1]\kappa_0=\int\limits_{\omega}\left[\lambda A(y', \nabla')-2\bar{k}(\lambda-1)\right]\mathcal{U}^*_{n,\bar{k}}(y', t)\, dy', \tag{4.48}$$

and we derive

$$g_{\bar{k}}(t)=-\frac{1}{\kappa_0}\int\limits_{\omega}\mathcal{U}^*_{n,\bar{k}}(y', t)\, dy'.$$

Note that all functions in the problems (4.41), (4.42) depend on time variable t as a parameter. Thus, using Theorems 2.1 and 2.5 we can formulate the following lemmas concerning the solvability of these problems.

Lemma 4.4. *The problem* (4.41) *admits a unique weak solution* $\boldsymbol{\mathcal{U}}'_k \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\boldsymbol{\mathcal{U}}'_k\|^2_{W^{1,2}(\omega)} \leq c\left(\|\boldsymbol{\mathcal{F}}'_{k-1}\|^2_{L^2(\omega)}+\|\mathcal{U}^*_{n,k}\|^2_{L^2(\omega)}\right). \tag{4.49}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_k \in L^2(\omega)$ *such that* $\int\limits_{\omega}\mathcal{P}_k(y')\, dy'=0$ *and*

$$\|\mathcal{P}_k\|^2_{L^2(\omega)} \leq c\left(\|\boldsymbol{\mathcal{F}}'_{k-1}\|^2_{L^2(\omega)}+\|\mathcal{U}^*_{n,k}\|^2_{L^2(\omega)}\right).$$

Since by construction all the data is smooth, the solution $\boldsymbol{\mathcal{U}}'_k$, $\mathcal{P}_k$ *is also smooth.*

Note that solvability condition (4.43) or (4.48) to the problem (4.41) is satisfied automatically due to the construction.

Lemma 4.5. *The problem* (4.42) *admits a unique weak solution* $\mathcal{U}^*_{n,k} \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathcal{U}^*_{n,k}\|_{W^{1,2}(\omega)} \leq c\|\mathcal{F}_{n,k-1}\|_{L^2(\omega)}. \tag{4.50}$$

Moreover, since by construction $\mathcal{F}_{n,k-1}$ *is infinitely smooth, the solution* $\mathcal{U}^*_{n,k}$ *is infinitely smooth up to the boundary.*

Remark 4.4. Since t is a parameter in the problems (4.41), (4.42) we can differentiate the equations with respect to time variable t and get the analogous results for the time derivatives of the solution.

Finally, we get

$$\begin{aligned}
&\mathbf{U}'^{[J]}(y',y_n,t) = y_n^{-\lambda(n-2)-1}\boldsymbol{\mathcal{U}}'_0(y',t) + \sum_{k=1}^{J} y_n^{-\lambda(n-2-2k)-2k-1}\boldsymbol{\mathcal{U}}'_k(y',t),\\
&U_n^{[J]}(y',y_n,t) = \frac{F(t)}{\kappa_0} y_n^{-\lambda(n-1)}\varphi(y') + \sum_{k=1}^{J} y_n^{-\lambda(n-1-2k)-2k}\mathcal{U}_{n,k}(y',t),\\
&P^{[J]}(y',y_n,t) = \frac{F(t)}{\kappa_0(1-\lambda(n+1))} y_n^{1-\lambda(n+1)} + y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y',t)\\
&+\sum_{k=1}^{J}\big[g_k(t) y_n^{-\lambda(n+1-2k)+1-2k}\big(1+\delta_{k,\bar k}\times(y_n^{\lambda(n+1-2k)-1+2k}\ln y_n - 1)\big)\\
&\quad + y_n^{-\lambda(n-1-2k)-1-2k}\mathcal{Q}_k(y',t)\big],
\end{aligned} \tag{4.51}$$

where $J \in \mathbb{N}$, $\delta_{k,\bar k}$ is Kronecker's delta.

Discrepancies. Case $\lambda = \frac{N+1}{N}$

The discrepancies $\mathbf{H}'_J(y',y_n,t)$, $H_{n,J}(y',y_n,t)$ that are left after $J+1$ steps can be written in the form

$$\begin{aligned}
\mathbf{H}'_J(y',y_n,t) = {} & y_n^{-\lambda(n-1)-2+(2J+1)(\lambda-1)}\widehat{\boldsymbol{\mathcal{F}}}'_J(y',t)\\
& + \sum_{k=\max\{0,J-\widehat{I}\}}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)}\widetilde{\boldsymbol{\mathcal{F}}}'_k(y',t),\\
H_{n,J}(y',y_n,t) = {} & y_n^{-\lambda(n-1)-2+2J(\lambda-1)}\widehat{\mathcal{F}}_{n,J}(y',t)\\
& + \sum_{k=\max\{0,J-\widehat{I}\}}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)}\widetilde{\mathcal{F}}_{n,k}(y',t).
\end{aligned} \tag{4.52}$$

Since $\lambda > 1$, after final number of steps $J^* > 0$ we arrive at

$$(\mathbf{H}'_{J^*-1}, H_{n,J^*-1}) \in L^2(0,2\pi;L^2(\Omega)). \tag{4.53}$$

From (4.52) one can see that (4.53) holds if and only if $J^* > \hat{I}$, since otherwise there is the term $y_n^{-\lambda(n-1)}\widetilde{\mathfrak{F}}_{n,1}(y',t)$ which is obviously not in $L^2(0,2\pi;L^2(\Omega))$. Remind that in the present case $\hat{I} = \dfrac{1}{\lambda-1} - 1$. Thus comparing the power exponents $-\lambda(n-1)-2+2(J^*-1)(\lambda-1)$ and $-\lambda(n-1)+2(J^*-1-\hat{I})(\lambda-1)$ of the most singular terms in $(4.52)_2$ we conclude that (4.53) is valid if the following relation

$$J^* = \min\left\{J \in \mathbb{N} : J > \frac{1}{4}\left[(n+2)\hat{I} + 2n + 5\right]\right\} \tag{4.54}$$

holds. It is easy to see that the number J^* is greater or equal to 3. In particular, $J^* = 3$, when $\hat{I} = 0$ (see (4.39)). This means that it is enough to construct three terms $(\mathbf{U}_k, P_k)$ in this case (i.e., to construct the approximate solution $(\mathbf{U}^{[2]}, P^{[2]})$).

Case $\lambda \neq \frac{N+1}{N}$

Now the right-hand sides have the form

$$\mathbf{F}'_k(y',y_n,t) = \begin{cases} y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)}\widehat{\boldsymbol{\mathfrak{F}}}'_k(y',t), & k \leq \widehat{I}, \\ y_n^{-\lambda(n-1)+(2j+1)(\lambda-1)}\widetilde{\boldsymbol{\mathfrak{F}}}'_j(y',t), & k = \widehat{I}+2j+1, \\ y_n^{-\lambda(n-1)-2+(2(\widehat{I}+j+1)-1)(\lambda-1)}\widehat{\boldsymbol{\mathfrak{F}}}'_{\widehat{I}+j+1}(y',t), & k = \widehat{I}+2j+2, \end{cases}$$

$$F_{n,k}(y',y_n,t) = \begin{cases} y_n^{-\lambda(n-1)-2+2k(\lambda-1)}\widehat{\mathfrak{F}}_{n,k}(y',t), & k \leq \widehat{I}, \\ y_n^{-\lambda(n-1)+2j(\lambda-1)}\widetilde{\mathfrak{F}}_{n,j}(y',t), & k = \widehat{I}+2j+1, \\ y_n^{-\lambda(n-1)-2+2(\widehat{I}+j+1)(\lambda-1)}\widehat{\mathfrak{F}}_{n,\widehat{I}+j+1}(y',t), & k = \widehat{I}+2j+2, \end{cases}$$

$k = 0,1,2,...$, $j = 0,1,2,....$ We look for the approximate solution $(\mathbf{U}^{[J]}, P^{[J]})$ to the problem $(4.30)_1$-$(4.30)_4$ in the form

$$\begin{aligned} \mathbf{U}'^{[J]}(y',y_n,t) = y_n^{-\lambda(n-2)-1}\boldsymbol{\mathfrak{U}}'_0(y',t) \\ + \sum_{k=1}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)}\boldsymbol{\mathfrak{U}}'_k(y',t) \\ + \sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+2+(2k+1)(\lambda-1)}\widetilde{\boldsymbol{\mathfrak{U}}}'_k(y',t), \end{aligned}$$

$$U_n^{[J]}(y',y_n,t) = \frac{F(t)}{\kappa_0}y_n^{-\lambda(n-1)}\varphi(y')+$$

$$
\begin{aligned}
&+\sum_{k=1}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)+2k(\lambda-1)}\mathcal{U}_{n,k}(y',t)\\
&+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+2+2k(\lambda-1)}\widetilde{\mathcal{U}}_{n,k}(y',t),\\
P^{[J]}(y',y_n,t) = \frac{F(t)}{\kappa_0(1-\lambda(n+1))}&y_n^{1-\lambda(n+1)+y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y',t)}\\
&+\sum_{k=1}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} g_k(t)y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)}\\
&+\sum_{k=1}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)-1+2k(\lambda-1)}\mathcal{Q}_k(y',t)\\
&+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} \widetilde{g}_k(t)y_n^{-\lambda(n-1)+1+2(k-1)(\lambda-1)}\\
&+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+1+2k(\lambda-1)}\widetilde{\mathcal{Q}}_k(y',t),
\end{aligned}
\tag{4.55}
$$

where $J \in \mathbb{N}$, functions $(\widetilde{\boldsymbol{\mathcal{U}}}'_k, \widetilde{\mathcal{Q}}_k)$, $k = 1, 2, ...$, are solutions to the problems

$$
\begin{cases}
-\nu\Delta'\widetilde{\boldsymbol{\mathcal{U}}}'_k + \nabla'\widetilde{\mathcal{Q}}_k = \widetilde{\boldsymbol{\mathcal{F}}}'_{k-1}, & y' \in \omega,\\
\operatorname{div}'\widetilde{\boldsymbol{\mathcal{U}}}'_k = [\lambda A(y',\nabla') + 2 - 2k(\lambda-1)]\,\widetilde{\mathcal{U}}_{n,k}, & y' \in \omega,\\
\widetilde{\boldsymbol{\mathcal{U}}}'_k = 0, & y' \in \partial\omega,
\end{cases}
\tag{4.56}
$$

$$
\widetilde{\mathcal{U}}_{n,k}(y',t) = \widetilde{g}_k(t)(-\lambda(n-1)+1+2k(\lambda-1))\varphi(y') + \widetilde{\mathcal{U}}^*_{n,k}(y',t),
$$

φ solves (4.10), $\widetilde{\mathcal{U}}^*_{n,k}$, $k = 1, 2, ...$, satisfy the equations (4.42) with the right-hand sides $\widetilde{\mathcal{F}}_{n,k-1}$, functions $\widetilde{g}_k$, $k = 1, 2, ...$, are determined from the solvability condition for the problem (4.56) which is equivalent to the equation

$$
\begin{aligned}
&2\widetilde{g}_k(t)\kappa_0[-\lambda(n-1)+1+2k(\lambda-1)][-1+k(\lambda-1)]\\
&\quad= \int\limits_{\omega} \left[\lambda A(y',\nabla') + 2 - 2k(\lambda-1)\right]\widetilde{\mathcal{U}}^*_{n,k}(y',t)\,dy'.
\end{aligned}
\tag{4.57}
$$

For all $k \in \mathbb{N}$ the terms in brackets in the left hand side of (4.57) are nonzero. It means that functions $\widetilde{g}_k$, $k = 1, 2, ...$, are uniquely determined from (4.57).

The functions $\boldsymbol{\mathcal{U}}'_k, \mathcal{Q}_k, \mathcal{U}_{n,k}$, $k = 1, 2, ...$, are solutions to the problems (4.41), (4.42) with the right-hand sides $\widehat{\boldsymbol{\mathcal{F}}}'_{k-1}, \widehat{\mathcal{F}}_{n,k-1}$. If $-\lambda(n+1-2k)+$

$1 - 2k \neq 0$, the functions g_k, $k = 1, 2, ...$, are uniquely determined from the solvability condition (4.44). If $-\lambda(n+1-2k)+1-2k = 0$ then, analogically to Section 4.1.2, we look for $(\mathbf{U}_{\bar{k}}, P_{\bar{k}})$ in the special form (see (4.47)).

Note that all functions in the problem (4.56) depend on time variable t as a parameter. Thus, using Theorem 2.5 we can formulate the following lemma concerning the solvability of this problem.

Lemma 4.6. *The problem* (4.56) *admits a unique weak solution* $\widetilde{\boldsymbol{\mathfrak{U}}}'_k \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\widetilde{\boldsymbol{\mathfrak{U}}}'_k\|^2_{W^{1,2}(\omega)} \leq c\left(\|\widetilde{\boldsymbol{\mathfrak{F}}}'_{k-1}\|^2_{L^2(\omega)} + \|\widetilde{\mathfrak{U}}^*_{n,k}\|^2_{L^2(\omega)}\right). \tag{4.58}$$

Moreover, there exists a corresponding pressure function $\widetilde{\mathfrak{P}}_k \in L^2(\omega)$ *such that* $\int\limits_\omega \widetilde{\mathfrak{P}}_k(y')\,dy' = 0$ *and*

$$\|\widetilde{\mathfrak{P}}_k\|^2_{L^2(\omega)} \leq c\left(\|\widetilde{\boldsymbol{\mathfrak{F}}}'_{k-1}\|^2_{L^2(\omega)} + \|\widetilde{\mathfrak{U}}^*_{n,k}\|^2_{L^2(\omega)}\right).$$

Since by construction all the data is smooth, the solution $\widetilde{\boldsymbol{\mathfrak{U}}}'_k$, $\widetilde{\mathfrak{P}}_k$ *is also smooth.*

Note that solvability condition (4.57) to the problem (4.56) is satisfied automatically due to the construction.

Remark 4.5. Since t is a parameter in the problem (4.56) we can differentiate the equations with respect to time variable t and get the analogous results for the time derivatives of the solution.

Finally, we find the approximate solution $(\mathbf{U}^{[J]}, P^{[J]})$:

$$\mathbf{U}'^{[J]}(y', y_n, t) = y_n^{-\lambda(n-2)-1}\boldsymbol{\mathfrak{U}}'_0(y', t) + \sum_{k=1}^{\min\{J, \lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)}\boldsymbol{\mathfrak{U}}'_k(y', t)$$
$$+ \sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+2+(2k+1)(\lambda-1)}\widetilde{\boldsymbol{\mathfrak{U}}}'_k(y', t),$$

$$U_n^{[J]}(y', y_n, t) = \frac{F(t)}{\kappa_0} y_n^{-\lambda(n-1)}\varphi(y') + \sum_{k=1}^{\min\{J, \lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)+2k(\lambda-1)}\mathfrak{U}_{n,k}(y', t)$$
$$+ \sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+2+2k(\lambda-1)}\widetilde{\mathfrak{U}}_{n,k}(y', t), \tag{4.59}$$

$$P^{[J]}(y', y_n, t) = \frac{F(t)}{\kappa_0(1-\lambda(n+1))} y_n^{1-\lambda(n+1)+y_n^{-\lambda(n-1)-1}\mathcal{Q}_0(y',t)}$$
$$+ \sum_{k=1}^{\min\{J, \lfloor\frac{J+\widehat{I}}{2}\rfloor\}} \Big[g_k(t) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)} \big(1+$$
$$+\delta_k^{\bar{k}}(y_n^{\lambda(n-1)+1-2(k-1)(\lambda-1)} \ln\, y_n - 1)\big) + y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathcal{Q}_k(y', t)\Big]$$
$$+ \sum_{k=1}^{\lfloor\frac{J-\widehat{I}-1}{2}\rfloor} \Big[\widetilde{g}_k(t) y_n^{-\lambda(n-1)+1+2k(\lambda-1)} + y_n^{-\lambda(n-1)+1+2(k+1)(\lambda-1)} \widetilde{\mathcal{Q}}_k(y', t)\Big],$$
where $J \in \mathbb{N}$.

Discrepancies. Case $\lambda \neq \frac{N+1}{N}$

The discrepancies $\mathbf{H}'_J(y', y_n, t)$, $H_{n,J}(y', y_n, t)$ that are left after $J+1$ steps can be written in the form

$$\begin{aligned}
\mathbf{H}'_J(y', y_n, t) &= \sum_{k=\min\{J, \lfloor\frac{J+\widehat{I}}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)} \widehat{\boldsymbol{\mathfrak{F}}}'_k(y', t) \\
&+ \sum_{k=\max\{0, \lfloor\frac{J-\widehat{I}+1}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \widetilde{\boldsymbol{\mathfrak{F}}}'_k(y', t), \\
H_{n,J}(y', y_n, t) &= \sum_{k=\min\{J, \lfloor\frac{J+\widehat{I}}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)-2+2k(\lambda-1)} \widehat{\mathfrak{F}}_{n,k}(y', t) \\
&+ \sum_{k=\max\{0, \lfloor\frac{J-\widehat{I}+1}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \widetilde{\mathfrak{F}}_{n,k}(y', t).
\end{aligned} \tag{4.60}$$

By the same argument as in Subsection 4.1.2 one can prove that

$$J^* = \widetilde{J} + \widehat{J}, \tag{4.61}$$

where
$$\widetilde{J} = \min\left\{ J \in \mathbb{N} :\ J > \frac{1}{4}\left[\frac{n-2}{\lambda-1} + n + 3\right]\right\},$$
$$\widehat{J} = \min\left\{ J \in \mathbb{N} :\ J > \frac{1}{4}\left[\frac{n+2}{\lambda-1} + n + 3\right]\right\}.$$
It is easy to see that the number J^* is greater or equal to 4.

Estimates of the higher-order terms

The estimates of this section are valid for both cases $\lambda = \frac{N+1}{N}$ and $\lambda \neq \frac{N+1}{N}$, i.e. for all $\lambda > 1$.

Let us come back to the variables x, t and define

$$\mathbf{u}^{[J]}(x,t) = \mathbf{U}^{[J]}(x'/x_n^\lambda, x_n, t),$$

$$p^{[J]}(x,t) = P^{[J]}(x'/x_n^\lambda, x_n, t),$$

where $\mathbf{U}^{[J]}$, $P^{[J]}$ are defined by (4.51) or by (4.59) depending on the value of λ. By the construction

$$\operatorname{div} \mathbf{u}^{[J]}(x,t) = 0 \ \text{ in } \ \Omega_H, \quad \mathbf{u}^{[J]}(x,t) = 0 \ \text{ on } \ \partial\Omega_H \cap \partial\Omega,$$

$$\mathbf{u}^{[J]}(x,0) = \mathbf{u}^{[J]}(x,2\pi) \ \text{ in } \ \Omega_H, \quad \int\limits_{\sigma(h)} \mathbf{u}^{[J]} \cdot \mathbf{n}\, dx' = F(t).$$

Functions $\mathbf{u}^{[J]}$, $p^{[J]}$ satisfy the Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u}^{[J]} + \nabla p^{[J]} &= \mathbf{H}^{J-1}, \quad x \in \Omega_H, \\ \operatorname{div}\mathbf{u}^{[J]} &= 0, \qquad\quad x \in \Omega_H, \\ \mathbf{u}^{[J]} &= 0, \qquad\quad x \in \partial\Omega_H \cap \partial\Omega, \end{cases} \tag{4.62}$$

where the right-hand side $\mathbf{H}^{J-1}(x,t) = \mathbf{H}_J(x'/x_n^\lambda, x_n, t)$ is described by formulas (4.52) or (4.60) depending on the value of λ. Then by construction we deduce

$$\begin{aligned} |D_x^\alpha D_t^\beta \mathbf{H}^{J-1\prime}(x,t)| \le C \sum_{k=0}^{J} \left| \frac{\partial^{k+\beta} F(t)}{\partial t^{k+\beta}} \right| \\ \cdot x_n^{-\lambda(n-1+|\alpha|)-2+(2J-1)(\lambda-1)-\alpha_n(\lambda-1)}, \end{aligned} \tag{4.63}$$

$$\begin{aligned} |D_x^\alpha D_t^\beta H_n^{J-1}(x,t)| \le C \sum_{k=0}^{J} \left| \frac{\partial^{k+\beta} F(t)}{\partial t^{k+\beta}} \right| \\ \cdot x_n^{-\lambda(n-1+|\alpha|)-2+2(J-1)(\lambda-1)-\alpha_n(\lambda-1)}, \end{aligned} \tag{4.64}$$

if $\lambda = \frac{N+1}{N}$. On the other hand, if $\lambda \neq \frac{N+1}{N}$, we have

$$\begin{aligned} |D_x^\alpha D_t^\beta \mathbf{H}^{J-1\prime}(x,t)| \le C \sum_{k=0}^{J} \left| \frac{\partial^{k+\beta} F(t)}{\partial t^{k+\beta}} \right| \Big(x_n^{-\lambda(n-1)-2+(2i+1)(\lambda-1)} \\ + x_n^{-\lambda(n-1)+(2j+1)(\lambda-1)} \Big) x_n^{-|\alpha|\lambda+\alpha_n(\lambda-1)}, \end{aligned} \tag{4.65}$$

$$\begin{aligned} |D_x^\alpha D_t^\beta H_n^{J-1}(x,t)| \le C \sum_{k=0}^{J} \left| \frac{\partial^{k+\beta} F(t)}{\partial t^{k+\beta}} \right| \Big(x_n^{-\lambda(n-1)-2+2i(\lambda-1)} \\ + x_n^{-\lambda(n-1)+2j(\lambda-1)} \Big) x_n^{-|\alpha|\lambda+\alpha_n(\lambda-1)}, \end{aligned} \tag{4.66}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\beta \in \mathbb{N}$, $i = \min\{J, \lfloor \frac{J+\widehat{I}}{2} \rfloor\}$ and $j = \max\{0, \lfloor \frac{J-\widehat{I}+1}{2} \rfloor\}$.

Finally, for all $\lambda > 1$, there hold the following estimates for the functions $\mathbf{u}^{[J]}$, $p^{[J]}$

$$|D_x^\alpha D_t^\beta \mathbf{u}'^{[J]}(x,t)| \leq C \sum_{k=0}^{J} \left| \frac{\partial^k F(t)}{\partial t^k} \right| x_n^{-(n-2+|\alpha|)\lambda - 1 + \alpha_n(\lambda-1)}, \tag{4.67}$$

$$|D_x^\alpha D_t^\beta u_n^{[J]}(x,t)| \leq C \sum_{k=0}^{J} \left| \frac{\partial^{k+\beta} F(t)}{\partial t^{k+\beta}} \right| x_n^{-(n-1+|\alpha|)\lambda + \alpha_n(\lambda-1)}, \tag{4.68}$$

$$|D_x^\alpha D_t^\beta p^{[J]}(x,t)| \leq C \sum_{k=0}^{J} \left| \frac{\partial^{k+\beta} F(t)}{\partial t^{k+\beta}} \right| x_n^{-(n+1+|\alpha|)\lambda + 1 + \alpha_n(\lambda-1)}, \tag{4.69}$$

where $|\alpha| \geq 0$, $\beta \in \mathbb{N}$.

4.1.3 Regularity conditions

Constructing the asymptotic expansion of the solution we have supposed that all needed derivatives exist. Now consider the asymptotic expansion $(\mathbf{U}^{[J]}, P^{[J]})$ of order $J \in \mathbb{N}$ (see (4.40), (4.55), (4.59)) and the corresponding discrepancies $\mathbf{H}_J(y', y_n, t)$ (see (4.52), (4.60)). In order to get the asymptotic representations we had to solve the problems (4.33), (4.42) and (4.16), (4.41), if $\lambda = \frac{N+1}{N}$ (plus the problem (4.56) if $\lambda \neq \frac{N+1}{N}$). Analyzing the right-hand sides of these equations we see that the regularity of asymptotic expansions depends on the regularity of the flux $F(t)$ in a recursive way, i.e., one time derivative is "lost" on each step of the construction. Thus, if we want to construct the asymptotic expansion up to the order J, we have to assume that the flux $F(t)$ satisfies the following regularity condition

$$F \in W^{J+1,2}(0, 2\pi).$$

Since the flux $F(t)$ is the integral of the normal component of the boundary value $\mathbf{a}(x,t)$ over $\partial\Omega$, we have to assume the following regularity conditions:

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0, 2\pi; W^{1/2,2}(\partial\Omega)), \ l = 0, 1, 2, ..., J+1. \tag{4.70}$$

4.2 Existence of the solution

Let $\xi \in C^\infty[0, \infty)$ be a nonnegative cut-off function, described in Chapter 2, Eq. (2.3). We look for the solution $(\mathbf{u}, p)$ of the problem (4.1) in the form

$$\mathbf{u}(x', x_n, t) = \xi(x_n)\mathbf{U}^{[J^*-1]}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) + \mathbf{V}(x', x_n, t) + \widehat{\mathbf{U}}(x', x_n, t), \tag{4.71}$$

$$p(x', x_n, t) = \xi(x_n)P^{[J^*-1]}\left(\frac{x'}{x_n^\lambda}, x_n, t\right) + \widehat{P}(x', x_n, t), \tag{4.72}$$

where $(\mathbf{U}^{[J^*-1]}, P^{[J^*-1]})$ is the approximate solution, given by the asymptotic formula (4.51) with J^* defined by (4.54) if $\lambda = \frac{N+1}{N}$ (in the case $\lambda \neq \frac{N+1}{N}$, $(\mathbf{U}^{[J^*-1]}, P^{[J^*-1]})$ and J^* are given respectively by (4.59) and (4.61)), $\mathbf{V}(x', x_n, t) = 0$ for $x_n \leq H/2$, while for $x_n \geq H/2$ the function $\mathbf{V}$ is the solution of the following "divergence problem":

$$\begin{cases} \operatorname{div}\mathbf{V} &= -\xi' U_n^{[J^*-1]}, & x \in \Omega_{H/2,H}, \\ \mathbf{V} &= \mathbf{a}, & x \in \partial\Omega_{H/2,H}, \end{cases} \tag{4.73}$$

where $\Omega_{H/2,H}$ is a Lipschitz domain described in (2.2) and $\mathbf{a}$ is extended on the plane $x_n = H/2$ by zero (and denote by the same letter $\mathbf{a}$) (recall that $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega$ (see Figure 2.1)). Finally, $(\widehat{\mathbf{U}}, \widehat{P})$ is a solution to the problem

$$\begin{cases} \widehat{\mathbf{U}}_t - \nu\Delta\widehat{\mathbf{U}} + \nabla\widehat{P} &= \mathbf{f}^*, & x \in \Omega, \\ \operatorname{div}\widehat{\mathbf{U}} &= 0, & x \in \Omega, \\ \widehat{\mathbf{U}} &= 0, & x \in \partial\Omega, \\ \widehat{\mathbf{U}}(x,0) &= \widehat{\mathbf{U}}(x,2\pi), & x \in \Omega, \end{cases} \tag{4.74}$$

where $\mathbf{f}^* = \mathbf{f} + \xi\mathbf{H}_{J^*-1} - \mathbf{V}_t + \nu\Delta\mathbf{V} + \nu(2\nabla\xi\cdot\nabla\mathbf{U}^{[J^*-1]} + \Delta\xi\mathbf{U}^{[J^*-1]}) - \nabla\xi P^{[J^*-1]}$, $\mathbf{H}_J$ is described by (4.52) (or (4.60), if $\lambda \neq \frac{N+1}{N}$), and $J^* \in \mathbb{N}$ is the number which ensures that $\mathbf{H}_{J^*-1}$ belongs to $L^2(0,2\pi; L^2(\Omega))$ (see (4.54), (4.61)). If the regularity conditions (4.70) with $J = J^* - 1$ are satisfied, all terms in the expression for $\mathbf{f}^* - \mathbf{f}$ are well defined.

Consider the divergence problem (4.73). Since

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx = -\int\limits_{\sigma(H/2)} U_n^{[J^*-1]}\,dS = -F(t),$$

the necessary compatibility condition

$$-\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx = \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a}\cdot\mathbf{n}\,dS \tag{4.75}$$

is satisfied and the problem (4.73) is solvable (e.g. [18]).

Lemma 4.7. *Let* $\mathbf{a}, \mathbf{a}_t \in L^2(0,2\pi; W^{1/2,2}(\partial\Omega_{H/2,H}))$ *and there holds the compatibility condition* (4.75). *Then the problem* (4.73) *has at least one solution* $\mathbf{V} \in W^{1,2}(\Omega_{H/2,H})$ *satisfying the following estimates*

$$\|\mathbf{V}\|_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} \leq C\|\mathbf{a}\|_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}, \tag{4.76}$$

$$\|\mathbf{V}_t\|_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} \leq C\|\mathbf{a}_t\|_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}. \tag{4.77}$$

Proof. Consider the linear extension operator $E : W^{1/2,2}(\partial\Omega_{H/2,H}) \to W^{1,2}(\Omega_{H/2,H})$ given by the formula $E\mathbf{a} = \mathbf{w}$, where $\mathbf{w}|_{\partial\Omega_{H/2,H}} = \mathbf{a}$. The operator E is bounded:

$$\|E\mathbf{a}\|^2_{W^{1,2}(\Omega_{H/2,H})} = \|\mathbf{w}\|^2_{W^{1,2}(\Omega_{H/2,H})} \le C\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega_{H/2,H})}, \tag{4.78}$$

see, e.g. [1].

If $\mathbf{a} = \mathbf{a}(x,t)$ depends on t and $\mathbf{a}_t \in W^{1/2,2}(\partial\Omega_{H/2,H})$, then due to the fact that the operator E is linear, we have $E\mathbf{a}_t = \mathbf{w}_t$ and

$$\|E\mathbf{a}_t\|^2_{W^{1,2}(\Omega_{H/2,H})} \le C\|\mathbf{a}_t\|^2_{W^{1/2,2}(\partial\Omega_{H/2,H})}. \tag{4.79}$$

If $\mathbf{a}$, $\mathbf{a}_t \in L^2(0,2\pi; W^{1/2,2}(\partial\Omega_{H/2,H}))$, then integrating (4.78), (4.79) by t we get

$$\begin{aligned} \|\mathbf{w}\|^2_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} &\le C\|\mathbf{a}\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}, \\ \|\mathbf{w}_t\|^2_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} &\le C\|\mathbf{a}_t\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}. \end{aligned} \tag{4.80}$$

see, e.g., [1]. We look for the solution V of the problem (4.73) in the form $\mathbf{V} = \mathbf{w} + \mathbf{v}$, where the function $\mathbf{v}$ is a solution to

$$\begin{cases} \operatorname{div}\mathbf{v} = -\xi' U_n^{[J^*-1]} - \operatorname{div}\mathbf{w} := h, & x \in \Omega_{H/2,H}, \\ \mathbf{v} = 0, & x \in \partial\Omega_{H/2,H}. \end{cases} \tag{4.81}$$

Note, that the variable t in the equations (4.81) plays the role of a parameter. Therefore, functions $\mathbf{v}$ and h depend on t as a parameter. Thus, we can differentiate with respect to time variable t. Then, from (4.75) it follows that $\int\limits_{\Omega_{H/2,H}} h\,dx = 0$ and $\int\limits_{\Omega_{H/2,H}} h_t\,dx = 0$ for all t. Therefore the problem (4.81) has a solution $\mathbf{v}$ satisfying the estimates

$$\|\nabla\mathbf{v}\|_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} \le C\|h\|_{L^2(0,2\pi;L^2(\Omega_{H/2,H}))},$$

$$\|\nabla\mathbf{v}_t\|_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} \le C\|h_t\|_{L^2(0,2\pi;L^2(\Omega_{H/2,H}))}.$$

(see [19] or the corresponding subsection about divergence problem in Chapter 2). Collecting the obtained estimates we get

$$\begin{aligned} &\|\mathbf{V}\|_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} \le C\big(\|\mathbf{a}\|_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))} \\ &+\|h\|_{L^2(0,2\pi;L^2(\Omega_{H/2,H}))}\big) \le C\|\mathbf{a}\|_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}, \end{aligned} \tag{4.82}$$

$$\begin{aligned} &\|\mathbf{V}_t\|_{L^2(0,2\pi;W^{1,2}(\Omega_{H/2,H}))} \le C\big(\|\mathbf{a}_t\|_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))} \\ &+\|h_t\|_{L^2(0,2\pi;L^2(\Omega_{H/2,H}))}\big) \le C\|\mathbf{a}_t\|_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}. \quad \square \end{aligned} \tag{4.83}$$

Thus, it follows from the estimates (4.76), (4.63)-(4.69) and the fact that the flux F can be estimated by the L^2 norm of the function $\mathbf{a}$ we have proved the following estimate for the right-hand side $\mathbf{f}^*$ of the problem (4.74)

$$\begin{aligned}\|\mathbf{f}^*\|^2_{L^2(0,2\pi;L^2(\Omega))} \leq C\Big(&\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(\Omega))} + \|\mathbf{a}\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))} \\ &+ \sum_{k=1}^{J^*} \Big\|\frac{\partial^k \mathbf{a}}{\partial t^k}\Big\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}\Big).\end{aligned} \tag{4.84}$$

Then solvability of the problem (4.74) follows from Theorem 2.6, i.e. the following lemma holds.

Lemma 4.8. *Let $\mathbf{f}^* \in L^2(0,2\pi;L^2(\Omega))$ be a time-periodic function. Then the problem (4.74) admits a unique time-periodic weak solution $\widehat{\mathbf{U}} \in L^2(0,2\pi;$ $\mathring{W}^{1,2}(\Omega))$ with $\widehat{\mathbf{U}}_t \in L^2(0,2\pi;L^2(\Omega))$ and there holds the estimate*

$$\begin{aligned}\sup_{t\in[0,2\pi]} \|\widehat{\mathbf{U}}(\cdot,t)\|^2_{W^{1,2}(\Omega)} + \|\widehat{\mathbf{U}}\|^2_{L^2(0,2\pi;W^{1,2}(\Omega))} + \|\widehat{\mathbf{U}}_t\|^2_{L^2(0,2\pi;L^2(\Omega))} \\ \leq c\|\mathbf{f}^*\|^2_{L^2(0,2\pi;L^2(\Omega))}.\end{aligned} \tag{4.85}$$

4.2.1 Existence theorem

Definition 4.1. *By a weak solution of problem (4.1) we understand a solenoidal time-periodic vector field $\mathbf{u} \in L^2(0,2\pi;W^{1,2}_{loc}(\Omega) \cap W^{1,2}(\Omega_{H/2,H}))$ with $\mathbf{u}_t \in L^2(0,2\pi;L^2_{loc}(\Omega) \cap L^2(\Omega_{H/2,H}))$ satisfying the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$ and the integral identity*

$$\int_0^{2\pi}\int_\Omega \mathbf{u}_\tau(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau + \nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\,dxd\tau$$

$$= \int_0^{2\pi}\int_\Omega \mathbf{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau,$$

for every time-periodic solenoidal $\boldsymbol{\eta} \in L^2\left(0,2\pi;C_0^\infty(\Omega)\right)$.

From what was proved in Sections 4.1–4.2 follows the main result of the paper concerning the solvability of the problem (4.1).

Theorem 4.1. *Let $\mathbf{f} \in L^2(0,2\pi;L^2(\Omega))$, $\mathbf{a}$, $\mathbf{a}_t \in L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))$ be given time-periodic functions,* $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_{H/2,H} \cap \partial\Omega$ *and there hold the regularity conditions (4.70). Then the problem (4.1) admits at least one time-periodic weak solution $\mathbf{u} \in L^2(0,2\pi;W^{1,2}_{loc}(\Omega)\cap W^{1,2}(\Omega_{H/2,H}))$, $\mathbf{u}_t \in L^2(0,2\pi;L^2_{loc}(\Omega) \cap L^2(\Omega_{H/2,H}))$ which can be represented as the sum*

(4.71), *where* $\mathbf{U}^{[J^*-1]}$ *is the constructed in Section* 4.1.2 *asymptotic expansion* (*see formulas* (4.40), (4.51), (4.55), (4.59)), $\mathbf{V}$ *is a solution to the problem* (4.73), *and* $\widehat{\mathbf{U}}$ *is the weak solution of the problem* (4.74). *Moreover, the following estimate*

$$\begin{aligned}
&\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t)-\xi(\cdot)\mathbf{U}^{[J^*-1]}(\cdot,t)\|^2_{W^{1,2}(\Omega)}\\
&+\|\mathbf{u}-\xi\mathbf{U}^{[J^*-1]}\|^2_{L^2(0,2\pi;W^{1,2}(\Omega))}+\|\mathbf{u}_t-\xi\mathbf{U}_t^{[J^*-1]}\|^2_{L^2(0,2\pi;L^2(\Omega))}\\
&\leq C\Big(\|\mathbf{f}\|^2_{L^2(0,2\pi;L^2(\Omega))}+\|\mathbf{a}\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}\\
&+\sum_{k=1}^{J^*+1}\Big\|\frac{\partial^k\mathbf{a}}{\partial t^k}\Big\|^2_{L^2(0,2\pi;W^{1/2,2}(\partial\Omega_{H/2,H}))}\Big)
\end{aligned}\tag{4.86}$$

holds.

Proof. The difference $\mathbf{u}-(\xi\mathbf{U}^{[J^*-1]}+\mathbf{V})=\widehat{\mathbf{U}}$ is a weak solution of the problem (4.74). Therefore estimate (4.86) follows from (2.33), (4.82), (4.83). $\square$

Chapter 5

The nonstationary Stokes problem

In this section we consider the initial boundary value problem for the Stokes system in domains Ω (see Fig. 1.1 or Fig. 2.1) having a power cusp (peak) type singular point on the boundary

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \qquad x \in \Omega, \\ \operatorname{div}\mathbf{u} &= 0, \qquad x \in \Omega, \\ \mathbf{u} &= \mathbf{a}, \qquad x \in \partial\Omega, \\ \mathbf{u}(x,0) &= \mathbf{b}(x), \quad x \in \Omega. \end{cases} \tag{5.1}$$

We assume that $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega$ and that the flux of $\mathbf{a}$ is nonzero, i.e.,

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS = -F(t). \tag{5.2}$$

Further, we suppose that the initial velocity $\mathbf{b}$ is represented as a sum

$$\mathbf{b}(x) = \zeta(x_n)\mathbf{u}_{sing}(x) + \mathbf{u}_0(x), \tag{5.3}$$

where ζ is a smooth cut-off function with $\zeta(t) = 1$ for $t \leq H/2$ and $\zeta(t) = 0$ for $t \geq H$,

$$\mathbf{u}_{sing}(x) = \left(x_n^{-\lambda(n-2)-1}\boldsymbol{u}'_s\big(x'x_n^{-\lambda}\big),\ x_n^{-\lambda(n-1)}u_{n,s}\big(x'x_n^{-\lambda}\big)\right),$$

and $\boldsymbol{u}'_s, u_{n,s} \in \mathring{W}^{1,2}(\omega)$, $\omega = \{y' \in \mathbb{R}^{n-1} : |y'| < \gamma_0\}$, $\mathbf{u}_0 \in W^{1,2}(\Omega)$, $y' = x'x_n^{-\lambda}$. Moreover, the initial velocity $\mathbf{b}$ and the boundary value $\mathbf{a}$ have to satisfy the necessary compatibility conditions

$$\operatorname{div}\mathbf{b}(x) = 0, \quad \mathbf{b}(x)|_{\partial\Omega} = \mathbf{u}_0(x)|_{\partial\Omega} = \mathbf{a}(x,0). \tag{5.4}$$

The singular part $\mathbf{u}_{sing}$ of the initial condition $\mathbf{b}$ is assumed to be solenoidal and it is "responsible" for the flux in the cusp point O at the time moment $t=0$:

$$\operatorname{div}\mathbf{u}_{sing}=0,\qquad \int\limits_{\sigma(h)}\mathbf{u}_{sing}\cdot\mathbf{n}\,dS=F(0)\quad\forall h\in(0,H/2). \tag{5.5}$$

From (5.2) it also follows that

$$\int\limits_{\partial\Omega}\mathbf{u}_0\cdot\mathbf{n}\,dS=-F(0). \tag{5.6}$$

In coordinates y the flux condition in (5.5) takes the form

$$\int\limits_{\omega}u_{n,s}(y)\,dy=F(0). \tag{5.7}$$

Note that the conditions (5.4) and (5.5) imply $\operatorname{div}\mathbf{u}_0=0$ in $\Omega_{H/2}$, and from the inclusion $\mathbf{u}_0\in W^{1,2}(\Omega)$ it follows that $\int\limits_{\sigma(h)}\mathbf{u}_0\cdot\mathbf{n}\,dS=0\ \forall h\le H/2$. Indeed, obviously,

$$\int\limits_{\sigma(h)}\mathbf{u}_0\cdot\mathbf{n}\,dS-\int\limits_{\sigma(H)}\mathbf{u}_0\cdot\mathbf{n}\,dS=0\quad\forall h\le H/2.$$

The flux of $\mathbf{u}_0$ is constant, say F_0, i.e.

$$\int\limits_{\sigma(h)}\mathbf{u}_0\cdot\mathbf{n}\,dS=F_0\quad\forall h\le H/2.$$

Then

$$F_0^2=\Bigg(\int\limits_{\sigma(h)}\mathbf{u}_0\cdot\mathbf{n}\,dS\Bigg)^2\le\int\limits_{\sigma(h)}\mathbf{u}_0^2\,dS\le cx_n^{2\lambda}\int\limits_{\sigma(h)}|\nabla\mathbf{u}_0|^2\,dS\quad\forall h\le H/2.$$

Hence

$$\int\limits_{\varepsilon}^{h}F_0^2x_n^{-2\lambda}\,dx_n\le c\int\limits_{\varepsilon}^{h}\int\limits_{\sigma(h)}|\nabla\mathbf{u}_0|^2\,dS\,dx_n<\infty\quad\forall h\le H/2.$$

The integral on the left-hand side is diverging as $\varepsilon\to 0$. Therefore, $F_0=0$.

We consider the case where the flux $-F(t)$ of $\mathbf{a}(x,t)$ is nonzero (see (5.2)), and we look for a solution $\mathbf{u}(x,t)$ of problem (5.1) satisfying the additional flux condition:

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS = F(t), \tag{5.8}$$

so that the following necessary condition

$$\int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS + \int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = 0$$

holds for any $h\in(0,H]$.

The results of this chapter were published in [7].

Remark 5.1. If $\mathbf{u}(x,0)=0$ and the functions $\mathbf{f}$ and $\mathbf{a}$ are equal to zero in a neighbourhood of the point $t=0$ all results of Chapter 4 remain valid.

5.1 Formal asymptotic decomposition

In this section we construct a formal asymptotic decomposition of the solution, which is the sum of the outer asymptotic decomposition and the boundary-layer-in-time, needed to satisfy the initial condition. The outer asymptotic decomposition is constructed in the same way as in the case of time-periodic problem (see Chapter 4). Therefore, we mostly omit the derivation of these formulas, but we construct in details the boundary-layer-in-time decomposition. We consider the case of the homogeneous ($\mathbf{f}=0$) equations (5.1).

5.1.1 The leading-order term of the asymptotic decomposition

The leading-order term for outer asymptotic decomposition

Let us remind shortly the construction of the main asymptotic term (for details see Section 4.1.1). Consider the problem (5.1) with homogeneous boundary conditions in the domain Ω_H (remind that $\mathbf{u}|_{\partial\Omega_H\cap\partial\Omega}=0$). We rewrite the problem (5.1) in coordinates $y'=x'x_n^{-\lambda}, y_n=x_n, t=t:$

$$\left\{\begin{array}{rcll} \mathbf{u}'_t-\nu(y_n^{-2\lambda}\Delta'+\mathfrak{D}^2)\mathbf{u}'+y_n^{-\lambda}\nabla' p &=& 0, & y\in\Pi,\\ u_{n,t}-\nu(y_n^{-2\lambda}\Delta'+\mathfrak{D}^2)u_n+\mathfrak{D}p &=& 0, & y\in\Pi,\\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}'+\mathfrak{D}u_n &=& 0, & y\in\Pi,\\ \mathbf{u} &=& 0, & y\in\partial\Pi,\\ \mathbf{u}(y',y_n,0) &=& 0, & y\in\Pi, \end{array}\right. \tag{5.9}$$

where $\Pi = \{y \in \mathbb{R}^n : |y'| < \gamma_0, \, y_n \in (0, H)\}$. For the reader convenience we remind that

$$\mathfrak{D} = \partial_n - \lambda y_n^{-1} y' \cdot \nabla', \quad \mathbf{u}' = (u_1, ..., u_{n-1}), \quad \partial_k = \frac{\partial}{\partial y_k}, \quad k = 1, ..., n,$$

$$\nabla' = (\partial_1, ..., \partial_{n-1}), \quad \operatorname{div}' \mathbf{u}' = \nabla' \cdot \mathbf{u}', \quad \Delta' = \nabla' \cdot \nabla'.$$

It is shown in Chapter 4 that the leading-order asymptotic term has the form

$$\begin{aligned} U_{n,0}(y', y_n, t) &= \frac{F(t)}{\kappa_0} y_n^{-\lambda(n-1)} \varphi(y'), \\ \mathbf{U}_0'(y', y_n, t) &= y_n^{-\lambda(n-2)-1} \mathfrak{U}_0'(y', t), \\ P_0(y', y_n, t) = \frac{F(t)}{\kappa_0(1 - \lambda(n+1))} & y_n^{1-\lambda(n+1)} + y_n^{-\lambda(n-1)-1} \mathfrak{Q}_0(y', t), \end{aligned} \tag{5.10}$$

where the function φ is the solution to the Poisson problem

$$\left\{ \begin{array}{rcll} \nu \Delta' \varphi(y') &=& 1, & y' \in \omega, \\ \varphi(y') &=& 0, & y' \in \partial\omega, \end{array} \right. \tag{5.11}$$

and is described in Chapter 2 (see Subsection *Poisson problem*), ω is a bounded domain: $\omega = \left\{ y' \in \mathbb{R}^{n-1} : |y'| < \gamma_0 \right\}$ and $(\mathfrak{U}_0', \mathfrak{Q}_0)$ is the solution of

$$\left\{ \begin{array}{rcll} -\nu \Delta' \mathfrak{U}_0'(y', t) + \nabla' \mathfrak{Q}_0(y', t) &=& 0, & y' \in \omega, \\ \operatorname{div}' \mathfrak{U}_0'(y', t) &=& \mathfrak{G}_0(y', t), & y' \in \omega, \\ \mathfrak{U}_0'(y', t) &=& 0, & y' \in \partial\omega, \end{array} \right. \tag{5.12}$$

with

$$\mathfrak{G}_0(y', t) = \lambda \kappa_0^{-1} F(t)(n - 1 + y' \cdot \nabla')\varphi(y').$$

Note, that by construction, the following compatibility condition for the problem (5.12)

$$\int\limits_{\omega} \mathfrak{G}_0(y', t) \, dy' = 0$$

must hold. The solvability of the problem (5.12) follows from classical results in [18]. Since time variable t is only a parameter in the problem (5.12), using Theorem 2.5 we can formulate the following lemma concerning the solvability of the problem.

Lemma 5.1. *The problem* (5.12) *admits a unique weak solution* $\mathfrak{U}_0' \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathfrak{U}_0'\|^2_{W^{1,2}(\omega)} \leq c \|\mathfrak{G}_0\|^2_{L^2(\omega)}. \tag{5.13}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_0 \in L^2(\omega)$ *such that* $\int\limits_\omega \mathcal{P}_0(y')\,dy' = 0$ *and the following estimate holds*

$$\|\mathcal{P}_0\|^2_{L^2(\omega)} \leq c\|\mathcal{G}_0\|^2_{L^2(\omega)}.$$

Since by construction the data is smooth, solution $\boldsymbol{\mathcal{U}}_0'$*,* $\mathcal{P}_0$ *is also smooth, i.e. is infinitely differentiable up to the boundary.*

Remark 5.2. Since t is a parameter in the problem (5.12) we can differentiate the equation with respect to time variable t and get the analogous results for the time derivatives of the solution.

The leading-order term for the boundary-layer-in-time

Since in (5.12) the time variable t is included only as a parameter, in general, the vector function $(\mathbf{U}_0', U_{n,0})$ does not satisfy the initial condition. In order to compensate the discrepancy $\mathbf{u}(x,0) = \mathbf{u}_s(x) - \mathbf{U}_0(y', y_n, 0)$, we have to construct a boundary layer near the point $t = 0$[1]. Rewriting (5.1) in Ω_H in new coordinates

$$y' = x' x_n^{-\lambda}, y_n = x_n, \tau = t/x_n^{2\lambda},$$

we get

$$\left\{\begin{array}{rclr} y_n^{-2\lambda}\mathbf{u}'_\tau - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}_b^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p &=& 0, & y \in \Pi, \\ y_n^{-2\lambda}u_{n,\tau} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}_b^2)u_n + \mathfrak{D}_b p &=& 0, & y \in \Pi, \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}_b u_n &=& 0, & y \in \Pi, \\ \mathbf{u} &=& 0, & y \in \partial\Pi, \\ \mathbf{u}(y,0) = \mathbf{u}_s(y) &-& \mathbf{U}_0(y,0), & y \in \Pi, \end{array}\right. \tag{5.14}$$

where

$$\mathfrak{D}_b = \partial_n - \lambda y_n^{-1} y' \cdot \nabla' - 2\lambda y_n^{-1}\tau\partial_\tau.$$

Remind, that $y = (y', y_n)$ and $\mathbf{u}_s(y) = \Big(y_n^{-\lambda(n-2)-1}\boldsymbol{u}_s'(y'), y_n^{-\lambda(n-1)}u_{n,s}(y')\Big)$.

We look for the solution $(\mathbf{U}_0^b, P_0^b)$ of (5.14) in the form

$$\begin{aligned} \mathbf{U}_0^b(y', y_n, \tau) &= \left(\mathbf{U}_0'^b(y', y_n, \tau), U_{n,0}^b(y', y_n, \tau)\right), \\ P_0^b(y', y_n, \tau) &= y_n^{1-\lambda(n+1)} g_0^b(\tau) + Q_0^b(y', y_n, \tau), \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \mathbf{U}_0'^b(y', y_n, \tau) &= y_n^{-\lambda(n-2)-1}\boldsymbol{\mathcal{U}}_0'^b(y', \tau), \\ Q_0^b(y', y_n, \tau) &= y_n^{-\lambda(n-1)-1}\mathcal{Q}_0^b(y', \tau), \\ U_{n,0}^b(y', y_n, \tau) &= y_n^{-\lambda(n-1)}\mathcal{U}_{n,0}^b(y', \tau), \end{aligned}$$

[1]Note that on this step we satisfy only the singular part of the initial condition.

with

$$\mathcal{U}^b_{n,0}(y',\tau) = (1-\lambda(n+1))\,\Phi^b_0(y',\tau).$$

Substituting $(\mathbf{U}^b_0, P^b_0)$ into equations (5.14) and selecting the leading terms as $y_n \to 0$ terms, we get

$$\begin{cases} y_n^{-2\lambda}\mathbf{U}'^b_{0,\tau} - \nu y_n^{-2\lambda}\Delta'\mathbf{U}'^b_0 + y_n^{-\lambda}\nabla' Q^b_0 &= 0, \qquad y' \in \omega,\\ y_n^{-\lambda}\mathrm{div}'\mathbf{U}'^b_0 &= -\mathfrak{D}_b U^b_{n,0}, \quad y' \in \omega,\\ \mathbf{U}'^b_0 &= 0, \qquad y' \in \partial\omega,\\ \mathbf{U}'^b_0(y', y_n, 0) &:= \mathbf{u}'^b_0(y', y_n), \quad y' \in \omega, \end{cases}$$

and

$$\begin{cases} y_n^{-2\lambda}U^b_{n,0,\tau} - \nu y_n^{-2\lambda}\Delta' U^b_{n,0} + (1-\lambda(n+1))\, y_n^{-\lambda(n+1)} g^b_0 \\ \qquad -2\lambda y_n^{-\lambda(n+1)}\tau\dfrac{dg^b_0}{d\tau} = 0, \qquad y' \in \omega,\\ U^b_{n,0} = 0, \qquad y' \in \partial\omega,\\ U^b_{n,0}(y', y_n, 0) := \mathrm{u}^b_{n,0}(y', y_n), \qquad y' \in \omega, \end{cases}$$

where

$$\mathbf{u}'^b_0(y', y_n) = \mathbf{u}'_s(y', y_n) - \mathbf{U}'_0(y', y_n, 0)$$

and

$$\mathrm{u}^b_{n,0}(y', y_n) = \mathrm{u}_{n,s}(y', y_n) - U_{n,0}(y', y_n, 0).$$

Two previous problems can be equivalently rewritten as follows

$$\begin{cases} \boldsymbol{\mathcal{U}}'^b_{0,\tau} - \nu\Delta'\boldsymbol{\mathcal{U}}'^b_0 + \nabla'\mathcal{Q}^b_0 = 0, & y' \in \omega,\\ \mathrm{div}'\boldsymbol{\mathcal{U}}'^b_0 = \lambda A_b(y', \tau, \nabla', \partial_\tau)\mathcal{U}^b_{n,0}(y', \tau), & y' \in \omega,\\ \boldsymbol{\mathcal{U}}'^b_0 = 0, & y' \in \partial\omega,\\ \boldsymbol{\mathcal{U}}'^b_0(y', 0) = \boldsymbol{u}'^b_0(y'), & y' \in \omega, \end{cases} \tag{5.16}$$

and

$$\begin{cases} \partial_\tau \Phi^b_0(y', \tau) - \nu\Delta'\Phi^b_0(y', \tau) := s^b_0(\tau), & y' \in \omega,\\ \Phi^b_0(y', \tau) = 0, & y' \in \partial\omega,\\ \Phi^b_0(y', 0) := u^b_{n,0}(y'), & y' \in \omega,\\ \int\limits_\omega \Phi^b_0(y', \tau)dy' = 0, \end{cases} \tag{5.17}$$

where

$$A_b(y', \tau, \nabla', \partial_\tau) = n - 1 + y'\cdot\nabla' + 2\tau\partial_\tau, \tag{5.18}$$

$$s^b_0(\tau) = -g^b_0(\tau) - 2\frac{\lambda}{\lambda(n+1)-1}\tau\frac{dg^b_0(\tau)}{d\tau} \tag{5.19}$$

and

$$u_{n,0}^{b}(y') = u_{n,s}(y') - \mathcal{U}_{n,0}(y',0).$$

Note, that (5.17) is the inverse problem, the function $s_0^b(\tau)$ is not known and we have to choose it in order to satisfy the flux condition $(5.17)_4$.

By construction the following compatibility condition

$$\int\limits_{\omega} \Phi_0^b(y',0)dy' = \int\limits_{\omega} u_{n,0}^b(y')\,dy' = 0 \tag{5.20}$$

holds (see (5.7) and (5.10)). Therefore, the solvability of the problem (5.17) follows from Theorem 2.7. Behavior of the solution to (5.17) as $\tau \to \infty$ is described by Theorem 2.8, which states that the solution (Φ_0^b, s_0^b) is, in fact, exponentially decaying. In other words there hold following lemmas.

Lemma 5.2. *There exists a unique weak solution* (Φ_0^b, s_0^b) *of the problem* (5.17) *such that*

$$\begin{aligned}\sup_{\tau\in[0,\infty)} \|\Phi_0^b(\cdot,\tau)\|_{W^{1,2}(\omega)} + \|\Phi_0^b\|_{L^2(0,\infty;W^{1,2}(\omega))} \\ +\|\partial_\tau\Phi_0^b\|_{L^2(0,\infty;L^2(\omega))} + \|s_0^b\|_{L^2(0,\infty)} \le c\|u_{n,0}^b\|_{W^{1,2}(\omega)}.\end{aligned} \tag{5.21}$$

Lemma 5.3. *The solution* (Φ_0^b, s_0^b) *of the problem* (5.17) *satisfies the estimate*

$$\begin{aligned}&\max_{\tau\in[0,\infty)} \left[exp(\nu_*\tau)\left(\int\limits_{\omega} |\Phi_0^b(y,\tau)|^2\,dy + \nu\int\limits_{\omega} |\nabla\Phi_0^b(y,\tau)|^2\,dy\right)\right] \\ &+\int\limits_0^\infty exp(\nu_*\tau)\int\limits_{\omega} \left|\partial_\tau\Phi_0^b(y,\tau)\right|^2\,dy\,d\tau + \int\limits_0^\infty exp(\nu_*\tau)|s_0^b(\tau)|^2\,d\tau \\ &\qquad\le c\|u_{n,0}^b\|^2_{W^{1,2}(\omega)},\end{aligned} \tag{5.22}$$

where $\nu_* = \min\{\lambda_1, 1\}$. *Here* λ_1 *is the first eigenvalue of the Dirichlet problem for the Laplace equation*

$$\begin{cases} -\nu\Delta u(x) = \lambda u(x), \quad x\in\omega, \\ u(x)|_{\partial\omega} = 0. \end{cases}$$

Solvability condition for the problem (5.16)

$$\int\limits_{\omega} A_b(y',\tau,\nabla',\partial_\tau)\mathcal{U}_{n,0}^b(y',\tau)\,dy' = 0$$

is equivalent to

$$(1-\lambda(n+1))2\tau\partial_\tau \int\limits_{\omega} \Phi_0^b(y',\tau)\,dy' = 0$$

and is satisfied automatically (see $(5.17)_4$). Then the solvability of the problem (5.16) and the exponential decay of its solution as $\tau \to \infty$ follows from Theorem 2.10.

Lemma 5.4. *The problem* (5.16) *admits a unique weak solution* $\mathcal{U}_0'^b$ *such that*

$$
\begin{aligned}
&\max_{\tau\in[0,\infty)} \|\mathcal{U}_0'^b(\cdot,\tau)\|^2_{W^{1,2}(\omega)} + \|\mathcal{U}_0'^b\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\mathcal{U}_{0,\tau}'^b\|^2_{L^2(0,\infty;L^2(\omega))} \\
&\le c\Big(\|\Phi_0^b\|_{L^2(0,\infty;W^{1,2}(\omega))} + \|\partial_\tau\Phi_0^b\|_{L^2(0,\infty;L^2(\omega))} + \|\boldsymbol{u}_0'^b\|^2_{W^{1,2}(\omega)}\Big)
\end{aligned} \tag{5.23}
$$

There exists a corresponding pressure function $\mathcal{P}_0^b \in L^2(0,\infty;L^2(\omega))$ *such that* $\int\limits_\omega \mathcal{P}_0^b(y')\,dy' = 0$ *and the following estimate holds*

$$
\begin{aligned}
\|\mathcal{P}_0^b\|^2_{L^2(0,\infty;L^2(\omega))} \le c\Big(&\|\Phi_0^b\|_{L^2(0,\infty;W^{1,2}(\omega))} + \|\partial_\tau\Phi_0^b\|_{L^2(0,\infty;L^2(\omega))} \\
&+\|\boldsymbol{u}_0'^b\|^2_{W^{1,2}(\omega)}\Big).
\end{aligned}
$$

Note that solvability condition for problem (5.16)

$$
\int\limits_\omega \lambda A_b(y',\tau,\nabla',\partial_\tau)\mathcal{U}_{n,0}^b(y',\tau)\,dy' = 0
$$

holds due to the construction.

The function g_0^b is the solution to the ODE (5.19) and has the form

$$
g_0^b(\tau) = \left(-M_0\int\limits_0^\tau s_0^b(t)t^{M_0-1}\,dt + C\right)\tau^{-M_0}, \quad M_0 = \frac{n+1}{2} - \frac{1}{2\lambda} > 1.
$$

We set the constant $C = 0$ in order to have finite boundary layer pressure P_0^b at point $\tau = 0$. Moreover, there hold the following properties

$$
\lim_{\tau\to 0} g_0^b(\tau) = -s_0^b(0), \quad \lim_{\tau\to\infty} g_0^b(\tau) = 0. \tag{5.24}
$$

The equalities (5.24) follow from Theorem 2.8 using L'Hospital's rule.

Discrepancies

Functions $\mathbf{U}_0, Q_0, \mathbf{U}_0^b, Q_0^b$ leave in the equations $(5.9)_1$, $(5.9)_2$, $(5.14)_1$, $(5.14)_2$ the discrepancies $\mathbf{H}_0'(y',y_n,t,\tau)$, $H_{n,0}(y',y_n,t,\tau)$:

$$
\begin{aligned}
\mathbf{H}_0'(y',y_n,t,\tau) = {}& \nu\mathfrak{D}^2\mathbf{U}_0'(y,t) - \mathbf{U}_{0,t}'(y,t) + \nu\mathfrak{D}_b^2\mathbf{U}_0'^b(y,\tau) \\
= {}& y_n^{-\lambda(n-2)-3}\widehat{\mathfrak{F}}_0'(y',t) + y_n^{-\lambda(n-2)-1}\widetilde{\mathfrak{F}}_0'(y',t) \\
&+y_n^{-\lambda(n-2)-3}\mathfrak{F}_0'^b(y',\tau)
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
&= \mathbf{F}_0'^{o}(y', y_n, t) + \mathbf{F}_0'^{b}(y', y_n, \tau),\\
H_{n,0}(y', y_n, t, \tau) = &\ \nu\mathfrak{D}^2 U_{n,0}(y,t) - U_{n,0,t}(y,t)\\
&-\mathfrak{D}\left(y_n^{-\lambda(n-2)-1}\mathfrak{Q}_0(y',t)\right)\\
&+\nu\mathfrak{D}_b^2 U_{n,0}^b(y', y_n, \tau) - \mathfrak{D}_b\mathfrak{Q}_0^b(y', y_n, \tau)\\
= &\ y_n^{-\lambda(n-1)-2}\widehat{\mathfrak{F}}_{n,0}(y',t) + y_n^{-\lambda(n-1)}\widetilde{\mathfrak{F}}_{n,0}(y',t)\\
&+y_n^{-\lambda(n-1)-2}\mathfrak{F}_{n,0}^b(y',\tau)\\
= &\ F_{n,0}^o(y', y_n, t) + F_{n,0}^b(y', y_n, \tau).
\end{aligned}
$$

The discrepancy $\mathbf{H}_0(y', y_n, t, \tau)$ is represented as a sum $\mathbf{F}_0^o(y,t) + \mathbf{F}_0^b(y,\tau)$, where $\mathbf{F}_0^o(y', y_n, t)$ is a collection of the discrepancies arising from the construction of the leading-order term of the outer asymptotic expansion, while $\mathbf{F}_0^b(y', y_n, \tau)$ is the discrepancy arising from the boundary layer construction. Moreover, $\mathbf{F}_0^o(y', y_n, t)$ consists of two parts: terms $\widetilde{\boldsymbol{\mathfrak{F}}}$ denotes the discrepancies appearing due to $\mathbf{U}_{0,t}$, and $\widehat{\boldsymbol{\mathfrak{F}}}$ - the rest of the discrepancies.

Our goal is to construct the asymptotic decomposition of the solution so that the discrepancy belongs to the space $L^2(0,T; L^2(\Omega))$. However, since $\lambda > 1$, neither $\mathbf{F}_0^o$ nor $\mathbf{F}_0^b$ belongs to $L^2(0,T; L^2(\Omega))$ and we need to construct higher-order terms of the asymptotic decomposition.

Estimates of the leading-order term

Let us come back to the variables x, t and define

$$
\mathbf{u}^0(x,t) = \mathbf{U}_0(x'/x_n^\lambda, x_n, t) + \mathbf{U}_0^b(x'/x_n^\lambda, x_n, t/x_n^{2\lambda}),
$$

$$
p^0(x,t) = P_0(x'/x_n^\lambda, x_n, t) + P_0^b(x'/x_n^\lambda, x_n, t/x_n^{2\lambda}).
$$

By construction

$$
\operatorname{div}\, \mathbf{u}^0(x,t) = 0 \ \text{ in } \ \Omega_H, \quad \mathbf{u}^0(x,t) = 0 \ \text{ on } \ \partial\Omega_H \cap \partial\Omega,
$$

$$
\mathbf{u}^0(x,0) = \mathbf{u}_{sing}(x) \ \text{ in } \ \Omega_H, \qquad \int\limits_{\sigma(h)} \mathbf{u}^0 \cdot \mathbf{n}\, dx' = F(t).
$$

Functions $\mathbf{u}^0$, p^0 satisfy the Stokes equations

$$
\begin{cases}
\mathbf{u}_t^0 - \nu\Delta\mathbf{u}^0 + \nabla p^0 = \mathbf{H}^0 + \mathbf{U}_{0,t}, & x \in \Omega_H,\\
\operatorname{div}\mathbf{u}^0 = 0, & x \in \Omega_H,\\
\mathbf{u}^0 = 0, & x \in \partial\Omega_H \cap \partial\Omega,\\
\mathbf{u}^0(x,0) = \mathbf{u}_{sing}(x), & x \in \Omega_H,
\end{cases}
\tag{5.26}
$$

where function $\mathbf{H}^0(x,t) = \mathbf{H}_0(x'/x_n^\lambda, x_n, t, t/x_n^{2\lambda})$ is defined by formula (5.25). Note that the function $\mathbf{U}_{0,t}$ is included in the expression of the discrepancy $\mathbf{H}^0$ (see formula (5.25)) and therefore the right-hand side expression $\mathbf{H}^0 + \mathbf{U}_{0,t}$ does not contain time derivatives.

By construction we get

$$\|\mathbf{H}_0\|^2_{L^2(\omega)} \le c y_n^{-2\lambda(n-1)-4}\Big(\|\widehat{\boldsymbol{\mathfrak{F}}}_0\|^2_{L^2(\omega)} + \|\widetilde{\boldsymbol{\mathfrak{F}}}_0\|^2_{L^2(\omega)} + \|\boldsymbol{\mathfrak{F}}_0\|^2_{L^2(\omega)}\Big).$$

5.1.2 Higher-order terms of the asymptotic decomposition

In order to obtain the discrepancy in $L^2(0,T;L^2(\Omega))$, we have, first, to compensate the most singular term in $\mathbf{F}_0^o$, that is

$$(y_n^{-\lambda(n-2)-3}\widehat{\boldsymbol{\mathfrak{F}}}'_0(y',t),\, y_n^{-\lambda(n-1)-2}\widehat{\mathfrak{F}}_{n,0}(y',t)) = (\widehat{\mathbf{F}}'_0(y',y_n,t), \widehat{F}_{n,0}(y',y_n,t)).$$

So, we have to find functions $(\mathbf{U}_1, P_1)$ which solve the nonhomogeneous equations $(5.9)_1$–$(5.9)_4$ with the right-hand sides $\widehat{\mathbf{F}}'_0$, $\widehat{F}_{n,0}$, i.e. we compensate the most singular terms $\widehat{\mathbf{F}}'_0$, $\widehat{F}_{n,0}$. Functions $(\mathbf{U}_1, P_1)$ leave some new discrepancies $\mathbf{F}_1^o$ in (5.9), etc. We shall keep constructing the functions $(\mathbf{U}_k, P_k)$, $k \in \mathbb{N}$, which satisfy the nonhomogeneous equations $(5.9)_1$–$(5.9)_4$ with the right-hand sides $\mathbf{F}'_{k-1}$, $F_{n,k-1}$, until we get the discrepancies belonging to $L^2(0,T;L^2(\Omega))$.

To explain this algorithm heuristically, consider the equation $(5.9)_2$. In order to compensate the term $y_n^\mu \mathfrak{F}_{n,k}$ in the right-hand side, we construct a function $y_n^{\mu+2\lambda}\mathfrak{U}_{n,k+1}$ which produces a new discrepancy $y_n^{\mu+2(\lambda-1)}\mathfrak{F}_{n,k+1}$, i.e. after every step the power exponents μ of y_n (describing the singularity) are changing by the following rule

$$\mu \longrightarrow \mu + 2(\lambda - 1). \tag{5.27}$$

At every step of the construction we select the most singular terms in the discrepancies $\mathbf{F}'^o_k$, $F^o_{n,k}$. The process of selecting right-hand sides can be illustrated by the scheme

$$\begin{array}{c} \widehat{\mathbf{F}}_0 \Rightarrow \ldots \Rightarrow \widehat{\mathbf{F}}_{\hat{I}} \\ \Downarrow \\ \left\{ \begin{array}{ll} \widehat{\mathbf{F}}_{\hat{I}+1} + \widetilde{\mathbf{F}}_0 \Rightarrow \ldots \Rightarrow \widehat{\mathbf{F}}_{\hat{I}+1+k} + \widetilde{\mathbf{F}}_k \Rightarrow \ldots, & \lambda = \frac{N+1}{N}, \\ \widetilde{\mathbf{F}}_0 \Rightarrow \widehat{\mathbf{F}}_{\hat{I}+1} \Rightarrow \ldots \Rightarrow \widetilde{\mathbf{F}}_k \Rightarrow \widehat{\mathbf{F}}_{\hat{I}+1+k} \Rightarrow \ldots, & \lambda \ne \frac{N+1}{N}. \end{array} \right. \end{array} \tag{5.28}$$

The number $\hat{I}$ is given by

$$\widehat{I} = \min\left\{ I \in \mathbb{N} : I \ge \frac{1}{\lambda - 1} - 1 \right\} = \left\{ \begin{array}{ll} \frac{1}{\lambda-1} - 1, & \lambda = \frac{N+1}{N}, \\ \left\lfloor \frac{1}{\lambda-1} \right\rfloor, & \lambda \ne \frac{N+1}{N}. \end{array} \right. \tag{5.29}$$

The analogous algorithm is used to construct the boundary layer asymptotic decomposition: at every step k we select the most singular term in the discrepancies $\mathbf{F}^b_k(y', y_n, \tau)$, $k \in \mathbb{N}$, and compensate it. Moreover, at every step we compensate the corresponding discrepancies in the initial condition (since obviously $\mathbf{U}_k(y', y_n, 0) \neq 0$).

Case: $\lambda = \frac{N+1}{N}$

Outer asymptotics. Case $\lambda = \frac{N+1}{N}$. If $\lambda = \frac{N+1}{N}$, then up to the number $\hat{I}$, given by (5.29), the terms with $(\widehat{\boldsymbol{F}}'_k, \widehat{F}_{n,k})$, $k = 0, 1, ..., \widehat{I}$, are "more singular" than $(\widetilde{\boldsymbol{F}}'_k, \widetilde{F}_{n,k})$. When we reach the number $\hat{I}$ the terms $\widehat{\mathbf{F}}_{\hat{I}+1}$ and $\widetilde{\mathbf{F}}_0$ have the same power exponents of y_n, and so we compensate the sum $\widetilde{\mathbf{F}}_0 + \widehat{\mathbf{F}}_{\hat{I}+1}$ and continue doing so till we reach the satisfactory discrepancy from $L^2(0, T; L^2(\Omega))$ (see (5.28)). Therefore, the "most singular" terms in the discrepancies are:

$$\mathbf{F}'_k(y', y_n, t) = y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)} \boldsymbol{\mathfrak{F}}'_k(y', t),$$

$$F_{n,k}(y', y_n, t) = y_n^{-\lambda(n-1)-2+2k(\lambda-1)} \mathfrak{F}_{n,k}(y', t),$$

where

$$\boldsymbol{\mathfrak{F}}_k(y', t) = (\boldsymbol{\mathfrak{F}}'_k(y', t), \mathfrak{F}_{n,k}) = \begin{cases} \widehat{\boldsymbol{\mathfrak{F}}}_k(y', t), & k \leq \widehat{I}, \\ \widehat{\boldsymbol{\mathfrak{F}}}_k(y', t) + \widetilde{\boldsymbol{\mathfrak{F}}}_{k-\widehat{I}-1}(y', t), & k > \widehat{I}, \end{cases}$$

$k = 0, 1, 2, ...$ Thus, we can look for the approximate solution $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ in the form of series in powers of y_n:

$$\begin{aligned}
\mathbf{U}'^{O,[J]}(y, t) &= y_n^{-\lambda(n-2)-1} \boldsymbol{\mathfrak{U}}'_0(y', t) + \sum_{k=1}^{J} y_n^{-\lambda(n-2-2k)-2k-1} \boldsymbol{\mathfrak{U}}'_k(y', t), \\
U_n^{O,[J]}(y, t) &= \frac{F(t)}{\kappa_0} y_n^{-\lambda(n-1)} \varphi(y') + \sum_{k=1}^{J} y_n^{-\lambda(n-1-2k)-2k} \mathfrak{U}_{n,k}(y', t), \\
P^{O,[J]}(y, t) &= \frac{F(t)}{\kappa_0(1-\lambda(n+1))} y_n^{1-\lambda(n+1)} + y_n^{-\lambda(n-1)-1} \mathfrak{Q}_0(y', t) \\
&\quad + \sum_{k=1}^{J} \left[y_n^{-\lambda(n+1-2k)+1-2k} g_k(t) + y_n^{-\lambda(n-1-2k)-1-2k} \mathfrak{Q}_k(y', t) \right],
\end{aligned} \tag{5.30}$$

where $J \in \mathbb{N}$ and $(\boldsymbol{\mathfrak{U}}'_k, \mathfrak{Q}_k)$, $k = 1, 2, ...$, are solutions to the problems

$$\begin{cases} -\nu \Delta' \boldsymbol{\mathfrak{U}}'_k + \nabla' \mathfrak{Q}_k &= \boldsymbol{\mathfrak{F}}'_{k-1}, & y' \in \omega, \\ \operatorname{div}' \boldsymbol{\mathfrak{U}}'_k &= [\lambda A(y', \nabla') - 2k(\lambda - 1)] \, \mathfrak{U}_{n,k}, & y' \in \omega, \\ \boldsymbol{\mathfrak{U}}'_k &= 0, & y' \in \partial\omega, \end{cases} \tag{5.31}$$

$$\mathcal{U}_{n,k}(y',t) = g_k(t)(-\lambda(n+1-2k)+1-2k)\varphi(y') + \mathcal{U}^*_{n,k}(y',t),$$

$A(y',\nabla') = n-1+y'\cdot\nabla'$, the function φ is the solution to the problem (5.11), $\mathcal{U}^*_{n,k}$ satisfy the equations

$$\left\{\begin{array}{rcll} -\nu\Delta'\mathcal{U}^*_{n,k} & = & \mathcal{F}_{n,k-1}, & y'\in\omega, \\ \mathcal{U}^*_{n,k} & = & 0, & y'\in\partial\omega, \end{array}\right. \tag{5.32}$$

and the functions g_k, $k=1,2,...$, are uniquely determined from the solvability condition for the problem (5.31) with $-\lambda(n+1-2k)+1-2k\neq 0$, i.e.

$$g_k(t) = -\frac{1}{\kappa_0(-\lambda(n+1-2k)+1-2k)}\int\limits_\omega \mathcal{U}^*_{n,k}(y',t)\,dy'. \tag{5.33}$$

If $-\lambda(n+1-2k)+1-2k=0$, we have

$$k = \frac{n+1}{2} + \frac{n}{2(\lambda-1)} := \bar{k}. \tag{5.34}$$

Note that such $\bar{k}$ not necessarily exists, since $k\in\mathbb{N}$ (for example, it is easy to verify that $\bar{k}$ does not exists when $n=2$; in this case $\frac{n}{2(\lambda-1)}$ is a natural number (see (5.29))). However, if it does exist, we have to look for $(\mathbf{U}_{\bar{k}}, P_{\bar{k}})$ in a special form (see Section 4.1.2). We were not able to solve the same "$\bar{k}$" problem for the boundary layer. Therefore, hereafter we assume that

$$-\lambda(n+1-2k)+1-2k\neq 0. \tag{5.35}$$

Note that all functions in the problems (5.31), (5.32) depend on time variable t as a parameter. Thus, from Theorems 2.5, 2.1 we get the following Lemmas concerning the solvability of these problems.

Lemma 5.5. *The problem* (5.31) *admits a unique weak solution* $\boldsymbol{\mathcal{U}}'_k\in\overset{\circ}{W}{}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\boldsymbol{\mathcal{U}}'_k\|^2_{W^{1,2}(\omega)} \le c\left(\|\boldsymbol{\mathcal{F}}'_{k-1}\|^2_{L^2(\omega)} + \|\mathcal{U}^*_{n,k}\|^2_{L^2(\omega)}\right). \tag{5.36}$$

Moreover, there exists a corresponding pressure function $\mathcal{P}_k\in L^2(\omega)$ *such that* $\int\limits_\omega \mathcal{P}_k(y')\,dy'=0$ *and the following estimate holds*

$$\|\mathcal{P}_k\|^2_{L^2(\omega)} \le c\left(\|\boldsymbol{\mathcal{F}}'_{k-1}\|^2_{L^2(\omega)} + \|\mathcal{U}^*_{n,k}\|^2_{L^2(\omega)}\right).$$

Since by construction all the data is smooth, the solution $\boldsymbol{\mathcal{U}}'_k$, $\mathcal{P}_k$ *is also smooth.*

Note that solvability condition (5.33) to the problem (5.31) is satisfied automatically due to the construction.

Lemma 5.6. *The problem* (5.32) *admits a unique weak solution* $\mathcal{U}^*_{n,k} \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\mathcal{U}^*_{n,k}\|_{W^{1,2}(\omega)} \leq c\|\mathcal{F}_{n,k-1}\|_{L^2(\omega)}. \tag{5.37}$$

Moreover, since by construction $\mathcal{F}_{n,k-1}$ *is infinitely smooth, the solution* $\mathcal{U}^*_{n,k}$ *is infinitely smooth up to the boundary.*

Remark 5.3. Since t is a parameter in the problems (5.31), (5.32) we can differentiate the equations with respect to time variable t and get the analogous results for the time derivatives of the solution.

Boundary layer. Case $\lambda = \frac{N+1}{N}$. Let $\lambda = \frac{N+1}{N}$ and consider the problem (5.14) with the right-hand sides having the special form

$$\begin{cases} y_n^{-2\lambda}\mathbf{u}'_\tau - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}_b^2)\mathbf{u}' + y_n^{-\lambda}\nabla' p \; = \; y_n^{-\lambda(n-2)-3}\boldsymbol{\mathcal{F}}'^b_0(y',\tau), \\ y_n^{-2\lambda}u_{n,\tau} - \nu(y_n^{-2\lambda}\Delta' + \mathfrak{D}_b^2)u_n + \mathfrak{D}_b p \; = \; y_n^{-\lambda(n-1)-2}\mathcal{F}^b_{n,0}(y',\tau), \\ y_n^{-\lambda}\mathrm{div}'\mathbf{u}' + \mathfrak{D}_b u_n \; = \; 0, \\ \mathbf{u}|_{\partial\Pi} \; = \; 0, \\ \mathbf{u}(y', y_n, 0) \; = \; -\mathbf{U}_1(y', y_n, 0), \end{cases} \tag{5.38}$$

where $y \in \Pi$, $\boldsymbol{\mathcal{F}}'^b_0$, $\mathcal{F}^b_{n,0}$ are described in (5.25). We put

$$\begin{aligned} &P^b_1(y', y_n, \tau) = g^b_1(\tau)y_n^{-\lambda(n-1)-1} + y_n^{-\lambda(n-3)-3}\mathcal{Q}^b_1(y',\tau), \\ &\mathbf{U}'^b_1(y', y_n, \tau) = y_n^{-\lambda(n-4)-3}\boldsymbol{\mathcal{U}}'^b_1(y',\tau), \\ &U^b_{n,1}(y', y_n, \tau) = y_n^{-\lambda(n-3)-2}\mathcal{U}^b_{n,1}(y',\tau), \end{aligned} \tag{5.39}$$

with

$$\mathcal{U}^b_{n,1}(y',\tau) = (-\lambda(n-1)-1)\left(\Phi^b_1(y',\tau) + \mathcal{U}^\diamond_{n,1}(y',\tau)\right).$$

Function $\mathcal{U}^\diamond_{n,1}$ satisfies the equations

$$\begin{cases} \partial_\tau\mathcal{U}^\diamond_{n,1} - \nu\Delta'\mathcal{U}^\diamond_{n,1} \; = \; (-\lambda(n-1)-1)^{-1}\mathcal{F}^b_{n,0}, & y' \in \omega, \\ \mathcal{U}^\diamond_{n,1} \; = \; 0, & y' \in \partial\omega, \\ \mathcal{U}^\diamond_{n,1}(y',0) \; = \; 0, & y' \in \omega, \end{cases} \tag{5.40}$$

and the pair (Φ^b_1, s^b_1) is the solution to the inverse problem

$$
\begin{cases}
\partial_\tau \Phi_1^b(y',\tau) - \nu\Delta'\Phi_1^b(y',\tau) & := & s_1^b(\tau), & y' \in \omega, \\
\Phi_1^b(y',\tau) & = & 0, & y' \in \partial\omega, \\
\Phi_1^b(y',0) & = & -\mathfrak{U}_{n,1}(y',0), & y' \in \omega, \\
\int\limits_\omega \Phi_1^b(y',\tau)\,dy' & = & -\int\limits_\omega \mathfrak{U}_{n,1}^\diamond(y',\tau)\,dy'. &
\end{cases}
\tag{5.41}
$$

The function $g_1^b(\tau)$ is related to $s_1^b(\tau)$ by the following ODE

$$
s_1^b(\tau) = -g_1^b(\tau) - 2\frac{\lambda}{\lambda(n-1)+1}\tau\frac{dg_1^b(\tau)}{d\tau}. \tag{5.42}
$$

Remind that, due to the construction (see Section 5.1), $\int\limits_\omega \mathfrak{U}_{n,1}(y',t)\,dy' = 0$ and, therefore, the solvability condition

$$
\int\limits_\omega \mathfrak{U}_{n,1}^\diamond(y',0)\,dy' = \int\limits_\omega \mathfrak{U}_{n,1}(y',0)\,dy'
$$

holds automatically (see also (5.40)).

From (5.42) we find that

$$
g_1^b(\tau) = \left(-M_1\int\limits_0^\tau s_1^b(t)t^{M_1-1}\,dt\right)\tau^{-M_1}, \quad M_1 = \frac{n-1}{2} + \frac{1}{2\lambda} > 1/2.
$$

The right-hand side of the problem (5.40) is exponentially decaying. Consequently, functions $\mathfrak{U}_{n,1}^\diamond$, $\int\limits_\omega \mathfrak{U}_{n,1}^\diamond(y',\tau)\,dy'$ also decay exponentially as $\tau \to \infty$ and we can use Theorem 2.8. Thus, as in (5.24), using Theorem 2.8 and L'Hospital's rule, we derive

$$
\lim_{\tau\to 0} g_1^b(\tau) = -s_1^b(0), \quad \lim_{\tau\to\infty} g_1^b(\tau) = 0. \tag{5.43}
$$

The pair $(\boldsymbol{\mathfrak{U}}_1'^b, \mathfrak{Q}_1^b)$ is the solution to

$$
\begin{cases}
\boldsymbol{\mathfrak{U}}_{1,\tau}'^b - \nu\Delta'\boldsymbol{\mathfrak{U}}_1'^b + \nabla'\mathfrak{Q}_1^b & = \boldsymbol{\mathfrak{F}}_0'^b, \\
\operatorname{div}'\boldsymbol{\mathfrak{U}}_1'^b & = \left[\lambda A_b(y',\tau,\nabla',\partial_\tau) - 2(\lambda-1)\right]\mathfrak{U}_{n,1}^b, \\
\boldsymbol{\mathfrak{U}}_1'^b|_{\partial\omega} & = 0, \\
\boldsymbol{\mathfrak{U}}_1'^b(y',0) & = -\boldsymbol{\mathfrak{U}}_1'(y',0),
\end{cases}
\tag{5.44}
$$

where $y' \in \omega$.

Notice that the solvability condition for the problem (5.44) is satisfied automatically due to the construction, i.e.

$$\int\limits_{\omega} \big[\lambda A_b(y',\tau,\nabla',\partial_\tau) - 2(\lambda-1)\big]\, \mathcal{U}^b_{n,1}(y',\tau)\, dy' = 0.$$

Functions $\mathbf{U}^b_1, Q^b_1$ leave in the equations $(5.38)_1$, $(5.38)_2$ the discrepancies $\mathbf{F}'^b_1(y',y_n,\tau)$, $F^b_{n,1}(y',y_n,\tau)$.

Recurrently we can write

$$\mathbf{F}'^b_k(y',y_n,\tau) = \nu\mathfrak{D}^2_b\mathbf{U}'^b_k(y',y_n,\tau) = y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)}\boldsymbol{\mathcal{F}}'^b_k(y',\tau),$$

$$\begin{aligned} F^b_{n,k}(y',y_n,\tau) &= \nu\mathfrak{D}^2_b U^b_{n,k}(y',y_n,\tau) - \mathfrak{D}_b\mathcal{Q}^b_k(y',y_n,\tau)\\ &= y_n^{-\lambda(n-1)-2+2k(\lambda-1)}\mathcal{F}^b_{n,k}(y',\tau), \end{aligned}$$

$k=0,1,2,\ldots$, and we can look for the boundary layer asymptotic expansion in the form:

$$\begin{aligned} \mathbf{U}'^{B,[J]}(y',y_n,\tau) = &\sum_{k=0}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)}\boldsymbol{\mathcal{U}}'^b_k(y',\tau),\\ U^{B,[J]}_n(y',y_n,\tau) = &\sum_{k=0}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)}\mathcal{U}^b_{n,k}(y',\tau),\\ P^{B,[J]}(y',y_n,\tau) = &\sum_{k=0}^{J} \Big(g^b_k(\tau) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)}\\ &+ y_n^{-\lambda(n-1)-1+2k(\lambda-1)}\mathcal{Q}^b_k(y',\tau)\Big), \end{aligned} \tag{5.45}$$

where $(\boldsymbol{\mathcal{U}}'^b_k, \mathcal{Q}^b_k)$, $k=1,2,\ldots$, are solutions to the problems

$$\begin{cases} \boldsymbol{\mathcal{U}}'^b_{k,\tau} - \nu\Delta'\boldsymbol{\mathcal{U}}'^b_k + \nabla'\mathcal{Q}^b_k &= \boldsymbol{\mathcal{F}}'^b_{k-1},\\ \mathrm{div}'\boldsymbol{\mathcal{U}}'^b_k &= [\lambda A_b(y',\tau,\nabla',\partial_\tau) - 2k(\lambda-1)]\,\mathcal{U}^b_{n,k},\\ \boldsymbol{\mathcal{U}}'^b_k|_{\partial\omega} &= 0,\\ \boldsymbol{\mathcal{U}}'^b_k(y',0) &= -\boldsymbol{\mathcal{U}}'_k(y',0), \end{cases} \tag{5.46}$$

$$\mathcal{U}^b_{n,k}(y',\tau) = (-\lambda(n+1-2k)+1-2k)\Big(\Phi^b_k(y',\tau) + \mathcal{U}^\diamond_{n,k}(y',\tau)\Big); \tag{5.47}$$

the functions $\mathcal{U}^\diamond_{n,k}$ satisfy the equations

$$\begin{cases} \partial_\tau\mathcal{U}^\diamond_{n,k} - \nu\Delta'\mathcal{U}^\diamond_{n,k} &= (-\lambda(n+1-2k)+1-2k)^{-1}\mathcal{F}^b_{n,k-1},\\ \mathcal{U}^\diamond_{n,k}|_{\partial\omega} &= 0,\\ \mathcal{U}^\diamond_{n,k}(y',0) &= 0, \end{cases} \tag{5.48}$$

where $y' \in \omega$, while the functions (Φ_k^b, s_k^b), $k = 1, 2, ...$, are solutions to the inverse problems

$$\left\{ \begin{array}{rcll} \partial_\tau \Phi_k^b(y',\tau) - \nu \Delta' \Phi_k^b(y',\tau) & := & s_k^b(\tau), & y' \in \omega, \\ \Phi_k^b(y',\tau) & = & 0, & y' \in \partial\omega, \\ \Phi_k^b(y',0) & = & -\mathfrak{U}_{n,k}(y',0), & y' \in \omega, \\ \int\limits_\omega \Phi_k^b(y',\tau)\,dy' & = & -\int\limits_\omega \mathfrak{U}^\diamond_{n,k}(y',\tau)\,dy'. & \end{array} \right. \tag{5.49}$$

Finally, the functions $g_k^b(\tau)$ are solutions to the following ODE

$$s_k^b(\tau) = -g_k^b(\tau) - 2c_k\tau \frac{dg_k^b(\tau)}{d\tau}, \tag{5.50}$$

where $c_k = \dfrac{\lambda}{\lambda(n-1) + 1 - 2(k-1)(\lambda - 1)}$. Note, that $\int\limits_\omega \mathfrak{U}_{n,k}(y',t)\,dy' = 0$ due to the construction. Therefore, the solvability condition

$$\int\limits_\omega \mathfrak{U}^\diamond_{n,k}(y',0)\,dy' = \int\limits_\omega \mathfrak{U}_{n,k}(y',0)\,dy' \tag{5.51}$$

holds automatically. Remind, that by assumption, there holds the condition (5.35). Hence, we find

$$g_k^b(\tau) = \left(-M_k \int\limits_0^\tau s_k^b(t) t^{M_k - 1}\,dt \right) \tau^{-M_k}, \quad \text{if} \ \ M_k > 0, \tag{5.52}$$

and

$$g_k^b(\tau) = \left(-M_k \int\limits_\tau^\infty s_k^b(t) t^{M_k - 1}\,dt \right) \tau^{-M_k}, \quad \text{if} \ \ M_k < 0, \tag{5.53}$$

where

$$M_k = \frac{n+1}{2} - k + \frac{2k-1}{2\lambda} > 3/2 - k.$$

Note, that $M_k \neq 0$ due to the assumption (5.35).

From Theorems 2.10, 2.2, 2.7 we get the following lemmas concerning the solvability of the problems (5.46), (5.48), (5.49).

Lemma 5.7. *The problem* (5.46) *admits a unique weak solution* $\boldsymbol{\mathfrak{U}}'^b_{k,\tau}$ *such that*

$$\begin{aligned} &\max_{\tau \in [0,\infty)} \|\boldsymbol{\mathfrak{U}}'^b_k(\cdot,\tau)\|^2_{W^{1,2}(\omega)} + \|\boldsymbol{\mathfrak{U}}'^b_k\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\boldsymbol{\mathfrak{U}}'^b_{k,\tau}\|^2_{L^2(0,\infty;L^2(\omega))} \\ &\leq c\Big(\|\boldsymbol{\mathfrak{F}}'^b_{k-1}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\Phi_k^b\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\Phi^b_{k,\tau}\|^2_{L^2(0,\infty;L^2(\omega))} \\ &+ \|\mathfrak{U}^\diamond_{n,k}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\mathfrak{U}^\diamond_{n,k,\tau}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\boldsymbol{\mathfrak{U}}'_k\|^2_{W^{1,2}(\omega)} \Big). \end{aligned} \tag{5.54}$$

There exists a corresponding pressure function $\mathcal{P}_k^b \in L^2(0,\infty; L^2(\omega))$ *such that* $\int\limits_\omega \mathcal{P}_k^b(y')\,dy' = 0$ *and the following estimate holds*

$$
\begin{aligned}
&\|\mathcal{P}_k^b\|^2_{L^2(0,\infty;L^2(\omega))} \le c \Big(\|\boldsymbol{\mathcal{F}}'^b_{k-1}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\Phi_k^b\|^2_{L^2(0,\infty;W^{1,2}(\omega))} \\
&+\|\Phi^b_{k,\tau}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\mathcal{U}^\diamond_{n,k}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\mathcal{U}^\diamond_{n,k,\tau}\|^2_{L^2(0,\infty;L^2(\omega))} \\
&+\|\boldsymbol{\mathcal{U}}'_k(y',0)\|^2_{W^{1,2}(\omega)} \Big).
\end{aligned}
$$

Note that solvability condition for problem (5.46)

$$
\int\limits_\omega \left[\lambda A_b(y',\tau,\nabla',\partial_\tau) - 2k(\lambda-1)\right] \mathcal{U}^b_{n,k} = 0
$$

holds due to the construction.

Lemma 5.8. *The problem* (5.48) *admits a unique weak solution* $\mathcal{U}^\diamond_{n,k} \in L^2(0,\infty; \mathring{W}^{1,2}(\omega))$ *with* $\partial_\tau \mathcal{U}^\diamond_{n,k} \in L^2(0,\infty;L^2(\omega))$ *and there holds the estimate*

$$
\begin{aligned}
&\max_{\tau\in[0,\infty)} \|\mathcal{U}^\diamond_{n,k}(\cdot,\tau)\|^2_{W^{1,2}(\omega)} + \|\mathcal{U}^\diamond_{n,k}\|^2_{L^2(0,\infty;W^{1,2}(\omega))} \\
&+\|\partial_\tau \mathcal{U}^\diamond_{n,k}\|^2_{L^2(0,\infty;L^2(\omega))} \le c\|\mathcal{F}^b_{n,k-1}\|_{L^2(0,\infty;L^2(\omega))}.
\end{aligned}
\tag{5.55}
$$

Lemma 5.9. *There exists a unique weak solution* (Φ_k^b, s_k^b) *of the problem* (5.49) *such that*

$$
\begin{aligned}
&\sup_{\tau\in[0,\infty)} \|\Phi_k^b(\cdot,\tau)\|_{W^{1,2}(\omega)} + \|\Phi_k^b\|_{L^2(0,\infty;W^{1,2}(\omega))} \\
&\qquad +\|\partial_\tau \Phi_k^b\|_{L^2(0,\infty;L^2(\omega))} + \|s_k^b\|_{L^2(0,\infty)} \\
&\le c \left(\|\int\limits_\omega \mathcal{U}^\diamond_{n,k}(y',\tau)\,dy'\|_{W^{1,2}(0,\infty)} + \|\mathcal{U}_{n,k}(y',0)\|_{W^{1,2}(\omega)} \right).
\end{aligned}
\tag{5.56}
$$

Discrepancies. Case $\lambda = \frac{N+1}{N}$. The discrepancies $\mathbf{H}'_J(y',y_n,t,\tau)$, $H_{n,J}(y',y_n,t,\tau)$ left by the sum $\mathbf{U}^{O,[J]}(y',y_n,t) + \mathbf{U}^{B,[J]}(y',y_n,\tau)$ can be written in the form

$$
\begin{aligned}
\mathbf{H}'_J(y',y_n,t,\tau) = \;& y_n^{-\lambda(n-1)-2+(2J+1)(\lambda-1)} \widehat{\boldsymbol{\mathcal{F}}}'_J(y',t) \\
&+ \sum_{k=\max\{0,J-\widehat{I}\}}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \widetilde{\boldsymbol{\mathcal{F}}}'_k(y',t) \\
&+y_n^{-\lambda(n-1)-2+(2J+1)(\lambda-1)} \boldsymbol{\mathcal{F}}'^b_J(y',\tau) \\
= \;& \mathbf{F}'^o_J(y',y_n,t) + \mathbf{F}'^b_J(y',y_n,\tau),
\end{aligned}
\tag{5.57}
$$

$$
\begin{aligned}
H_{n,J}(y', y_n, t, \tau) = \; & y_n^{-\lambda(n-1)-2+2J(\lambda-1)} \widehat{\mathfrak{F}}_{n,J}(y', t) \\
& + \sum_{k=\max\{0, J-\widehat{I}\}}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)} \widetilde{\mathfrak{F}}_{n,k}(y', t) \\
& + y_n^{-\lambda(n-1)-2+2J(\lambda-1)} \mathfrak{F}^b_{n,J}(y', \tau) \\
= \; & F^o_{n,J}(y', y_n, t) + F^b_{n,J}(y', y_n, \tau).
\end{aligned}
$$

The discrepancies (5.57) are represented as a sum $\mathbf{F}^o_J(y', y_n, t) + \mathbf{F}^b_J(y', y_n, \tau)$, where $\mathbf{F}^o_J(y', y_n, t)$ is a collection of the discrepancies arising from the main asymptotic term construction, and $\mathbf{F}^b_{n,J}(y', y_n, \tau)$ are the discrepancies arising from the boundary layer construction.

Since $\lambda > 1$, after final number of steps $J^* > 0$ we arrive at

$$
(\mathbf{H}'_{J^*-1}, H_{n,J^*-1}) \in L^2(0, T; L^2(\Omega)), \tag{5.58}
$$

where number J^* is given by the following relation

$$
J^* = \min \left\{ J \in \mathbb{N} : J > \frac{1}{4} \left[(n+2)\hat{I} + 2n + 5 \right] \right\}. \tag{5.59}
$$

Case $\lambda \neq \frac{N+1}{N}$

Outer asymptotics. Case $\lambda \neq \frac{N+1}{N}$. If $\lambda \neq \frac{N+1}{N}$, then up to the number $\hat{I}$, given by (5.29), the terms with $(\widehat{\boldsymbol{F}}'_k, \widehat{F}_{n,k})$, $k = 0, 1, ..., \widehat{I}$, are "more singular" than $(\widetilde{\boldsymbol{F}}'_k, \widetilde{F}_{n,k})$. After we reach this number $\hat{I}$ the terms $\hat{F}$ and $\tilde{F}$ are alternating (see (5.28) for more details), that is

$$
\mathbf{F}'_k(y', y_n, t) = \begin{cases} y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)} \widehat{\boldsymbol{\mathfrak{F}}}'_k(y', t), & k \leq \widehat{I}, \\ y_n^{-\lambda(n-1)+(2j+1)(\lambda-1)} \widetilde{\boldsymbol{\mathfrak{F}}}'_j(y', t), & k = \widehat{I} + 2j + 1, \\ y_n^{-\lambda(n-1)-2+(2(\widehat{I}+j+1)+1)(\lambda-1)} \widehat{\boldsymbol{\mathfrak{F}}}'_{\widehat{I}+j+1}(y', t), & k = \widehat{I} + 2j + 2, \end{cases}
$$

$$
F_{n,k}(y', y_n, t) = \begin{cases} y_n^{-\lambda(n-1)-2+2k(\lambda-1)} \widehat{\mathfrak{F}}_{n,k}(y', t), & k \leq \widehat{I}, \\ y_n^{-\lambda(n-1)+2j(\lambda-1)} \widetilde{\mathfrak{F}}_{n,j}(y', t), & k = \widehat{I} + 2j + 1, \\ y_n^{-\lambda(n-1)-2+2(\widehat{I}+j+1)(\lambda-1)} \widehat{\mathfrak{F}}_{n,\widehat{I}+j+1}(y', t), & k = \widehat{I} + 2j + 2, \end{cases}
$$

$k = 0, 1, ..., j = 0, 1, ...$ We look for the approximate solution $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ in the form

$$
\begin{aligned}
\mathbf{U}'^{O,[J]}(y', y_n, t) = \; & y_n^{-\lambda(n-2)-1} \boldsymbol{\mathfrak{U}}'_0(y', t) \\
& + \sum_{k=1}^{\min\{J, \lfloor \frac{J+\widehat{I}}{2} \rfloor\}} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)} \boldsymbol{\mathfrak{U}}'_k(y', t)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{k=1}^{\lfloor \frac{J-\widehat{I}+1}{2} \rfloor} y_n^{-\lambda(n-1)+2+(2k+1)(\lambda-1)} \widetilde{\mathbf{\mathfrak{U}}}'_k(y',t),\\
U_n^{O,[J]}(y',y_n,t) &= \frac{F(t)}{\kappa_0} y_n^{-\lambda(n-1)} \varphi(y') + \\
&\quad + \sum_{k=1}^{\min\{J,\lfloor \frac{J+\widehat{I}}{2} \rfloor\}} y_n^{-\lambda(n-1)+2k(\lambda-1)} \mathfrak{U}_{n,k}(y',t)\\
&\quad + \sum_{k=1}^{\lfloor \frac{J-\widehat{I}+1}{2} \rfloor} y_n^{-\lambda(n-1)+2+2k(\lambda-1)} \widetilde{\mathfrak{U}}_{n,k}(y',t),\\
P^{O,[J]}(y',y_n,t) &= \frac{F(t)}{\kappa_0(1-\lambda(n+1))} y_n^{1-\lambda(n+1)} + y_n^{-\lambda(n-1)-1} \mathfrak{Q}_0(y',t)\\
&\quad + \sum_{k=1}^{\min\{J,\lfloor \frac{J+\widehat{I}}{2} \rfloor\}} g_k(t) y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)}\\
&\quad + \sum_{k=1}^{\min\{J,\lfloor \frac{J+\widehat{I}}{2} \rfloor\}} y_n^{-\lambda(n-1)-1+2k(\lambda-1)} \mathfrak{Q}_k(y',t)\\
&\quad + \sum_{k=1}^{\lfloor \frac{J-\widehat{I}+1}{2} \rfloor} \widetilde{g}_k(t) y_n^{-\lambda(n-1)+1+2(k-1)(\lambda-1)}\\
&\quad + \sum_{k=1}^{\lfloor \frac{J-\widehat{I}+1}{2} \rfloor} y_n^{-\lambda(n-1)+1+2k(\lambda-1)} \widetilde{\mathfrak{Q}}_k(y',t),
\end{aligned}
\tag{5.60}
$$

where $J \in \mathbb{N}$, functions $(\widetilde{\mathbf{\mathfrak{U}}}'_k, \widetilde{\mathfrak{Q}}_k)$, $k = 1, 2, ...,$ are solutions to the problems

$$
\left\{
\begin{aligned}
-\nu\Delta'\widetilde{\mathbf{\mathfrak{U}}}'_k + \nabla'\widetilde{\mathfrak{Q}}_k &= \widetilde{\mathbf{\mathfrak{F}}}'_{k-1}, && y' \in \omega,\\
\operatorname{div}'\widetilde{\mathbf{\mathfrak{U}}}'_k &= [\lambda A(y',\nabla') + 2 - 2k(\lambda-1)]\, \widetilde{\mathfrak{U}}_{n,k}, && y' \in \omega,\\
\widetilde{\mathbf{\mathfrak{U}}}'_k|_{\partial\omega} &= 0, && y' \in \omega,
\end{aligned}
\right.
\tag{5.61}
$$

$$
\widetilde{\mathfrak{U}}_{n,k}(y',t) = \widetilde{g}_k(t)(-\lambda(n-1)+1+2k(\lambda-1))\varphi(y') + \widetilde{\mathfrak{U}}^*_{n,k}(y',t),
$$

φ solves (5.11), $\widetilde{\mathfrak{U}}^*_{n,k}$, $k = 1, 2, ...,$ satisfy the nonhomogeneous equations (5.32) with the right-hand sides $\widetilde{\mathfrak{F}}_{n,k-1}$. Since, by assumption, $-\lambda(n+1-2k)+1-2k \neq 0$ (see Subsection 5.1.2), the functions $\widetilde{g}_k$, $k = 1, 2, ...,$ are uniquely determined from the solvability condition for the problem (5.61) which is equivalent to the equation

$$
\begin{aligned}
&2\widetilde{g}_k(t)\kappa_0[-\lambda(n-1)+1+2k(\lambda-1)][-1+k(\lambda-1)]\\
&\quad = \int\limits_{\omega} \left[\lambda A(y',\nabla') + 2 - 2k(\lambda-1)\right] \widetilde{\mathfrak{U}}^*_{n,k}(y',t)\, dy'.
\end{aligned}
\tag{5.62}
$$

Note that all functions in the problem (5.61) depend on time variable t as a parameter. Thus, from Theorem 2.5 we get the following lemma concerning the solvability of this problem.

Lemma 5.10. *The problem* (5.61) *admits a unique weak solution* $\widetilde{\mathfrak{U}}'_k \in \mathring{W}^{1,2}(\omega)$ *and there holds the estimate*

$$\|\widetilde{\mathfrak{U}}'_k\|^2_{W^{1,2}(\omega)} \leq c\left(\|\widetilde{\mathfrak{F}}'_{k-1}\|^2_{L^2(\omega)} + \|\widetilde{\mathfrak{U}}^*_{n,k}\|^2_{L^2(\omega)}\right). \tag{5.63}$$

Moreover, there exists a corresponding pressure function $\widetilde{\mathfrak{P}}_k \in L^2(\omega)$ *such that* $\int\limits_\omega \widetilde{\mathfrak{P}}_k(y')\,dy' = 0$ *and*

$$\|\widetilde{\mathfrak{P}}_k\|^2_{L^2(\omega)} \leq c\left(\|\widetilde{\mathfrak{F}}'_{k-1}\|^2_{L^2(\omega)} + \|\widetilde{\mathfrak{U}}^*_{n,k}\|^2_{L^2(\omega)}\right).$$

Since by construction all the data is smooth, the solution $\widetilde{\mathfrak{U}}'_k$, $\widetilde{\mathfrak{P}}_k$ *is also smooth.*

Remark 5.4. Since t is a parameter in the problem (5.61) we can differentiate the equation with respect to time variable t and get the analogous results for the time derivatives of the solution.

Note that solvability condition (5.62) to the problem (5.61) is satisfied automatically due to the construction.

Boundary layer. Case $\lambda \neq \frac{N+1}{N}$. Since, in the case $\lambda \neq \frac{N+1}{N}$, the discrepancies $\hat{F}$ and $\tilde{F}$ arising in outer asymptotics construction procedure are alternating (see the scheme (5.28)), we construct both functions $\mathbf{U}_k(y', y_n, t)$ and $\widetilde{\mathbf{U}}_k(y', y_n, t)$ (which are also alternating). Therefore, now we have to compensate both initial values $\mathbf{U}_k(y', y_n, 0)$ and $\widetilde{\mathbf{U}}_k(y', y_n, 0)$. Thus, the right-hand sides for the boundary layer problems are alternating in a similar way as for the outer asymptotics

$$\mathbf{F}'^b_k(y,\tau) = \begin{cases} \nu\mathfrak{D}^2_b\mathbf{U}'^b_k(y,\tau), & k \leq \widehat{I}, \\ \nu\mathfrak{D}^2_b\widetilde{\boldsymbol{U}}'^b_j(y,\tau), & k = \widehat{I}+2j+1, \\ \nu\mathfrak{D}^2_b\mathbf{U}'^b_{\widehat{I}+j+1}(y,\tau), & k = \widehat{I}+2j+2, \end{cases}$$

$$= \begin{cases} y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)}\widehat{\mathfrak{F}}'^b_k(y',\tau), & k \leq \widehat{I}, \\ y_n^{-\lambda(n-1)+(2j+1)(\lambda-1)}\widetilde{\mathfrak{F}}'^b_j(y',\tau), & k = \widehat{I}+2j+1, \\ y_n^{-\lambda(n-1)-2+(2(\widehat{I}+j+1)+1)(\lambda-1)}\widehat{\mathfrak{F}}'^b_{\widehat{I}+j+1}(y',\tau), & k = \widehat{I}+2j+2, \end{cases}$$

$$F^b_{n,k}(y,\tau)=\begin{cases}\nu\mathfrak{D}^2_b U^b_{n,k}(y,\tau)-\mathfrak{D}_b Q^b_k(y,\tau), & k\le \widehat{I},\\ \nu\mathfrak{D}^2_b \widetilde{U}^b_{n,j}(y,\tau)-\mathfrak{D}_b \widetilde{Q}^b_j(y,\tau), & k=\widehat{I}+2j+1,\\ \nu\mathfrak{D}^2_b U^b_{n,\widehat{I}+j+1}(y,\tau)-\mathfrak{D}_b Q^b_{\widehat{I}+j+1}(y,\tau), & k=\widehat{I}+2j+2,\end{cases}$$

$$=\begin{cases} y_n^{-\lambda(n-1)-2+2k(\lambda-1)}\widehat{\mathfrak{F}}^b_{n,k}(y',\tau), & k\le\widehat{I},\\ y_n^{-\lambda(n-1)+2j(\lambda-1)}\widetilde{\mathfrak{F}}^b_{n,j}(y',\tau), & k=\widehat{I}+2j+1,\\ y_n^{-\lambda(n-1)-2+2(\widehat{I}+j+1)(\lambda-1)}\widehat{\mathfrak{F}}^b_{n,\widehat{I}+j+1}(y',\tau), & k=\widehat{I}+2j+2,\end{cases}$$

$k=0,1,2,...$, $j=0,1,2,...$, and we set $\widetilde{\boldsymbol{\mathfrak{F}}}^b_0\equiv 0$ for consistency. We look for the approximate solution $(\mathbf{U}^{B,[J]},P^{B,[J]})$ in the form

$$\begin{aligned}\mathbf{U}'^{B,[J]}(y',y_n,\tau)&=\sum_{k=0}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)}\boldsymbol{\mathfrak{U}}'^b_k(y',\tau)\\&\quad+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+2+(2k+1)(\lambda-1)}\widetilde{\boldsymbol{\mathfrak{U}}}'^b_k(y',\tau),\\ U_n^{B,[J]}(y',y_n,\tau)&=\sum_{k=0}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)+2k(\lambda-1)}\mathfrak{U}^b_{n,k}(y',\tau)\\&\quad+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+2+2k(\lambda-1)}\widetilde{\mathfrak{U}}^b_{n,k}(y',\tau),\end{aligned}\tag{5.64}$$

$$\begin{aligned}P^{B,[J]}(y',y_n,\tau)&=\sum_{k=0}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} g^b_k(\tau)y_n^{-\lambda(n-1)-1+2(k-1)(\lambda-1)}\\&\quad+\sum_{k=0}^{\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}} y_n^{-\lambda(n-1)-1+2k(\lambda-1)}\mathfrak{Q}^b_k(y',\tau)\\&\quad+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor}\widetilde{g}^b_k(\tau)y_n^{-\lambda(n-1)+1+2(k-1)(\lambda-1)}\\&\quad+\sum_{k=1}^{\lfloor\frac{J-\widehat{I}+1}{2}\rfloor} y_n^{-\lambda(n-1)+1+2k(\lambda-1)}\widetilde{\mathfrak{Q}}^b_k(y',\tau),\end{aligned}$$

where the pair $(\boldsymbol{\mathfrak{U}}'^b_k,\mathfrak{Q}^b_k)$, $k=1,2,...$, solves problems (5.46) with the right-hand sides $\widehat{\boldsymbol{\mathfrak{F}}}'^b_{k-1}$, the functions $\mathfrak{U}^b_{n,k}$ are described by (5.47), where $\mathfrak{U}^\diamond_{n,k}$ are the solutions to the problems (5.48) with the right-hand sides $(-\lambda(n+1-2k)+1-2k)^{-1}\widehat{\mathfrak{F}}^b_{n,k-1}$ and Φ^b_k together with s^b_k solve the inverse problems (5.49). Since it is assumed that $-\lambda(n+1-2k)+1-2k\neq 0$ (see (5.35)), the functions g^b_k, $k=1,2,...$, are described by (5.52).

Functions $(\widetilde{\boldsymbol{\mathfrak{U}}}_k'^b, \widetilde{\mathfrak{Q}}_k^b)$, $k = 1, 2, \ldots$, are solutions to

$$\begin{cases} \widetilde{\boldsymbol{\mathfrak{U}}}_{k,\tau}'^b - \nu\Delta'\widetilde{\boldsymbol{\mathfrak{U}}}_k'^b + \nabla'\widetilde{\mathfrak{Q}}_k^b &= \widetilde{\boldsymbol{\mathfrak{F}}}_{k-1}'^b, \\ \mathrm{div}'\widetilde{\boldsymbol{\mathfrak{U}}}_k'^b &= [\lambda A_b(y', \tau, \nabla', \partial_\tau) - 2k(\lambda - 1)]\, \widetilde{\mathfrak{U}}_{n,k}^b, \\ \widetilde{\boldsymbol{\mathfrak{U}}}_k'^b|_{\partial\omega} &= 0, \\ \widetilde{\boldsymbol{\mathfrak{U}}}_k'^b(y', 0) &= -\widetilde{\boldsymbol{\mathfrak{U}}}_k'(y', 0), \end{cases} \tag{5.65}$$

where $y' \in \omega$ and

$$\widetilde{\mathfrak{U}}_{n,k}^b(y', \tau) = (-\lambda(n + 1 - 2k) + 1 - 2k) \left(\widetilde{\Phi}_k^b(y', \tau) + \widetilde{\mathfrak{U}}_{n,k}^\diamond(y', \tau)\right),$$

where the functions $(\widetilde{\Phi}_k^b, \widetilde{s}_k^b)$ solve the inverse problem

$$\begin{cases} \partial_\tau\widetilde{\Phi}_k^b(y', \tau) - \nu\Delta'\widetilde{\Phi}_k^b(y', \tau) &:= \widetilde{s}_k^b(\tau), & y' \in \omega, \\ \widetilde{\Phi}_k^b(y', \tau) &= 0, & y' \in \partial\omega, \\ \widetilde{\Phi}_k^b(y', 0) &= -\widetilde{\mathfrak{U}}_{n,k}(y', 0), & y' \in \omega, \\ \int\limits_\omega \widetilde{\Phi}_k^b(y', \tau)\, dy' &= -\int\limits_\omega \widetilde{\mathfrak{U}}_{n,k}^\diamond(y', \tau)\, dy', \end{cases} \tag{5.66}$$

the functions $\widetilde{\mathfrak{U}}_{n,k}^\diamond$, $k = 1, 2, \ldots$, satisfy the equations (5.48) with the right-hand sides $(-\lambda(n + 1 - 2k) + 1 - 2k)^{-1}\widetilde{\mathfrak{F}}_{n,k}$; the functions $\widetilde{g}_k$, $k = 1, 2, \ldots$, are solutions to

$$\widetilde{s}_k^b(\tau) = -\widetilde{g}_k^b(\tau) - 2c_k\tau\frac{d\widetilde{g}_k^b(\tau)}{d\tau}, \tag{5.67}$$

$c_k = \dfrac{\lambda}{\lambda(n-1) + 1 - 2(k-1)(\lambda - 1)}$, and are described by (5.52) (with $\widetilde{s}_k^b$ instead of s_k^b).

Note, that $\int\limits_\omega \widetilde{\mathfrak{U}}_{n,k}(y', t)\, dy' = 0$ (see (5.61), (5.62)). Therefore, the solvability condition for the problem (5.66) holds automatically:

$$\int\limits_\omega \widetilde{\mathfrak{U}}_{n,k}^\diamond(y', 0)\, dy' = \int\limits_\omega \widetilde{\mathfrak{U}}_{n,k}(y', 0)\, dy' \tag{5.68}$$

(see also (5.48)).

From Theorems 2.10, 2.7 we get the following lemmas concerning the solvability of the problems (5.65), (5.66).

Lemma 5.11. *The problem* (5.65) *admits a unique weak solution* $\widetilde{\mathfrak{U}}_k^{\prime b}$ *such that*

$$\begin{aligned}&\max_{\tau\in[0,\infty)} \|\widetilde{\mathfrak{U}}_k^{\prime b}(\cdot,\tau)\|^2_{W^{1,2}(\omega)} + \|\widetilde{\mathfrak{U}}_k^{\prime b}\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\widetilde{\mathfrak{U}}_{k,\tau}^{\prime b}\|^2_{L^2(0,\infty;L^2(\omega))}\\ &\leq c\Big(\|\widetilde{\mathfrak{F}}_{k-1}^{\prime b}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\widetilde{\Phi}_k^b\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\widetilde{\Phi}_{k,\tau}^b\|^2_{L^2(0,\infty;L^2(\omega))}\\ &+ \|\widetilde{\mathfrak{U}}_{n,k}^{\diamond}\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\widetilde{\mathfrak{U}}_{n,k,\tau}^{\diamond}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\widetilde{\mathfrak{U}}_k^{\prime}\|^2_{W^{1,2}(\omega)}\Big).\end{aligned} \tag{5.69}$$

There exists a corresponding pressure function $\widetilde{\mathcal{P}}_k^b \in L^2(0,\infty;L^2(\omega))$ *such that* $\int\limits_\omega \widetilde{\mathcal{P}}_k^b(y')\,dy' = 0$ *and the following estimate holds*

$$\begin{aligned}&\|\widetilde{\mathcal{P}}_k^b\|^2_{L^2(0,\infty;L^2(\omega))} \leq c\Big(\|\widetilde{\mathfrak{F}}_{k-1}^{\prime b}\|^2_{L^2(0,\infty;L^2(\omega))} + \|\widetilde{\Phi}_k^b\|^2_{L^2(0,\infty;W^{1,2}(\omega))}\\ &+\|\widetilde{\Phi}_{k,\tau}^b\|^2_{L^2(0,\infty;L^2(\omega))} + \|\widetilde{\mathfrak{U}}_{n,k}^{\diamond}\|^2_{L^2(0,\infty;W^{1,2}(\omega))} + \|\widetilde{\mathfrak{U}}_{n,k,\tau}^{\diamond}\|^2_{L^2(0,\infty;L^2(\omega))}\\ &+\|\widetilde{\mathfrak{U}}_k^{\prime}(y',0)\|^2_{W^{1,2}(\omega)}\Big).\end{aligned}$$

Note that compatibility condition for problem (5.65)

$$\int\limits_\omega \left[\lambda A_b(y',\tau,\nabla',\partial_\tau) - 2k(\lambda-1)\right]\widetilde{\mathfrak{U}}_{n,k}^b\,dy' = 0$$

holds automatically due to the construction.

Lemma 5.12. *There exists a unique weak solution* $(\widetilde{\Phi}_k^b, \widetilde{s}_k^b)$ *of the problem* (5.66) *such that*

$$\begin{aligned}&\sup_{\tau\in[0,\infty)} \|\widetilde{\Phi}_k^b(\cdot,\tau)\|_{W^{1,2}(\omega)} + \|\widetilde{\Phi}_k^b\|_{L^2(0,\infty;W^{1,2}(\omega))}\\ &\qquad+\|\partial_\tau\widetilde{\Phi}_k^b\|_{L^2(0,\infty;L^2(\omega))} + \|\widetilde{s}_k^b\|_{L^2(0,\infty)}\\ &\leq c\left(\|\int\limits_\omega \widetilde{\mathfrak{U}}_{n,k}^{\diamond}(y',\tau)\,dy'\|_{W^{1,2}(0,\infty)} + \|\widetilde{\mathfrak{U}}_{n,k}(y',0)\|_{W^{1,2}(\omega)}\right).\end{aligned} \tag{5.70}$$

Discrepancies. Case $\lambda \neq \frac{N+1}{N}$. The discrepancies $\mathbf{H}_J'(y',y_n,t,\tau)$, $H_{n,J}$ (y',y_n,t,τ) left by the sum $\mathbf{U}^{O,[J]}(y',y_n,t)+\mathbf{U}^{B,[J]}(y',y_n,\tau)$ can be written in the form

$$\begin{aligned}\mathbf{H}_J'(y',y_n,t,\tau) = &\sum_{k=\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)-2+(2k+1)(\lambda-1)}\widehat{\mathfrak{F}}_k^{\prime}(y',t)\\ &+\sum_{k=\max\{0,\lfloor\frac{J-\widehat{I}+1}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)+(2k+1)(\lambda-1)}\widetilde{\mathfrak{F}}_k^{\prime}(y',t)\end{aligned}$$

$$+y_n^{-\lambda(n-1)-2+(2\widehat{j}+1)(\lambda-1)}\widehat{\mathfrak{F}}'^b_{\widehat{j}}(y',\tau)$$

$$+y_n^{-\lambda(n-1)+(2\widetilde{j}+1)(\lambda-1)}\widetilde{\mathfrak{F}}'^b_{\widetilde{j}}(y',\tau)+\mathbf{F}'_J(y',y_n,t,\tau)$$

$$=\mathbf{F}'^o_J(y',y_n,t)+\mathbf{F}'^b_J(y',y_n,\tau)+\mathbf{F}'_J(y',y_n,t,\tau),$$

$$\begin{aligned} H_{n,J}(y',y_n,t,\tau) = & \sum_{k=\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)-2+2k(\lambda-1)}\widehat{\mathfrak{F}}_{n,k}(y',t) \\ & + \sum_{k=\max\{0,\lfloor\frac{J-\widehat{I}+1}{2}\rfloor\}}^{J} y_n^{-\lambda(n-1)+2k(\lambda-1)}\widetilde{\mathfrak{F}}_{n,k}(y',t) \\ & +y_n^{-\lambda(n-1)-2+2\widehat{j}(\lambda-1)}\widehat{\mathfrak{F}}^b_{n,\widehat{j}}(y',\tau) \\ & +y_n^{-\lambda(n-1)+2\widetilde{j}(\lambda-1)}\widetilde{\mathfrak{F}}^b_{n,\widetilde{j}}(y',\tau)+F_{n,J}(y',y_n,t,\tau) \\ & = F^o_{n,J}(y',y_n,t)+F^b_{n,J}(y',y_n,\tau)+F_{n,J}(y',y_n,t,\tau), \end{aligned} \tag{5.71}$$

where $\widehat{j}=\min\{J,\lfloor\frac{J+\widehat{I}}{2}\rfloor\}$ and $\widetilde{j}=\lfloor\frac{J-\hat{I}}{2}\rfloor$, $\tilde{j}>\hat{I}$.

Since $\lambda>1$, after final number of steps $J^*>0$ we arrive at (5.58). Here the number J^* is given by

$$J^*=\widetilde{J}+\widehat{J}, \tag{5.72}$$

where

$$\widetilde{J}=\min\left\{J\in\mathbb{N}:\ J>\frac{1}{4}\left[\frac{n-2}{\lambda-1}+n+3\right]\right\},$$

$$\widehat{J}=\min\left\{J\in\mathbb{N}:\ J>\frac{1}{4}\left[\frac{n+2}{\lambda-1}+n+3\right]\right\}.$$

The discrepancies in (5.71) are represented as the sum $\mathbf{F}^o_J(y,t)+\mathbf{F}^b_J(y,\tau)+\mathbf{F}_J(y,t,\tau)$, where $\mathbf{F}^o_J(y,t)$ is a collection of the discrepancies arising from construction of the outer asymptotics, $\mathbf{F}^b_{n,J}(y,\tau)$ are the discrepancies arising from the boundary layer construction and $\mathbf{F}_J(y,t,\tau)$ is the collection of discrepancies which already belong to $L^2(0,T;L^2(\Omega))$ (arising from both the outer and the boundary layer asymptotics construction).

Estimates of the higher-order terms

The estimates of this section are valid for both cases $\lambda=\frac{N+1}{N}$ and $\lambda\neq\frac{N+1}{N}$, i.e. for all $\lambda>1$.

Let us come back to the variables x, t and define

$$\mathbf{u}^{[J]}(x,t)=\mathbf{U}^{O,[J]}(x'/x_n^\lambda,x_n,t)+\mathbf{U}^{B,[J]}(x'/x_n^\lambda,x_n,t/x_n^{2\lambda}),$$

$$p^{[J]}(x,t)=P^{O,[J]}(x'/x_n^\lambda,x_n,t)+P^{B,[J]}(x'/x_n^\lambda,x_n,t/x_n^{2\lambda}).$$

where $\mathbf{U}^{O,[J]}$, $\mathbf{U}^{B,[J]}$, $P^{O,[J]}$, $P^{B,[J]}$ are defined by (5.30), (5.45) or by (5.60), (5.64) depending on the value of λ. By construction

$$\operatorname{div}\,\mathbf{u}^{[J]}(x,t)=0 \ \text{ in } \ \Omega_H, \quad \mathbf{u}^{[J]}(x,t)=0 \ \text{ on } \ \partial\Omega_H\cap\partial\Omega,$$

$$\mathbf{u}^{[J]}(x,0)=\mathbf{u}_{sing}(x) \ \text{ in } \ \Omega_H, \quad \int\limits_{\sigma(h)} \mathbf{u}^{[J]}\cdot\mathbf{n}\,dx'=F(t).$$

Functions $\mathbf{u}^{[J]}$, $p^{[J]}$ satisfy the Stokes equations

$$\begin{cases} \mathbf{u}_t^{[J]}-\nu\Delta\mathbf{u}^{[J]}+\nabla p^{[J]} &= \mathbf{H}^{J-1}+\mathbf{U}_t^{O,[J]}, \quad x\in\Omega_H,\\ \operatorname{div}\mathbf{u}^{[J]} &= 0, \quad x\in\Omega_H,\\ \mathbf{u}^{[J]} &= 0, \quad x\in\partial\Omega_H\cap\partial\Omega,\\ \mathbf{u}^{[J]}(x,0) &= \mathbf{u}_{sing}(x), \quad x\in\Omega_H, \end{cases} \tag{5.73}$$

where the right-hand side $\mathbf{H}^{J-1}(x,t)=\mathbf{H}'_J(x'/x_n^\lambda,x_n,t,t/x_n^{2\lambda})$ is described by formula (5.57) or (5.71) depending on the value of λ.

By construction we deduce

$$\|\mathbf{H}_{J-1}\|^2_{L^2(\omega)}\le cy_n^{-2\lambda(n-1)-4}\Big(\|\widehat{\mathfrak{F}}_{J-1}\|^2_{L^2(\omega)}+\sum_{k=\max\{0,J-1-\hat{I}\}}^{J-1}\|\widetilde{\mathfrak{F}}_k\|^2_{L^2(\omega)} +\|\mathfrak{F}^b_{J-1}\|^2_{L^2(\omega)}\Big),$$

if $\lambda=\frac{N+1}{N}$, and

$$\|\mathbf{H}_{J-1}\|^2_{L^2(\omega)}\le cy_n^{-2\lambda(n-1)-4}\Big(\sum_{k=\min\{J-1,\lfloor\frac{J-1+\widehat{I}}{2}\rfloor\}}^{J-1}\|\widehat{\mathfrak{F}}_k\|^2_{L^2(\omega)} + \sum_{k=\max\{0,\lfloor\frac{J-\widetilde{I}}{2}\rfloor\}}^{J-1}\|\widetilde{\mathfrak{F}}_k\|^2_{L^2(\omega)}+\|\widehat{\mathfrak{F}}^b_{\widehat{j}}\|^2_{L^2(\omega)}+\|\widetilde{\mathfrak{F}}^b_{\widetilde{j}}\|^2_{L^2(\omega)}+\|\mathbf{F}_{J-1}\|^2_{L^2(\omega)}\Big),$$

if $\lambda\neq\frac{N+1}{N}$; where $\widehat{j}=\min\{J-1,\lfloor\frac{J-1+\widehat{I}}{2}\rfloor\}$ and $\widetilde{j}=\lfloor\frac{J-1-\hat{I}}{2}\rfloor$, $\tilde{j}>\hat{I}$.

5.1.3 Regularity conditions

Consider the asymptotic expansion

$$\begin{aligned} \mathbf{U}^{[J]}(x,t) &= \mathbf{U}^{O,[J]}\left(\tfrac{x'}{x_n^\lambda},x_n,t\right)+\mathbf{U}^{B,[J]}\left(\tfrac{x'}{x_n^\lambda},x_n,\tfrac{t}{x_n^{2\lambda}}\right),\\ P^{[J]}(x,t) &= P^{O,[J]}\left(\tfrac{x'}{x_n^\lambda},x_n,t\right)+P^{B,[J]}\left(\tfrac{x'}{x_n^\lambda},x_n,\tfrac{t}{x_n^{2\lambda}}\right), \end{aligned} \tag{5.74}$$

where $(\mathbf{U}^{O,[J]},P^{O,[J]})$ is the outer asymptotic expansion given by the formula (5.30) if $\lambda=\frac{N+1}{N}$ (or by the formula (5.60) if $\lambda\neq\frac{N+1}{N}$); $(\mathbf{U}^{B,[J]},P^{B,[J]})$

is the boundary layer expansion given by (5.45) if $\lambda = \frac{N+1}{N}$ (or by (5.64) if $\lambda \neq \frac{N+1}{N}$). $(\mathbf{U}^{[J]}, P^{[J]})$ is an approximate solution of the problem (5.1) and the corresponding discrepancies $\mathbf{H}_J(y', y_n, t, \tau)$ are given by the formulas (5.57), (5.71). Constructing the above asymptotic representations we were solving the problems (5.12), (5.16), (5.17), (5.31), (5.32), (5.46)–(5.49), (5.61), (5.65), (5.66). Therefore, it is necessary to have at each step sufficient regularity of the data which is needed for the solvability of the corresponding problems. Examining the right-hand sides of these problems we state the loss of one time derivative on each step of the outer asymptotic formula construction. Therefore, in order to ensure the existence of all terms of asymptotic expansion up to the order J we have to assume that the flux

$$F(t) \in W^{J+1,2}(0, T).$$

Since the flux $F(t)$ is the integral of the normal component of the boundary value $\mathbf{a}(x, t)$ over $\partial\Omega$, the last requirement imply, the following regularity conditions for $\mathbf{a}$:

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0, T; W^{1/2,2}(\partial\Omega)), \quad l = 0, 1, 2, ..., J+1.$$

The boundary layer construction does not cause any loss of regularity, and it is enough to suppose that $\mathbf{u}_0 \in W^{1,2}(\Omega)$, $\mathbf{u}'_s, u_{n,s} \in \mathring{W}^{1,2}(\omega)$.

Note that the above regularity conditions remain the same as in the case of the time-periodic Stokes problem, i.e., in the case without the boundary layer expansion (see Chapter 4).

5.2 Existence of the solution

Remind, that $\xi \in C^\infty[0, \infty)$ is a nonnegative cut-off function, described in Chapter 2. We look for the solution $(\mathbf{u}, p)$ of the problem (5.1) in the form

$$\mathbf{u}(x', x_n, t) = \xi(x_n)\mathbf{U}^{[J^*-1]}(x, t) + \widehat{\mathbf{U}}(x', x_n, t), \tag{5.75}$$

$$p(x', x_n, t) = \xi(x_n)P^{[J^*-1]}(x, t) + \widehat{P}(x', x_n, t), \tag{5.76}$$

where the pair $(\mathbf{U}^{[J^*-1]}, P^{[J^*-1]})$ is the approximate solution, described by the formula (5.74) with J^* defined by (5.59) if $\lambda = \frac{N+1}{N}$ (or J^* is given by (5.72) if $\lambda \neq \frac{N+1}{N}$), $(\widehat{\mathbf{U}}, \widehat{P})$ is a solution to the problem

$$\begin{cases} \widehat{\mathbf{U}}_t(x,t) - \nu\Delta\widehat{\mathbf{U}}(x,t) + \nabla\widehat{P}(x,t) = \mathbf{f}^*(x,t), \\ \operatorname{div}\widehat{\mathbf{U}}(x,t) = -\xi'(x_n)U_n^{[J^*-1]}(x,t), \\ \widehat{\mathbf{U}}(x,t)|_{\partial\Omega} = \mathbf{a}(x), \\ \widehat{\mathbf{U}}(x,0) = \mathbf{u}_0(x), \end{cases} \tag{5.77}$$

where $x \in \Omega$ and $\mathbf{f}^* = \mathbf{f} + \xi \mathbf{H}_{J^*-1} + \nu(2\nabla\xi \cdot \nabla \mathbf{U}^{O,[J^*-1]} + \Delta\xi \mathbf{U}^{O,[J^*-1]}) - \nabla\xi P^{O,[J^*-1]} + \nu(2\nabla\xi \cdot \nabla \mathbf{U}^{B,[J^*-1]} + \Delta\xi \mathbf{U}^{B,[J^*-1]}) - \nabla\xi P^{B,[J^*-1]}$. Here $\mathbf{H}_J$ is described by (5.57) (or (5.71)) and $J^* \in \mathbb{N}$ is the number which ensures that $\mathbf{H}_{J^*-1}$ belongs to $L^2(0,T;L^2(\Omega))$ (see (5.59), (5.72)) and $\mathbf{u}_0(x) \in W^{1,2}(\Omega)$ (see (5.1)). Finally, using the fact that the flux $F(t)$ is the integral of the normal component of the boundary value $\mathbf{a}(x,t)$ over $\partial\Omega$, we get

$$\|\mathbf{f}^*\|^2_{L^2(0,T;L^2(\Omega))} \leq C\Big(\|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{a}\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))} + \sum_{k=1}^{J^*} \Big\|\frac{\partial^k \mathbf{a}}{\partial t^k}\Big\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))}\Big) \tag{5.78}$$

(see paragraph *Estimates of the higher-order terms* in Subsection 5.1.2).

Since

$$\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx = \int\limits_{\Omega_{H/2,H}} \operatorname{div}(\xi U^{[J^*-1]})\,dx = \int\limits_{\sigma(H/2)} U_n^{[J^*-1]}\,dS = F(t),$$

the necessary compatibility condition

$$\int\limits_{\Omega_{H/2,H}} \xi' U_n^{[J^*-1]}\,dx + \int\limits_{\partial\Omega_{H/2,H}} \mathbf{a}\cdot\mathbf{n}\,dS = 0 \tag{5.79}$$

is satisfied (see (5.2)). The solvability results for the problem (5.77) are described in Theorem 2.10.

5.2.1 Existence theorem

Consider the problem (5.1).

Definition 5.1. *By a weak solution of the problem* (5.1) *we understand a solenoidal vector field* $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega \setminus \Omega_h))$ *with* $\mathbf{u}_t \in L^2(0,T;L^2(\Omega \setminus \Omega_h))$, $\forall h \in (0,H)$, *satisfying the boundary condition* $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$, *the initial condition* $\mathbf{u}(x,0) = \mathbf{u}_0(x) + \zeta(x_n)\mathbf{u}_{sing}(x)$ *and the integral identity*

$$\int_0^T\int_\Omega \mathbf{u}_\tau(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau + \nu\int_0^T\int_\Omega \nabla\mathbf{u}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)\,dxd\tau = \int_0^T\int_\Omega \mathbf{f}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)\,dxd\tau,$$

for every solenoidal $\boldsymbol{\eta} \in L^2(0,T;C_0^\infty(\Omega))$.

Theorem 5.1. *Assume that cusp's power* λ *satisfies the condition* (5.35). *Let functions* $\mathbf{f} \in L^2(0,T;L^2(\Omega))$, $\mathbf{u}_0 \in W^{1,2}(\Omega)$, $\mathbf{u}_s \in \mathring{W}^{1,2}(\omega)$ *and* $\mathbf{a}, \dfrac{\partial^k \mathbf{a}}{\partial t^k} \in$

$L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))$, $k=1,\dots,J^*+1$, *be given and satisfy the compatibility conditions* (5.4), (5.6), (5.5), $\operatorname{supp}\mathbf{a}\subset\partial\Omega_0\cap\partial\Omega\subset\partial\Omega_{H/2,H}\cap\partial\Omega$. *Then the problem* (5.1) *admits at least one weak solution* $\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega\setminus\Omega_h))$, $\mathbf{u}_t\in L^2(0,T;L^2(\Omega\setminus\Omega_h))$ $\forall h\in(0,H)$, *which can be represented as the sum* (5.75). *Moreover, the following estimate*

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\mathbf{u}(\cdot,t)-\xi(\cdot)\mathbf{U}^{[J^*-1]}(\cdot,t)\|^2_{W^{1,2}(\Omega)}\\
&+\|\mathbf{u}-\xi\mathbf{U}^{[J^*-1]}\|^2_{L^2(0,T;W^{1,2}(\Omega))}+\|\mathbf{u}_t-\xi\mathbf{U}_t^{[J^*-1]}\|^2_{L^2(0,T;L^2(\Omega))}\\
&\le C\Big(\|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))}+\|\mathbf{a}\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))}\\
&+\sum_{k=1}^{J^*+1}\|\frac{\partial^k\mathbf{a}}{\partial t^k}\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega_{H/2,H}))}+\|\mathbf{u}_0\|^2_{W^{1,2}(\Omega)}+\|\mathbf{u}_s\|^2_{W^{1,2}(\omega)}\Big)
\end{aligned}\tag{5.80}$$

holds.

Proof. The difference $\mathbf{u}-\xi\mathbf{U}^{[J^*-1]}=\widehat{\mathbf{U}}$ is a weak solution of the problem (5.77). The existence of $\widehat{\mathbf{U}}$ follows from Theorem 2.10, the inequality (5.80) is the consequence of the estimates (5.23) and (5.78). $\square$

Conclusions

The main objective of the dissertation was to prove the existence of solutions to the Stokes system (stationary, time-periodic and nonstationary) in the so-called power-cusp domains. Such domains have a singular point on the boundary. Therefore, there is a sink or a source in the cusp point O, i.e. the flux of the boundary value is nonzero.

- *The solutions to the stationary, time-periodic and nonstationary Stokes problems in a power cusp domains are necessarily singular and their singularity depends on a cusp's power λ.*

In order to represent such solutions, we had to construct the formal asymptotic expansion of the solution near the singular point of the boundary.

- *For the stationary and time-periodic Stokes problems the formal asymptotic expansion contains only the outer asymptotics terms;*
- *For the nonstationary Stokes problem the formal asymptotic expansion contains the outer and the inner asymptotics terms;*
- *The fast time in the boundary layer asymptotics terms depends on the space variable.*

The existence of at least one singular solution was proved.

- *For all three problems (stationary, time-periodic and nonstationary) the solution can be constructed as a sum of the asymptotic expansion and the term with finite dissipation of energy;*
- *The developed in the thesis methods can be used to solve analogical nonlinear problems, i.e. for the Navier-Stokes equations.*

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FOR NOTES

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