VILNIAUS UNIVERSITETAS

MATEMATIKOS IR INFORMATIKOS FAKULTETAS

**Magistro baigiamasis darbas**

# Žaliavų kainų ilgos atminties modeliavimas ir statistinė analizė

## Long memory modeling and statistical inference of commodity prices

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# Žaliavų kainų ilgos atminties modeliavimas ir statistinė analizė


### Santrauka

Žaliavų rinka dėl jos įtakos ekonomikai ir jos augimui yra svarbus tyrimų objektas. Šio darbo tikslas yra išanalizuoti pasirinktų žaliavų kainų kintamumą ir ilgos atminties savybes. Pirmiausia yra aprašyti 4 ilgos atminties modeliai: FIGARCH, HYGARCH, FIAPARCH, HYAPARCH, aprašytos literatūroje žinomos jų savybės, bei pateiktos pataisytos ir aprašytos paprastesnės sąlygos HYAPARCH modelio stacionarumui. Šie modeliai yra vėliau pritaikyti 12 žaliavų kainų duomenims ir yra išanalizuotos jų įverčių savybės.



**Raktiniai žodžiai :** Ilga atmintis, FIGARCH, HYGARCH, FIAPARCH, HYAPARCH, žaliavų kainos.


# Long memory modeling and statistical inference of commodity prices


### Abstract

Commodity market, because of it's impact on economy and growth, is important area of research. The main goal of this thesis is to analyse commodity price volatilities and their long memory properties. First, we are going to describe 4 long memory models: FIGARCH, HYGARCH, FIAPARCH, HYAPARCH and discuss their established properties from the literature. A corrected and simplified version of stationarity conditions for the HYAPARCH model is also provided. These models are applied to empirical data of 12 commodity futures returns and the estimated properties are discussed and compared.



**Key words :** Long memory, FIGARCH, HYGARCH, FIAPARCH, HYAPARCH, commodity prices

# Contents

# 1 Introduction

Commodity market and it's influence on other financial markets is important area of research in economics. Most of the literature studies the co-movements of commodity prices with other financial indicators. Because of events, such as financial crisis of 2008-2009, which caused a lot of market volatility, possible fall in commodity demand due to slowing economy of China and recent fall in oil prices due to overproduction, it is important to look also into the volatility of commodities to better understand their dynamics. Commodity market and its volatility is also important for countries which heavily rely on commodity production, Cavalcanti et al. (2015) showed that commodity volatility has a negative effect on economic growth which indicates importance of volatility estimation, but due to irregular properties of commodity prices the research in this sector only recently picked up the pace.

Two of the models often used in literature to model volatility are Autoregressive Conditional Heteroskedasticity (ARCH) model introduced in Engle (1982) and it's generalized version GARCH introduced in Bollerslev (1986). Later, to account for long-range volatility dependence, a more flexible Fractionally Integrated GARCH model was introduced in Baillie et al. (1996). Unfortunately, FIGARCH process is not covariance stationary. Because of this issue, a generalized version of FIGARCH model called Hyperbolic GARCH (HYGARCH) was introduced in Davidson (2004), which is weakly stationary. Another group of long memory models ,FIAPARCH and HYAPARCH, which were introduced in Tse (1998) and Schoffer (2003), allow to analyse asymmetric behaviour of volatilities.

The goal of this thesis is to to analyse volatility and long memory properties of 12 commodities by estimating and comparing 4 discussed long memory models and to select the best models. In Section 2, FIGARCH, HYGARCH, FIAPARCH and HYAPARCH models, their properties and results in the literature are presented. In Section 2.5 slightly corrected and simplified version of conditions for weak stationarity of HYAPARCH model are presented. In Section 3, general statistics of the data are provided and empirical application of models from Section 2 is presented. All of the series showed significant long memory properties and most of them had significant estimates of asymmetry parameter. In Section 3.3, analysis of possible structural breaks is performed on data split into 2 periods, the resulting changes in parameter estimates are then discussed. Finally, conclusions of the results are presented in Section 4.

# 2 Methodology

## 2.1 GARCH model

In order to analyse often observed non-constant variance in stock returns, inflation and other financial instruments, Generalized Autoregressive Conditional Heteroskedasticity (GARCH$(p, q)$) model was introduced in Bollerslev (1986). This model has the following form:

$$r_t = \sigma_t \epsilon_t,$$

where $\epsilon_t$ are independent and identically distributed random variables with mean 0 and variance 1 ($\epsilon_t \sim iid(0, 1)$) and

$$(1 - \sum_{i=1}^{q} \beta_i L^i)\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i L^i r_t^2.$$

Here L is the lag operator. Further on we will use this notation: $\beta(L) = \sum_{i=1}^{q} \beta_i L^i$, $\alpha(L) = \sum_{i=1}^{p} \alpha_i L^i$. Provided that zeroes of $B(L) = 1-\beta(L)$ polynomial lie outside the unit circle, this model can be rewritten into ARCH($\infty$) as provided in Conrad (2010) :

$$\sigma_t^2 = \omega(1 - \beta(1))^{-1} + \Psi^{GA}(L)r_t^2 = \omega(1 - \beta(1))^{-1} + \sum_{i=1}^{\infty} \psi_i^{GA} r_{t-i}^2.$$

Here $\Psi^{GA}(L) = (1 - \beta(L))^{-1}\alpha(L)$. In case of GARCH$(1, 1)$ the $\psi_i^{GA}$ coefficients are defined in this form: $\psi_1^{GA} = \alpha_1$, $\psi_i^{GA} = \beta_1 \psi_{i-1}^{GA}$, $i \geq 2$. To assure that GARCH process is weakly stationary we require that $\Psi^{GA}(1)<1$ which is equivalent to $\Phi^{GA}(1)>0$ where $\Phi^{GA}(L) = 1-\beta(L)-\alpha(L)$. In Figure 1, simulated GARCH$(1, 1)$ process is plotted with residuals $\epsilon_t$ generated according to standard normal distribution.

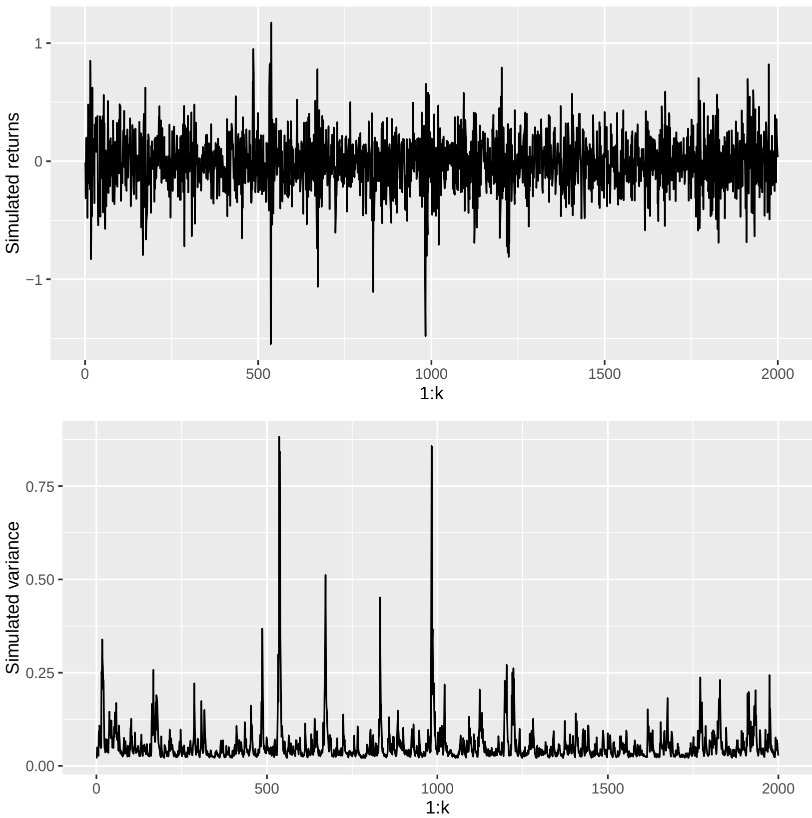



**Figure 1:** Simulated returns and conditional variance according to GARCH$(1, 1)$ with $\alpha = 0.3$, $\beta = 0.5$, $\omega = 0.02$ process

## 2.2 FIGARCH model

A more general version of GARCH models is the Fractionally integrated GARCH (FIGARCH) models, which allows the long range dependence by allowing coefficients to decay hyperbolically instead of geometrically as in GARCH model. FIGARCH$(p, d, q)$ models were first introduced in Baillie et al. (1996). They have the following form:

$$r_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \omega + (1 - (1 - \beta(L))^{-1}(1 - \phi(L))(1 - L)^d) r_t^2,$$

where $\sigma_t > 0, \epsilon_t \sim iid(0, 1)$. $\beta(L)$ and $\phi(L) = \alpha(L) + \beta(L)$ are polynomials of order $p$ and $q$ and $d$ is a fractional degree. This model includes both GARCH (when $d = 0$) and IGARCH (when $d = 1$). It is well known that

$$(1-L)^d = 1 - \sum_{j=1}^{\infty} g_j L^j,$$

where coefficients $g_j$ have the following representation:

$$g_j = \frac{d\Gamma(j-d)}{\Gamma(1-d)\Gamma(j+1)},\ j \geq 1.$$

To assure that the conditional variance produced by this model is non-negative, certain conditions on coefficients needs to be imposed which were derived in Conrad and Haag (2006). Because models estimated in Section 3 did not show significance for higher orders, only the non-negativity constrains of FIGARCH$(1, d, 1)$ model will be provided here. Higher order constrains can be found in Conrad and Haag (2006). Firstly, we need to introduce ARCH$(\infty)$ representation of FIGARCH$(1, d, 1)$:

$$\sigma_t^2 = \omega + \sum_{i=1}^{\infty} \psi_i r_{t-i}^2. \tag{1}$$

In FIGARCH$(1, d, 1)$ coefficients $\psi_i$ have the following expression:

$$\psi_1 = d + \phi_1 - \beta_1,$$

$$\psi_i = \beta_1 \psi_{i-1} + (f_i - \phi_1)(-g_{i-1})\ for\ i \geq 2.$$

For convenience, another form of coefficients $g_j$ is introduced:

$$g_j = f_j g_{j-1} = \prod_{i=1}^{j} f_i,\ f_j = \frac{j-1-d}{j},\ g_0 = 1.$$

The inequality constrains to assure that FIGARCH$(1, d, 1)$ model produces non-negative conditional variance are provided in Conrad and Haag (2006) Corollary 1:

**Corollary 1** *The conditional variance of the FIGARCH(*$1, d, 1$*) process is non-negative a.s. iff*
***Case 1****:* $0 < \beta < 1$
*either* $\psi_1 \geq 0$ *and* $\phi_1 \leq f_2$ *or for* $k > 2$ *with* $f_{k-1} < \phi_1 \leq f_k$*, it holds that* $\psi_{k-1} \geq 0$
***Case 2****:* $-1 < \beta < 0$
*either* $\psi_1 \geq 0, \psi_2 \geq 0$ *and* $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ *or for* $k > 3$ *with* $f_{k-2}(\beta_1 + f_{k-1})/(\beta_1 + f_{k-2} < \phi_1 \leq f_{k-1}(\beta_1 + f_k)/(\beta_1 + f_{k-1})$ *it holds that* $\psi_{k-1} \geq 0$ *and* $\psi_{k-2} \geq 0$*.*

Baillie et al. (1996) applied this model to Deutchmark-US dollar exchange rates and showed that it was superior to alternatives. Unfortunately, as explained in Davidson (2004), the process does not have defined set of parameters for which finite second moments exist, which is required for proof of covariance stationarity of the process. In figure 1, simulated FIGARCH$(1, d, 1)$ process is plotted with the same residuals as in previous section.

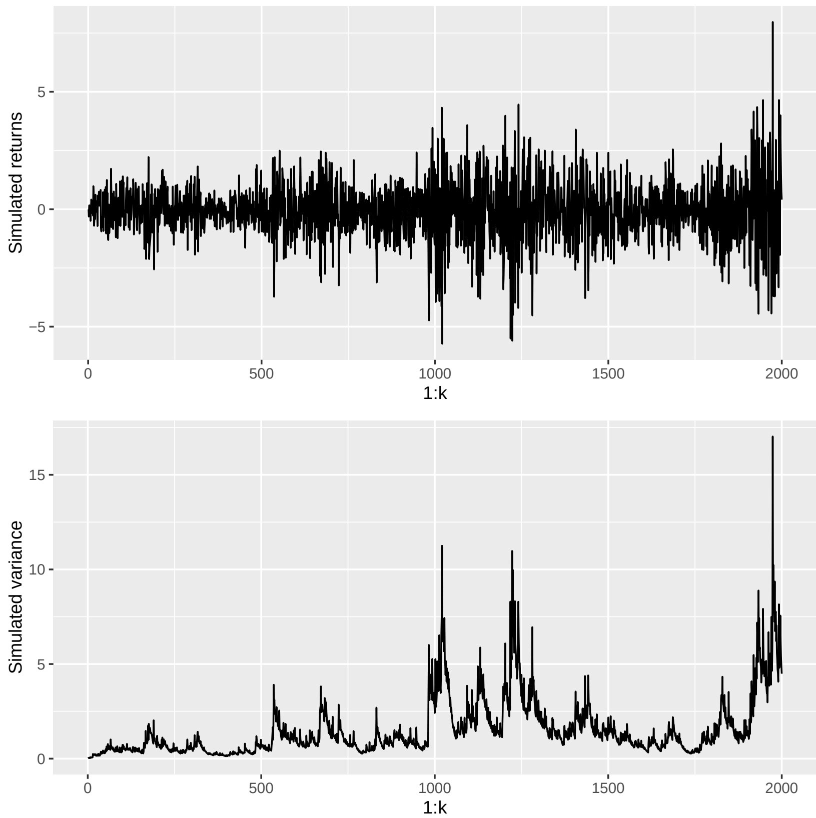



**Figure 2:** Simulated returns and conditional variance according to FIGARCH$(1, d, 1)$ process with $\alpha = -0.5$, $\beta = 0.8$, $\omega = 0.02$, $d = 0.7$

## 2.3 HYGARCH model

In order to account for the problem of infinite second moments and lack of weak stationarity a new model called Hyperbolic GARCH (HYGARCH) was introduced in Davidson (2004). The fractionally integrated part of the model $(1 - L)^d$ was replaced with $(1 - \tau) - \tau(1 - L)^d$. This form restricts the coefficients so that hyperbolic decay would still be present but it would also allow for defined parameter set for finite second moments and weak stationarity. The equation for HYGARCH$(1, d, 1)$ is the following:

$$\sigma_t^2 = \omega(1-\beta(L))^{-1} + (1-(1-\beta(L))^{-1}(1-\phi(L))((1-\tau)-\tau(1-L)^d))r_t^2. \quad (2)$$

The process (2) nests both GARCH (when $\tau = 0$ or $d = 0$) and IGARCH (when $\tau = 1$ and $d = 1$). To assure that conditional variance is positive and weakly stationary, the non-negativity and weak stationarity conditions of the model were introduced in Conrad (2010). For simplification of constrains, Conrad (2010) introduced the following form of HYGARCH model which shows that HYGARCH if linear combination of FIGARCH and GARCH with weights:

$$\sigma_t^2 = \frac{\omega}{B(1)} + \Psi^{HY}(L)r_t^2, \quad (3)$$

$$\Psi^{HY}(L) = \tau\Psi^{FI}(L) + (1-\tau)\Psi^{GA}(L),$$

here $\Psi^{GA}(L) = (B(L) - \Phi(L))B(L)^{-1}$ is GARCH component and $\Psi^{FI}(L) = 1 - ((1-L)^d\Phi(L))B(L)^{-1}$ is FIGARCH component. Covariance stationarity in this process, according to Conrad (2010), is shown trough behaviour of $\Psi^{HY}(1)$. Condition for weak stationarity when $\tau = 0$ is the same as in case of GARCH process. When $0 < \tau \leq 1$, the necessary condition is $\Psi^{HY}(1) = \tau + (1-\tau)\Psi^{GA}(1) < 1$ which means that, if GARCH conditions are fulfilled, the process is weakly stationary when $0 \leq \tau < 1$. The conditions for weak stationarity for special cases when $\tau > 1$ are also provided in Conrad (2010) and require relaxed assumption on residuals. The constrains to assure that the conditional variance is positive are provided for HYGARCH$(1, d, 1)$ process. The constrains in higher order models can also be found in Conrad (2010). In HYGARCH$(1, d, 1)$ case, the coefficients $\psi_i$ in (1) have the following form:

$$\psi_1^{HY} = \tau d + \phi_1 - \beta_1,$$

$$\psi_i^{HY} = \beta_1\psi_{i-1}^{HY} + \tau(f_i - \phi_1)(-g_{i-1}), \text{ for } i > 1.$$

The conditions for non-negative conditional variance is provided in Theorem 1 of Conrad (2010):

**Theorem 1** *The conditional variance of HYGARCH(*$1, d, 1$*) is non-negative a.s iff*
***Case 1:*** $0 < \beta_1 < 1$ *either* $\psi_1^{HY} \geq 0$ *and* $\phi_1 \leq f_2$ *or for* $k > 2$ *with* $f_{k-1} < \phi_1 \leq f_k$ *it holds that* $\psi_{k-1}^{HY} \geq 0$.
***Case 2:*** $-1 < \beta_1 < 0$ *either* $\psi_1^{HY} \geq 0, \psi_2^{HY} \geq 0$ *and* $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ *or for* $k > 3$ *with* $f_{k-2}(\beta_1 + f_{k-1})/(\beta_1 + f_{k-2} < \phi_1 \leq f_{k-1}(\beta_1 + f_k)/(\beta_1 + f_{k-1})$ *it holds that* $\psi_{k-1}^{HY} \geq 0$ *and* $\psi_{k-2}^{HY} \geq 0$.

Proof of this theorem is provided in the same paper. Restrictions for other orders of the model are also provided in Conrad (2010). In Figure 3, simulated HYGARCH$(1, d, 1)$ process is plotted with the same residuals as in previous sections.

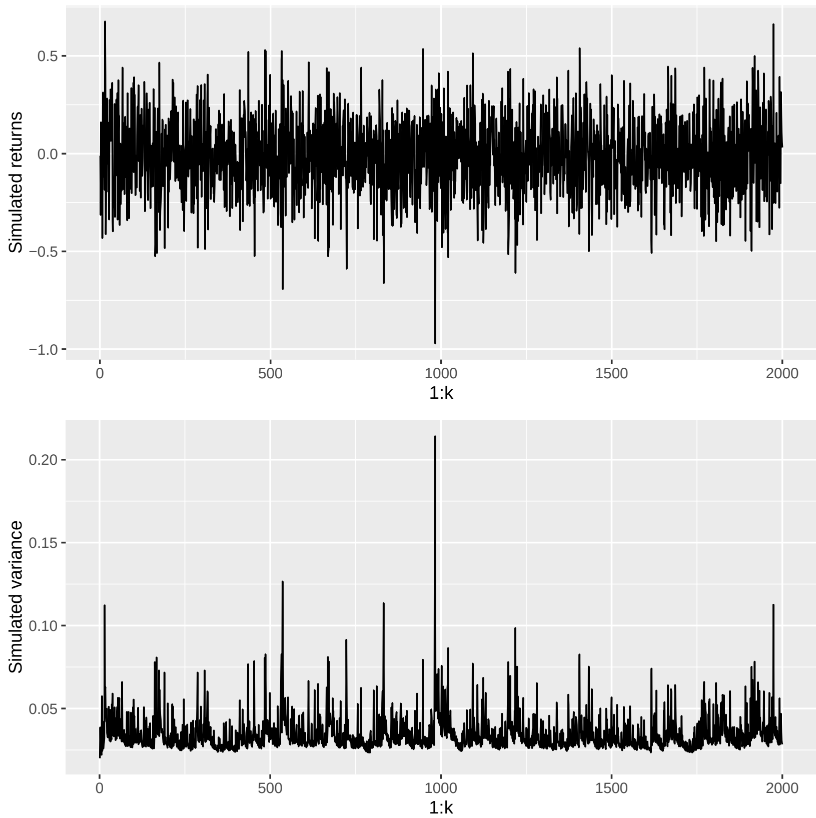


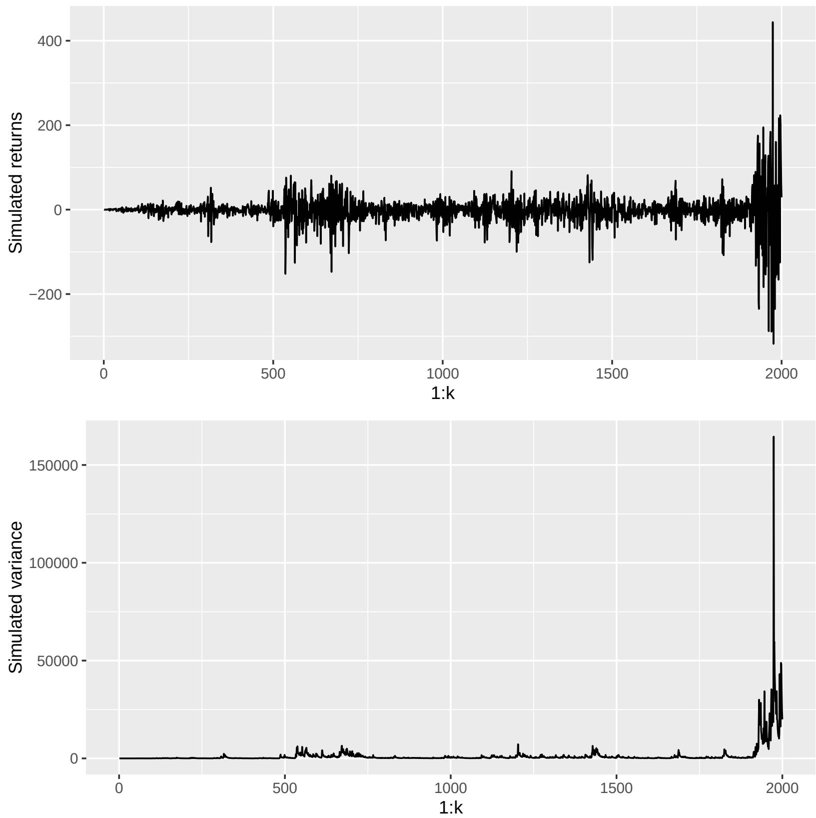

**Figure 3:** Simulated returns and conditional variance according to HYGARCH$(1, d, 1)$ process with $\alpha = -0.3$, $\beta = 0.7$, $\omega = 0.02$, $d = 0.7$, $\tau = 0.7$

## 2.4 FIAPARCH model

In order to account for often observed asymmetry in financial data, Fractionally Integrated Asymmetric Power ARCH model was introduced in Tse (1998). It has the following form:

$$\sigma_t^\delta = \omega(1 - \beta(1))^{-1} + (1 - (1 - \beta(L))^{-1}(1 - \phi(L))(1 - L)^d)(|r_t| - \gamma r_t)^\delta.$$

Where $\delta > 0$, $|\gamma| < 1$ and the rest of parameters are the same as in previous sections. Conditions for non-negativity are the same as in FIGARCH model. Unfortunately, as is the case with FIGARCH model, this model has no finite second moment and is not weakly stationary. In Figure, 4 simulated FIAPARCH$(1, d, 1)$ process is plotted with the same residuals as in previous sections.




[b]

**Figure 4:** Simulated returns and conditional variance according to FIAPARCH$(1, d, 1)$ process with $\alpha = -0.3$, $\beta = 0.7$, $\omega = 0.02$, $d = 0.7$, $\gamma = 0.33$, $\delta = 2.3$

## 2.5 HYAPARCH model

In order to solve the problem of infinite second moments in FIAPARCH, a new model was introduced in Schoffer (2003) in form of Hyperbolic Asymmetric Power ARCH (HYAPARCH) model:

$$\sigma_t^\delta = \omega(1-\beta(1))^{-1}+(1-(1-\beta(L))^{-1}(1-\phi(L))((1-\tau)-\tau(1-L)^d))(|r_t|-\gamma r_t)^\delta.$$

The parameters are the same as in previous sections. Non-negativity conditions are the same as in HYGARCH model. Unfortunately, the conditions for weak stationarity in Schoffer (2003) are derived from misspecified form of filter $\Psi$. This can be corrected by following Conrad (2010), Davidson (2004) and by following the proofs and conditions of weak stationarity in Seasonal HYAPARCH model, which were derived from Giraitis et al. (2000) in Diongue and Guegan (2007). Because the proofs for conditions do not depend on the form of the filter $\Psi$ in S-HYAPARCH, we can derive that the necessary condition for existence of second moments and weak stationarity are:

$$\sqrt{E((|\epsilon_t| - \gamma\epsilon_t)^{2\delta})}\Psi^{HY}(1) < 1.$$

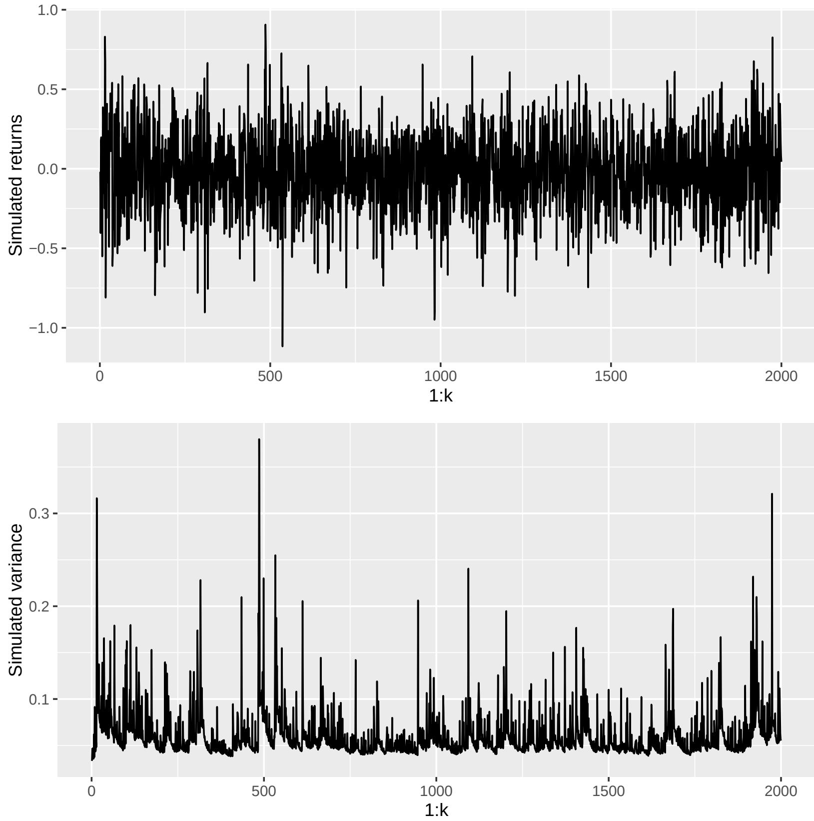
Following the simplification of HYGARCH case we can rewrite this condition in:

$$\sqrt{E((|\epsilon_t| - \gamma\epsilon_t)^{2\delta})}(\tau + (1-\tau)\Psi^{GA}(1)) < 1.$$

The form of $E(|\epsilon_t| - \gamma\epsilon_t)^{\delta}$ in the case of standard Student's t distribution with $v$ degrees of freedom can be found in Lambert and Laurent (2001) and has following form :

$$E((|\epsilon_t| - \gamma\epsilon_t)^{\delta}) = ((1+\gamma)^{\delta} + (1-\gamma)^{\delta})\frac{\Gamma(\frac{\delta+1}{2}\Gamma(\frac{v-\delta}{2})}{2\sqrt{(v-2)\pi}\Gamma(\frac{v}{2})}(v-2)^{\frac{\delta+1}{2}}.$$

In some literature APARCH model form has $(|r_t| - \gamma r_t)^{\delta}$ replaced with $(1+\gamma s_t)|r_t|^{\delta}$, where $s_t = 1$ if $r_t < 0$ and 0 otherwise. In this case the condition for weak stationarity would be $\sqrt{E((1+\gamma s_t)|r_t|^{\delta})^2}(\tau + (1-\tau)\Psi^{GA}(1)) < 1$. In Figure 5 simulated HYAPARCH$(1, d, 1)$ process is plotted with the same residuals as in previous sections.




[b]

**Figure 5:** Simulated returns and conditional variance according to HYAPARCH$(1, d, 1)$ process with $\alpha = -0.3$, $\beta = 0.7$, $\omega = 0.02$, $d = 0.7$, $\tau = 0.7$, $\gamma = 0.33$, $\delta = 2.3$

# 3 Empirical application on commodity prices data

## 3.1 Descriptive statistics

In this section, previously discussed models will be applied to estimate conditional variances of commodity price returns. Daily data sample on 12 commodities from 1995-01-03 to 2016-04-22, which was collected from www.quandl.com, will be used. The motivation behind particular series was to select a diverse set of commodities, which would allow us to analyse long memory properties in different industries. Description on what type of price indexes were used and how they will be refereed to in the following sections are presented in Table 1.

| | |
|---|---|
| corn | CBOT Chicago Corn Futures 1 |
| wheat | CBOT Wheat Futures 1 (W1) |
| gas | NYMEX Natural Gas Futures 1 (NG1) |
| oil | ICE Brent Crude Oil Futures 1 (B1) |
| sugar | ICE Sugar No. 11 Futures 1 (SB1) |
| coffee | ICE Coffee C Futures 1 (KC1) |
| cotton | ICE Cotton Futures 1 (CT1) |
| cattle | CME Live Cattle Futures 1 (LC1) |
| gold | NYMEX Gold Futures 1 (GC1) |
| silver | NYMEX Silver Futures 1 (SI1) |
| platinum | NYMEX Platinum Futures 1 (PL1) |
| copper | Copper Futures, Continuous Contract 1 (HG1) |

**Table 1:** Commodity futures names

Before estimation returns will be calculated by taking first differences of logarithm of the prices. The plots of original data and returns can be found in Appendix Figures 14–25. The descriptive statistics of all series are provided in Tables 2–4.

| | corn | wheat | sugar | coffee |
|---|---|---|---|---|
| Min | -0.0792939 | -0.100167 | -0.123658 | -0.150309 |
| Max | 0.0900753 | 0.0871062 | 0.104567 | 0.211999 |
| Mean | 9.25257e-005 | 3.15929e-005 | -1.61272e-006 | -5.93774e-005 |
| Median | 0 | 0 | 0 | 0 |
| Standard Deviation | 0.0169579 | 0.0189659 | 0.0205752 | 0.0236808 |
| Skewness | -0.0114753 | 0.0447287 | -0.194653 | 0.0998761 |
| Kurtosis | 5.18909 | 4.82083 | 5.1881 | 7.52818 |

**Table 2:** Descriptive statistics of corn, wheat, sugar, coffee returns

| | cotton | cattle | gas | oil |
|---|---|---|---|---|
| Min | -0.0892457 | -0.0939989 | -0.266429 | -0.144372 |
| Max | 0.116158 | 0.0505654 | 0.264486 | 0.128982 |
| Mean | -6.40499e-005 | 8.50425e-005 | 4.79139e-005 | 0.00019599 |
| Median | 0 | 0 | 0.000138962 | 0.000525799 |
| Standard Deviation | 0.0177185 | 0.00959017 | 0.0333915 | 0.0216567 |
| Skewness | -0.0628615 | -0.243925 | 0.0520468 | -0.122908 |
| Kurtosis | 5.21305 | 5.84294 | 6.5496 | 6.17395 |

**Table 3:** Descriptive statistics of cotton, cattle, gas, oil returns

| | gold | silver | platinum | copper |
|---|---|---|---|---|
| Min | -0.0981048 | -0.194976 | -0.0960331 | -0.116933 |
| Max | 0.0899886 | 0.124695 | 0.107617 | 0.11301 |
| Mean | 0.000222858 | 0.000240807 | 0.000168952 | 9.78707e-005 |
| Median | 2.83278e-005 | 0.00066608 | 0.000563653 | 0 |
| Standard Deviation | 0.010917 | 0.00066608 | 0.000563653 | 0.0177779 |
| Skewness | -0.0892878 | -0.768472 | -0.332582 | -0.261038 |
| Kurtosis | 9.84936 | 10.3555 | 7.2677 | 6.91435 |

**Table 4:** Descriptive statistics of gold, silver, platinum, copper returns

It can be observed from Tables 2–4 that all samples have mean close to 0 . We can also observe that the kurtosis for all samples is greater than 3, which indicates that distribution of the series have heavier tails than those of a normal distribution.

## 3.2 Model estimates

For each data sample, ARMA model with residuals modelled according to FIGARCH$(1, d, 1)$, HYGARCH$(1, d, 1)$, FIAPARCH$(1, d, 1)$ and HYAPARCH$(1, d, 1)$ will be estimated. Best ARMA order will be chosen by Akaike information criterion. The estimates are computed with conditional Maximum Likelihood Estimator (MLE) using Time Series Modelling 4.49 by James Davidson. The provided intercepts are in form of $\sqrt{\omega(1-\beta_1)^{-1}}$. Estimates of square root of Student's t degrees of freedom are also provided.

### 3.2.1 Estimates of ARMA-FIGARCH models

ARMA-FIGARCH model estimates for commodity returns can be found in Tables 5–7

| | corn | wheat | sugar | coffee |
|---|---|---|---|---|
| AR1 | – | – | -0.72033 (0.18399) | -0.47227(0.11996) |
| AR2 | – | – | – | -0.03824( 0.01388) |
| MA1 | -0.0626(0.01425) | – | -0.7365(0.17149) | -0.43825( 0.11941) |
| Student's t d.f. | 2.6325 (0.1121) | 3.30147 (0.2203) | 2.56487(0.1043) | 2.2641(0.0869) |
| GARCH Intercept | 0.00272 (0.0005) | 0.00379 (0.0008) | 0.00087(0.0002) | 0.00554(0.001) |
| FIGARCH $d$ | 0.50027 (0.0774) | 0.39368 (0.0814) | 1.01061(0.0432) | 0.32006 (0.0761 ) |
| $\alpha$ | -0.40238 (0.07703) | -0.35884 (0.08001) | -0.94548(0.04657) | -0.21262(0.07316) |
| $\beta$ | 0.6749 (0.07683) | 0.62529 (0.08612) | 0.96708(0.01186) | 0.64626(0.07246) |

**Table 5:** ARMA-FIGARCH estimates (standard deviation in brackets)

| | cotton | cattle | gas | oil |
|---|---|---|---|---|
| AR1 | – | 0.52761(0.08507) | -0.02836 (0.01297) | – |
| AR2 | – | – | – | – |
| MA1 | -0.04131(0.01507) | 0.46451(0.08757) | – | 0.04091(0.01429) |
| Student's t d.f. | 2.74468(0.1314) | 3.31422(0.3013) | 2.8227 (0.1611) | 2.69037( 0.1263) |
| GARCH Intercept | 0.00248(0.0004) | 0.00158(0.0003) | 0.00414 (0.0005) | 0.00268(0.0005) |
| FIGARCH $d$ | 0.49858( 0.0613) | 0.48609(0.1114) | 0.83823(0.0881) | 0.55231(0.08) |
| $\alpha$ | -0.38915( 0.06024) | -0.42835(0.10968) | -0.80548(0.09098) | -0.47515(0.07555) |
| $\beta$ | 0.71746(0.05579) | 0.7166(0.0814) | 0.83934(0.03732) | 0.74535(0.06591) |

**Table 6:** ARMA-FIGARCH estimates (standard deviation in brackets)

| | gold | silver | platinum | copper |
|---|---|---|---|---|
| AR1 | – | – | 0.56204 (0.13018) | – |
| AR2 | – | – | -0.04669 (0.01498) | – |
| MA1 | 0.02897(0.01327) | 0.05774(0.01301) | 0.53806 (0.13001) | 0.06451(0.01311) |
| Student's t d.f. | 2.24449 (0.0656) | 2.04701(0.0541) | 2.74444 (0.1233) | 2.34455(0.0871) |
| GARCH Intercept | 0.00121(0.0002) | 0.00249(0.0004) | 0.00184 (0.0004) | 0.00272 (0.0004) |
| FIGARCH $d$ | 0.43611(0.043) | 0.50226(0.0777) | 0.42379 (0.0529) | 0.49811 (0.0796) |
| $\alpha$ | -0.3968(0.04364) | -0.4141(0.0742) | -0.29904 (0.05216) | -0.44918(0.07984) |
| $\beta$ | 0.6851(0.04227) | 0.78285(0.04645) | 0.69052 (0.06337) | 0.71927(0.0598) |

**Table 7:** ARMA-FIGARCH estimates (standard deviation in brackets)

For estimation of conditional mean of corn, cotton, oil, gold, silver and copper returns, MA(1) was selected. For returns of gas, AR(1) was selected. For sugar returns ARMA(1,1) was selected. For returns of coffee and platinum, ARMA(2,1) was selected and for wheat, all of ARMA coefficients were insignificant. For all of the commodities FIGARCH$(1, d, 1)$ estimates match the conditions for conditional variance to be positive. Estimate for memory parameter of sugar exceeds 1 which suggest that it is integrated process. Largest memory parameter was observed for returns of gas which indicates weakest persistence of memory. All of the presented coefficients are statistically significant. In Figures 26–28 we present 1st–75th estimates of $\psi_i$ to represent the differences of information decay in each of the series.

### 3.2.2 Estimates of ARMA-HYGARCH models

In this section, estimates of ARMA-HYGARCH models are presented, coefficient estimates can be found in Tables 8–10.

| | corn | wheat | sugar | coffee |
|---|---|---|---|---|
| AR1 | – | – | -0.76665(0.26731) | -0.41379(0.12488) |
| AR2 | – | – | – | -0.03658(0.01482) |
| MA1 | -0.06255(0.01431) | – | -0.78131(0.249) | -0.38(0.12402) |
| Student's t d.f. | 2.65016(0.1192) | 3.36088(0.2215) | 2.52418(0.1086) | 2.32763(0.0885) |
| GARCH Intercept | 0.00283(0.0005) | 0.00447(0.0007) | 0.00225(0.0007) | 0.00732(0.001) |
| HYGARCH $d$ | 0.51378(0.0845) | 0.47856(0.0793) | 0.57413(0.2785) | 0.50761(0.0767) |
| HYGARCH $\tau$ | 0.98997(0.0199) | 0.94328(0.0253) | 1.00562(0.0235) | 0.86303(0.0425) |
| $\alpha$ | -0.41207(0.08174) | -0.41717(0.0763) | -0.50413(0.26437)* | -0.33451(0.07258) |
| $\beta$ | 0.68128(0.07734) | 0.66127(0.07269) | 0.79799(0.13841) | 0.68446(0.05378) |

**Table 8:** ARMA-HYGARCH estimates (standard deviation in brackets), *coefficient is not statistically significantly different from zero with 5% confidence

| | cotton | cattle | gas | oil |
|---|---|---|---|---|
| AR1 | – | 0.5243 (0.08654) | -0.02852(0.01333) | – |
| AR2 | – | – | – | – |
| MA1 | -0.04158(0.01516) | 0.46152 (0.0891) | – | 0.04095(0.01427) |
| Student's t d.f. | 2.77678(0.1348) | 3.41157(0.3076) | 2.94627(0.175) | 2.68216 (0.1328) |
| GARCH Intercept | 0.00285(0.0006) | 0.00135(0.0003) | 0.0053(0.0006) | 0.0026(0.0007) |
| HYGARCH $d$ | 0.52442(0.0653) | 0.94363 (0.078) | 0.93008(0.0535) | 0.54945(0.0789) |
| HYGARCH $\tau$ | 0.98009(0.021) | 0.98001 (0.0113) | 0.9766 (0.0072) | 1.00304( 0.016) |
| $\alpha$ | -0.40705(0.0623) | -0.88175 (0.08826) | -0.87821 (0.05699) | -0.47339(0.0744) |
| $\beta$ | 0.7219(0.05409) | 0.91379(0.03504) | 0.8644(0.02207) | 0.7453(0.06589) |

**Table 9:** ARMA-HYGARCH estimates (standard deviation in brackets)

| | gold | silver | platinum | copper |
|---|---|---|---|---|
| AR1 | – | – | 0.56079(0.12899) | – |
| AR2 | – | – | -0.04688(0.0149) | – |
| MA1 | 0.02967(0.013) | 0.05825(0.01293) | 0.53688(0.12881) | 0.06422(0.01322) |
| Student's t d.f. | 2.13865(0.0735) | 2.00393(0.0607) | 2.71428(0.1252) | 2.38504(0.093) |
| GARCH Intercept | 0.0009(0.0003) | 0.00206(0.0006) | 0.0016(0.0005) | 0.00319(0.0006) |
| HYGARCH $d$ | 0.37434(0.0603) | 0.47458(0.0812) | 0.40397(0.0588) | 0.53934(0.0962) |
| HYGARCH $\tau$ | 1.07201(0.0355) | 1.03032(0.0243) | 1.02171(0.0262) | 0.97238(0.0204) |
| $\alpha$ | -0.35759(0.05663) | -0.39652(0.07455) | -0.28484(0.05585) | -0.4778(0.0954) |
| $\beta$ | 0.65828(0.05721) | 0.78441(0.04833) | 0.68662(0.06818) | 0.72821(0.06517) |

**Table 10:** ARMA-HYGARCH estimates (standard deviation in brackets)

All the presented HYGARCH models have sufficient estimates for non-negativity of conditional variance. As it was in FIGARCH case, all of the $\alpha_1$ are negative and $\beta_1$ estimates are positive. All coefficient estimates except the $\alpha_1$ of sugar series are statistically significant. ARMA model order was chosen the same as previously. For sugar, oil, gold, silver, platinum and copper, we have $\tau$ exceeding 1 which result in non-stationarity. For the rest of the models, estimates match the stationarity conditions presented in Section 2. In Figures 29–31 we present 1st–75th estimates of $\psi_i$ to represent the differences of information decay in each of the series.

### 3.2.3 Estimates of ARMA-FIAPARCH models

In this section estimates, of ARMA-FIAPARCH models are presented which can be found in Tables 11–13.

| | corn | wheat | sugar | coffee |
|---|---|---|---|---|
| AR1 | – | – | -0.7213(0.19506) | -0.40349(0.11965) |
| AR2 | – | – | – | -0.03484(0.01497) |
| MA1 | -0.06232(0.01425) | – | -0.73749(0.18198) | -0.3698(0.1186) |
| Student's t d.f | 2.63933(0.1164) | 3.46831(0.2389) | 2.58735(0.1092) | 2.41369(0.095) |
| GARCH Intercept | 0.00561(0.0024) | 0.00501(0.0027) | 0.00213(0.0017) | 0.00833(0.0037) |
| FIGARCH $d$ | 1.02326(0.0584) | 0.49385(0.0848) | 0.99842(0.0488) | 0.61382(0.0865) |
| APARCH asymmetry $\gamma$ | 0.01603(0.07745)* | -0.426(0.12437) | 0.16055(0.16305)* | -0.80173(0.08086) |
| APARCH Power $\delta$ | 1.44856(0.174) | 1.99117(0.2676) | 1.66002(0.2803) | 1.94151(0.2231) |
| $\alpha$ | -0.93256(0.06144) | -0.44389(0.08533) | -0.93274(0.05179) | -0.45778(0.09033) |
| $\beta$ | 0.93346(0.01656) | 0.68214(0.08458) | 0.96405(0.01351) | 0.74831(0.05) |

**Table 11:** ARMA-FIAPARCH estimates (standard deviation in brackets), *coefficient is not statistically significantly different from zero with 5% confidence

| | cotton | cattle | gas | oil |
|---|---|---|---|---|
| AR1 | – | 0.51858(0.08872) | -0.02987(0.01343) | – |
| AR2 | – | – | – | – |
| MA1 | -0.04085(0.01518) | 0.45553(0.09093) | – | 0.0403(0.01429) |
| Student's t d.f | 2.76961(0.1319) | 3.33343(0.2931) | 2.99601(0.1771) | 2.67228(0.1296) |
| GARCH Intercept | 0.00721 (0.0037) | 0.00739(0.0023) | 0.00997(0.0051) | 0.00591(0.0035) |
| FIGARCH $d$ | 0.53646(0.0715) | 0.50299(0.1329) | 0.94591(0.0616) | 0.54923(0.0936) |
| APARCH asymmetry $\gamma$ | 0.0836(0.10417)* | 0.21062(0.11867)* | -0.27159(0.12468) | 0.25512(0.14145)* |
| APARCH Power $\delta$ | 1.60623(0.1935) | 1.4224(0.1172) | 1.66107(0.258) | 1.63822(0.2488) |
| $\alpha$ | -0.43031(0.07089) | -0.44794(0.13103) | -0.90537(0.06007) | -0.4694(0.08973) |
| $\beta$ | 0.73242(0.05678) | 0.71879(0.09428) | 0.87372(0.02153) | 0.74814(0.07289) |

**Table 12:** ARMA-FIAPARCH estimates (standard deviation in brackets), *coefficient is not statistically significantly different from zero with 5% confidence

| | gold | silver | platinum | copper |
|---|---|---|---|---|
| AR1 | – | – | 0.56492(0.13194) | – |
| AR2 | – | – | -0.04592(0.01506) | – |
| MA1 | 0.02883(0.01315) | 0.06003(0.01329) | 0.5415(0.13187) | 0.06485(0.01353) |
| Student's t d.f | 2.20313(0.0736) | 2.06999(0.0591) | 2.748(0.1249) | 2.37073(0.0923) |
| GARCH Intercept | 0.00072(0.0006) | 0.00106(0.0008) | 0.00079(0.0008) | 0.00917(0.0067) |
| FIGARCH $d$ | 0.41418(0.0556) | 0.54797(0.1083) | 0.43002(0.0585) | 0.61159(0.7179) |
| APARCH asymmetry $\gamma$ | -0.02409(0.17689)* | -0.37123(0.16658) | -0.26383(0.19663)* | 0.21788(0.15621)* |
| APARCH Power $\delta$ | 2.14119(0.1991) | 2.36901(0.2515) | 2.32895(0.2975) | 1.44789(0.5902) |
| $\alpha$ | -0.37386(0.05657) | -0.45159(0.107) | -0.29684(0.06078) | -0.56927(0.72577)* |
| $\beta$ | 0.66773(0.05155) | 0.80455(0.05393) | 0.69846(0.0657) | 0.78334(0.39454) |

**Table 13:** ARMA-FIAPARCH estimates (standard deviation in brackets), *coefficient is not statistically significantly different from zero with 5% confidence

Conclusions about $\alpha$ and $\beta$ coefficients are similar as in previous sections except that for copper the $\alpha_1$ parameter was found to be statistically insignificant. Order of ARMA model was selected the same as in previous sections. Coefficients $\alpha$ and $\beta$ are sufficient for conditional variance to be nonnegativee. Memory parameter $d$ for corn was estimated to be larger than 1 which suggest that process might be integrated. Asymmetry parameter was found significant only for wheat, coffee, gas, and silver returns. In all of these cases the asymmetry parameter was selected negative, which indicates that negative returns increases volatility more than positive returns.

### 3.2.4 Estimates of ARMA-HYAPARCH models

In this section, the results of ARMA-HYAPARCH models are presented, estimates can be found in Tables 14–16.

| | corn | wheat | sugar | coffee |
|---|---|---|---|---|
| AR1 | – | – | -0.72131(0.19516) | -0.41113(0.1129) |
| AR2 | – | – | – | -0.03335(0.01481) |
| MA1 | -0.06162(0.01421) | – | -0.73749(0.18208) | -0.37744(0.11179) |
| Student's t d.f | 2.61859(0.1151) | 3.48922(0.2485) | 2.58724(0.1101) | 2.38793(0.0956) |
| GARCH Intercept | 0.0075(0.0038) | 0.02054(0.0079) | 0.00214(0.0017) | 0.01479(0.0059) |
| HYGARCH $d$ | 1.01564(0.0679) | 0.40307(0.0741) | 0.9984(0.0493) | 0.51665(0.0692) |
| HYGARCH $\tau$ | 1.01001(0.0089) | 1.22053(0.0873) | 1.00004(0.0041) | 1.13049(0.0581) |
| APARCH asymmetry $\gamma$ | -0.14916(0.14727)* | -0.65696(0.09427) | 0.15938(0.20774)* | -0.83582(0.07294) |
| APARCH Power $\delta$ | 1.293(0.2349) | 1.27002(0.1733) | 1.65902(0.301) | 1.59834(0.2052) |
| $\alpha$ | -0.93032(0.06934) | -0.41167(0.06338) | -0.93272(0.05203) | -0.41958(0.06785) |
| $\beta$ | 0.93272(0.01735) | 0.70116(0.0743) | 0.96405(0.01354) | 0.74907(0.04532) |

**Table 14:** ARMA-HYAPARCH estimates (standard deviation in brackets),*coefficient is not statistically significantly different from zero with 5% confidence

| | cotton | cattle | gas | oil |
|---|---|---|---|---|
| AR1 | – | 0.52212(0.08549) | -0.03028(0.01331) | – |
| AR2 | – | – | – | – |
| MA1 | -0.03992(0.01529) | 0.4573(0.08795) | – | 0.0402(0.01435) |
| Student's t d.f | 2.75181(0.1322) | 3.48061(0.328) | 2.98617(0.1778) | 2.68569(0.1333) |
| GARCH Intercept | 0.00814(0.0037) | 0.00935(0.0046) | 0.01386(0.0058) | 0.00578(0.0038) |
| HYGARCH $d$ | 0.52492(0.0709) | 0.53033(0.1522) | 0.92253(0.071) | 0.54537(0.0877) |
| HYGARCH $\tau$ | 1.03603(0.0392) | 0.78852(0.135) | 1.01879(0.0119) | 0.98048(0.0405) |
| APARCH asymmetry $\gamma$ | -0.01329(0.14697)* | 1.38275(0.67651) | -0.34243(0.10658) | 0.33563(0.23709)* |
| APARCH Power $\delta$ | 1.51913(0.1808) | 1.47707(0.2092) | 1.44485(0.2227) | 1.67487(0.2818) |
| $\alpha$ | -0.43282(0.06966) | -0.38711(0.18058) | -0.89373(0.06673) | -0.45798(0.0927) |
| $\beta$ | 0.74127(0.05809) | 0.63917(0.12987) | 0.87273(0.02313) | 0.73655(0.07972) |

**Table 15:** ARMA-HYAPARCH estimates (standard deviation in brackets),*coefficient is not statistically significantly different from zero with 5% confidence

| | gold | silver | platinum | copper |
|---|---|---|---|---|
| AR1 | – | – | 0.56973(0.13142) | – |
| AR2 | – | – | -0.04581(0.01494) | – |
| MA1 | 0.03492(0.01334) | 0.06193(0.01288) | 0.54668(0.13126) | 0.06292(0.01326) |
| Student's t d.f | 2.1439(0.0747) | 2.01694(0.0596) | 2.71179(0.125) | 2.38935(0.0939) |
| GARCH Intercept | 0.00592(0.0033) | 0.00348(0.0022) | 0.0021(0.0014) | 0.00324(0.002) |
| HYGARCH $d$ | 0.45061(0.0881) | 0.9833(0.0666) | 0.42895(0.0697) | 0.31065(0.0356) |
| HYGARCH $\tau$ | 1.22133(0.0719) | 1.02727(0.0096) | 1.13771(0.0512) | 0.72847(0.1195) |
| APARCH asymmetry $\gamma$ | -0.43826(0.08721) | -0.55091(0.07825) | -0.34914(0.11575) | 0.49213(0.35025)* |
| APARCH Power $\delta$ | 1.31181(0.1788) | 1.40393(0.2457) | 1.86448(0.2329) | 2.29466(0.2888) |
| $\alpha$ | -0.50108(0.08231) | -0.92285(0.06705) | -0.33877(0.07048) | – |
| $\beta$ | 0.76825(0.06121) | 0.96294(0.00949) | 0.7396(0.06868) | 0.20341(0.04381) |

**Table 16:** ARMA-HYAPARCH estimates (standard deviation in brackets),*coefficient is not statistically significantly different from zero with 5% confidence

Order of ARMA was again selected the same as in previous sections. Estimates of $\alpha_1$ and $\beta_1$ have the same signs as before except that for copper returns the HYAPARCH$(1, d, 1)$ parameter search during model estimation did not converge and HYAPARCH$(0, d, 1)$ model was chosen. For wheat and gold models the conditions of non-negative conditional variance were not met and because of that these models are incorrect. Estimates for asymmetry were signifficant for coffee, cattle, gas, gold, silver and platinum. Asymmetry parameter for cattle was estimated to be larger than 1 which is not allowed by model specification, the rest of the asymmetry parameters are negative which indicates increase in volatility when returns are negative. For all of the models with significant asymmetry, the HYAPARCH parameter $\tau$ was larger than 1 which indicates that data generating process is not weakly stationary.

### 3.2.5 Model comparison

In Table 17 we provide Akaike information criterion (AIC) values. The values are reversed so that bigger AIC means that model captured more information.

| | FIGARCH | HYGARCH | FIAPARCH | HYAPARCH |
|---|---|---|---|---|
| corn | 14505 | 14504.1 | **14508.7** | **14508.7** |
| wheat | 13741.6 | 13743.1 | 13751.9 | **13759.4** |
| sugar | **13406.4** | 13397.3 | 13403.3 | 13402.3 |
| coffee | 12652.3 | 12656.2 | 12675.2 | **12676.8** |
| cotton | **14241** | 14240.4 | 14238.1 | 14237.4 |
| cattle | 17263.8 | 17267.9 | 17266.4 | **17276.3** |
| gas | 10898.1 | 10903.1 | 10906.8 | **10907.2** |
| oil | **13311.8** | 13310.8 | 13308.9 | 13308.1 |
| gold | 17209.8 | 17213.1 | 17206.7 | **17222** |
| silver | 14180.3 | 14180.2 | 14178.3 | **14193.6** |
| platinum | **15616.2** | 15615.5 | 15613.6 | 15615.6 |
| copper | **14396.4** | 14396.1 | 14394.4 | 14381.5 |

**Table 17:** Inverted AIC values

From Table 17 we can see that all AIC values are very similar which suggests that they retrieve similar amount of information. For corn the highest AIC value is for asymmetric models. From previous sections we can observe that the asymmetry parameter estimates are statistically insignificant and, since the FIGARCH model is non-stationary we should pick HYGARCH model since it has $\tau$ estimate smaller than 1 which suggests weak stationarity. For wheat and gold, we obtain that AIC is best for HYAPARCH specification, but, since this models exhibit negative conditional variance, we choose HYGARCH specification because it is weakly stationary. Similar decisions are made for coffee, cotton, cattle, gas, oil and copper models, and HYGARCH models are selected. For silver, since it gives best AIC and since all of the models are non-stationary, we chose HYAPARCH model. Lastly, for platinum AIC is largest for FIGARCH model, but since Conrad (2010) showed that when $\tau$ value of the data generating process is larger than 1, HYGARCH model estimates are better and, since we have significant estimate of asymmetry parameter, we choose HYAPARCH model as the final. For these chosen models Box-Pierce test of serial correlation was performed on residuals and squared residuals, the chosen lag was 16. For all, except cattle and gold, the null hypothesis of serial correlation in residuals is rejected. For squared residuals, null hypothesis is rejected for all models except for gold, silver and copper. For silver, this problem is solved by choosing HYGARCH model which rejects hypothesis of serial correlation in squared residuals. Unfortunately, for cattle, gold and copper, no specification results in rejection of serial correlation hypothesis, which suggest that other type of models should be used. In Figures 32–43, we provide estimates of conditional variance for chosen final models.

## 3.3 Parameter stability after economical crisis of 2008-2009

In this section we are going to check whether there is a significant change in parameters after the 2008-2009 crisis. For this purpose, we split data into 2 parts: first is from 1995-01-05 to 2009-06-01 and the second part is from 2009-06-02 to 2016-04-21. We then estimate the selected models in previous section and check whether there is a change in parameters. Unfortunately, for wheat, corn, gas and silver models in the post-crisis period, the parameter search using ML estimation did not converge, which suggest model misspecification. The converging specification was not found, so we leave those models out. In Tables 18–21, we can observe the estimates of the rest for the models.

| | Sugar I | Sugar II | Coffee I | Coffee II |
|---|---|---|---|---|
| AR1 | -0.76806(0.21436) | 0.95421(0.03426) | -0.27291(0.12413) | 0.42343(0.13783) |
| AR2 | – | – | -0.04559(0.016) | 0.19615(0.1063) |
| MA1 | -0.78988(0.19431) | 0.93982(0.03597) | -0.24138(0.12288) | 0.46506(0.13815) |
| Student's t d.f. | 2.43417(0.1182) | 2.73763(0.2334) | 2.19673(0.0918) | 2.80818(0.2581) |
| Garch intercept | 0 | 0 | 0.00792(0.0011) | 0 |
| HYGARCH $d$ | 0.52664(0.1922) | 0.40787(0.0662) | 0.56593(0.0827) | 0.19615(0.1063) |
| HYGARCH $\tau$ | 1.04037(0.0355) | 1.03109(0.0265) | 0.86168(0.046) | 1.1905(0.2174) |
| $\alpha$ | -0.44536(0.0355) | -0.39992(0.06665) | -0.37085(0.07927) | -0.14965(0.07607) |
| $\beta$ | 0.81987(0.07829) | 0.58182(0.09784) | 0.68961(0.05717) | 0.74915(0.09427) |

**Table 18:** Sugar and coffee model estimates for periods I and II

In Table 18 we can see that for both sugar and coffee model the estimates of ARMA part of the model are different between period I and II. This suggests that a structural break in data generating process should be taken into account. For sugar model, there seems to be a change in parameter $d$, though, for both of the periods, it is still within 2 sigma interval of the estimates in full model. For coffee model, the memory parameter $d$ was reduced by half in second period, which suggests that long memory properties of this series have changed. For both models $\alpha$ and $\beta$ parameters are also different between the periods, though the signs of these parameters stay the same. Combining all of these results, we conclude that there is significant evidence that there has been a structural break in the series which needs to be taken into account. In Figures 6 and 7 we can see first 75 coefficients $\psi_i$

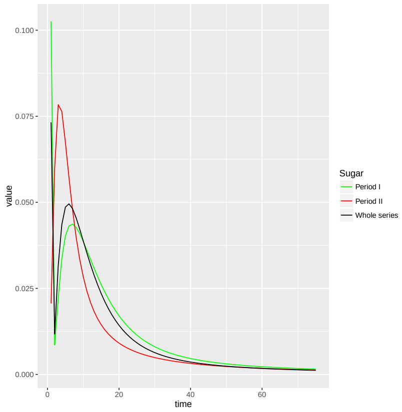



**Figure 6:** First 75 $\psi_i$ coefficients for periods I, II and complete series of sugar

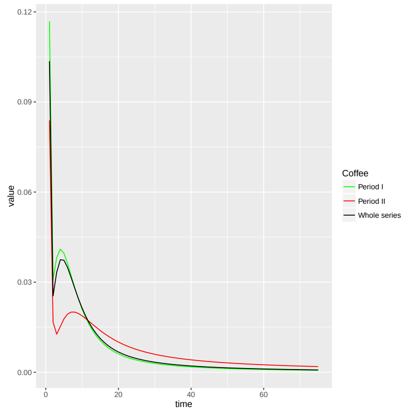



**Figure 7:** First 75 coefficients $\psi_i$ for periods I, II and complete series of coffee

From the figures, we can see that in sugar model the initial response to shock would be higher in second period, but it would decay faster and would be close to period 1 and full model after 50 steps. In coffee parameters, we can see that initial response to shocks is weaker but it also decays slower and is larger for more than 70 steps.

| | Cotton I | Cotton II | Cattle I | Cattle II |
|---|---|---|---|---|
| AR1 | | | 0.54793(0.11494) | 0.51889(0.11191) |
| AR2 | | | | |
| MA1 | -0.00856(0.01906)* | -0.0971(0.02486) | 0.49325(0.11837) | 0.44006(0.1148) |
| Student's t d.f. | 2.61635(0.1407) | 3.18621(0.3372) | 3.27922(0.3406) | 4.02546(0.7551) |
| Garch intercept | 0.00282(0.0008) | 0.00227(0.0024) | 0.00167(0.0004) | 0.00061(0.0003) |
| HYGARCH d | 0.49176(0.0969) | 0.41065(0.1372) | 0.95551(0.0808) | 0.91703(0.0935) |
| HYGARCH $\tau$ | 0.99794(0.0296) | 0.97156(0.0714) | 0.9703(0.0155) | 0.99684(0.0058) |
| $\alpha$ | -0.40367(0.08972) | -0.27483(0.11117) | -0.88038(0.09507) | -0.89728(0.10013) |
| $\beta$ | 0.72008(0.07791) | 0.62333(0.10646) | 0.90279(0.03895) | 0.94135(0.02854) |

**Table 19:** cotton and cattle model estimates for periods I and II

In Table 19 we can see that MA coefficient for cotton model in period I is insignificant and for period II it becomes significant, which suggests a structural break in the ARMA model. For cattle the AR and MA coefficients are within each others 2 sigma intervals, which suggests that structural break is insignificant. There does seem to be a small change in parameters $d$ and $\tau$

for both models, but they are within 2 standard deviations of the full model, which suggests that the long memory properties did not change significantly. For cotton there is significant change in $\alpha$ and $\beta$, this suggests that structural break might exist. For cattle model, $\alpha$ and $\beta$ are both within each others 2 sigma intervals, which suggest that significant change did not occur. In Figures 8 and 9 we can see first 75 coefficients $\psi_i$

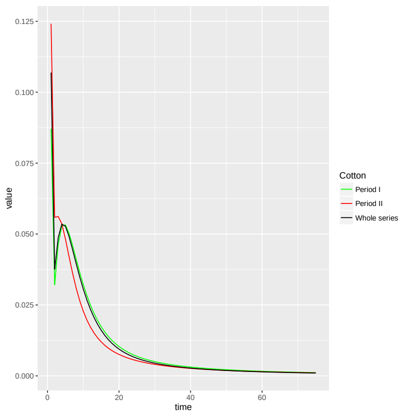



**Figure 8:** First 75 coefficients $\psi_i$ for periods I, II and complete series of cotton

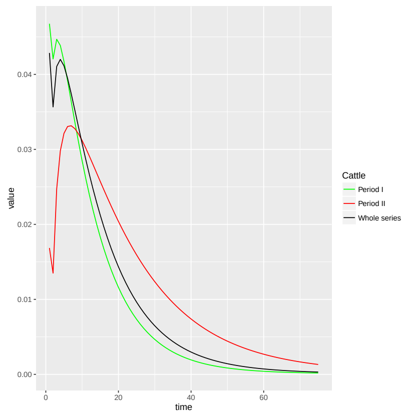



**Figure 9:** First 75 coefficients $\psi_i$ for periods I, II and complete series of cattle

From the figures we can see that parameters for cotton are very similar. For cattle model, the initial shocks for period II are weaker but they also decay slower and do not reach the coefficients of period I and full model for 70+ steps.

| | Oil I | Oil II | Gold I | Gold II |
|---|---|---|---|---|
| AR1 | | | | |
| AR2 | | | | |
| MA1 | 0.03899(0.0181) | 0.05012(0.02415) | 0.01671(0.01666)* | 0.05612(0.02084) |
| Student's t d.f. | 2.84706(0.1907) | 2.49792(0.2001) | 2.15105(0.0915) | 2.12751(0.1297) |
| Garch intercept | 0.00505(0.0008) | 0 | 0.00066(0.0004) | 0.00318(0.001) |
| HYGARCH $d$ | 0.34106(0.0751) | 0.46686(0.0905) | 0.31449(0.0852) | 0.43678(0.2305) |
| HYGARCH $\tau$ | 1.00549(0.0497) | 1.01062(0.0261) | 1.15214(0.0925) | 0.90181(0.0772) |
| $\alpha$ | -0.30327(0.06446) | -0.30936(0.0854) | -0.29914(0.07467) | -0.38838(0.20211) |
| $\beta$ | 0.60302(0.07562) | 0.72814(0.11991) | 0.62614(0.07811) | 0.64755(0.17052) |

**Table 20:** Oil and Gold model estimates for periods I and II

In Table 20, we can see that MA coefficient of oil models in both periods are similar and within one sigma interval of each other. In gold model, the MA coefficient in first period is insignificant and in II period it becomes significant, which suggests that structural break in the conditional mean model does occur. For both models the parameter $d$ is increased by about 0.1 going from period I to period II, which means that long memory properties of these series might have changed. Though, in gold case it is still within 2 sigma interval the full series model. Parameters $\tau$ for oil are within one sigma interval of each other in Period 1 and 2, for gold the model in period II becomes stationary because $\tau$ becomes smaller than 1. GARCH parameters retain same signs in both periods and is within 2 sigma interval of each other, though we do observe small decrease in $\alpha$ and $\beta$ in both models going from period I to period II. In Figures 10 and 11 we can see first 75 coefficients $\psi_i$.

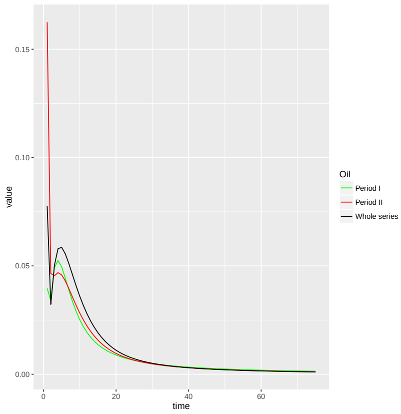



**Figure 10:** First 75 coefficients $\psi_i$ for periods I, II and complete series of oil

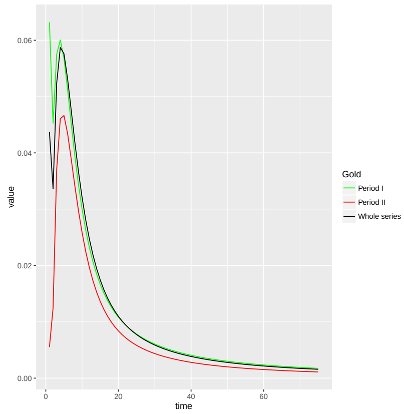



**Figure 11:** First 75 coefficients $\psi_i$ for periods I, II and complete series of gold

From these figures we can observe that in oil models the initial shock is the strongest in model for period II and it quickly converges to coefficients of period I. For steps 5–25, the full series model has stronger memory but it quickly decays and becomes similar to the periods I and II from step 25. For gold model the coefficients seem to be smaller in period II than in full series or period I for up to 75 steps which suggests that memory is weaker in this period.

| | Platinum I | Platinum II | Copper I | Copper II |
|---|---|---|---|---|
| AR1 | 0.55827(0.10847) | 0.99667(0.03642) | | |
| AR2 | -0.05853(0.01884) | -0.04423(0.0226)* | | |
| MA1 | 0.54569(0.10787) | 0.95535(0.02857) | 0.07003(0.01638) | 0.04956(0.02292) |
| Student's t d.f. | 2.52543(0.129) | 3.40702(0.3754) | 2.29818(0.0993) | 2.73761(0.2475) |
| Garch intercept | 0 | 0.01562(0.0268) | 0.0041(0.0008) | 0 |
| HYGARCH d | 0.54569(0.10787) | 0.37146(0.2589) | 0.39174(0.1001) | 0.42286(0.0958) |
| HYGARCH $\tau$ | 1.19182(0.0743) | 1.03896(0.1438) | 0.99354(0.0479) | 0.98214(0.022) |
| APARCH asymmetry $\gamma$ | -0.36112(0.12062) | 0.09259(0.55304)* | | |
| APARCH power $\delta$ | 1.97623(0.235) | 1.20645(0.8397) | | |
| $\alpha$ | -0.29628(0.07814) | -0.31627(0.19612)* | -0.34796(0.08837) | -0.36046(0.09911) |
| $\beta$ | 0.76094(0.08955) | 0.65599(0.16186) | 0.63031(0.09391) | 0.69947(0.07378) |

**Table 21:** platinum and copper model estimates for periods I and II

In Table 21 we can observe, that for Platinum model, in period II, due to increased standard deviations, the AR2 and $\alpha$ parameters become insignificant. Asymmetry parameter in second period also becomes insignificant. We can also observe that in period II, due to increased standard deviation, the memory parameter $d$ becomes close to 0 and $\tau$ is close to 1, which suggests that in second period proper data generating process does not show significant long memory properties.

In Figures 12 and 13 we can see first 75 coefficients $\psi_i$

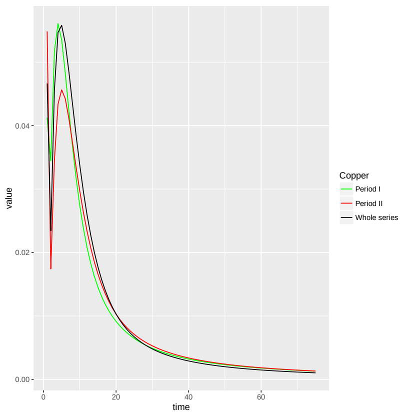



**Figure 12:** First 75 coefficients $\psi_i$ for periods I, II and complete series of copper

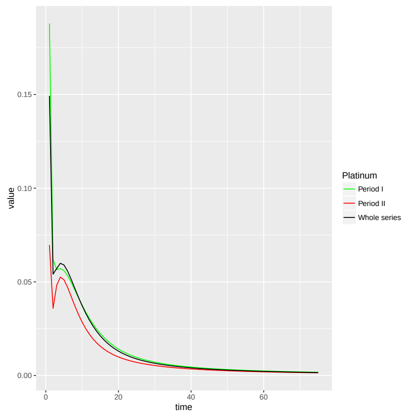



**Figure 13:** First 75 coefficients $\psi_i$ for periods I, II and complete series of platinum

From these figures we can see that for both commodities the series converge to the similar values quickly and initial coefficients for period II in both cases are smaller than for whole series and period I. For platinum, the coefficients are very similar but since we previously observed that standard deviation for period 2 parameters are high so this result might be misleading. For all of these models Box-Pierce serial correlation test with 16 lags was performed and no significant serial correlation in residuals and squared residuals was found.

# 4 Conclusion

In this thesis we have discussed 4 long memory models, FIGARCH, HYGARCH, FIAPARCH and HYAPARCH, and their properties. By using properties of HYGARCH model and established proofs of Seasonal HYAPARCH model, we have also introduced corrected form of stationarity constrains for HYAPARCH model. An empirical analysis of 12 commodity futures price returns data using these models was performed. We showed that all of the series presented excess kurtosis, so assumption of errors distributed by standard Student's t distribution was concluded. After estimation we found that the best models, based on AIC, in most cases were HYAPARCH. Unfortunately, due to negative condition variance for wheat and gold, and large parameter $\tau$ in other models, which suggested that chosen data generating process was not weakly stationary, we decided that the best model for all except platinum series was HYGARCH. For platinum model no weakly stationary model was found. Further suggestion would be to find a modification for HYAPARCH model which would widen the parameter set, that would allow us to take into account the asymmetric properties observed in some of the series. We also showed that all of the series have coefficient change after the global economical crisis of 2008-2009, which suggests that structural break might have happened. The memory properties slightly changed for most of the series. They all except platinum were still showing presence of long memory. In both periods no significant change in the signs of GARCH parameters were observed. Further analysis in order to take the change in parameters needs to be done. One of the suggestions is to apply a dynamic version of the models which would allow time varying coefficients.

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# 6 Appendix

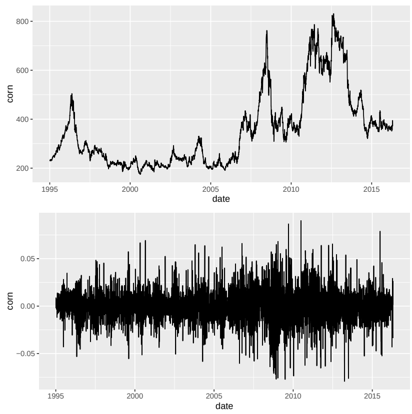


**Figure 14:** Prices of corn futures and their first log differences

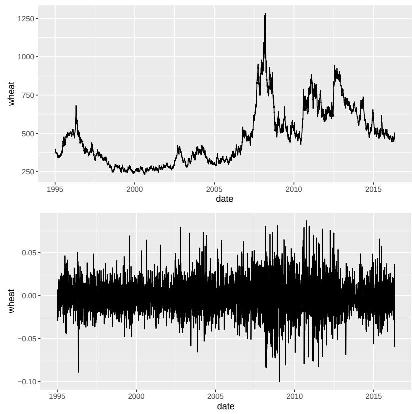


**Figure 15:** Prices of wheat futures and their first log differences

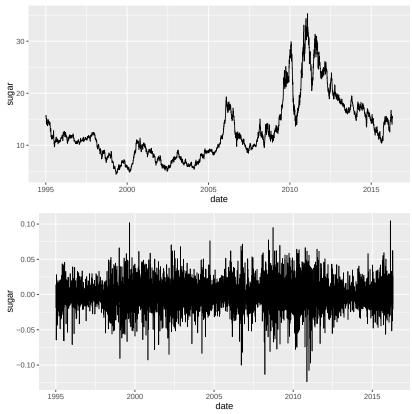


**Figure 16:** Prices of sugar futures and their first log differences

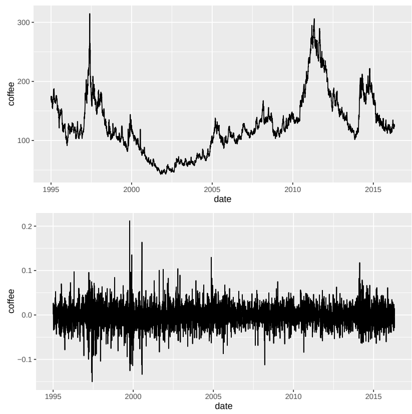


**Figure 17:** Prices of coffee futures and their first log differences

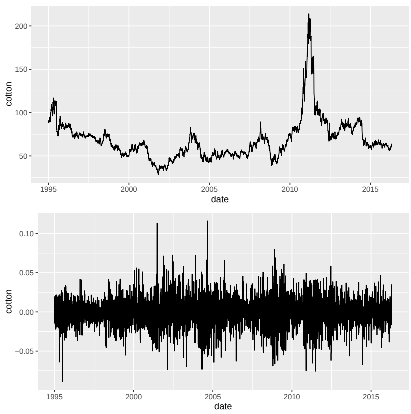


**Figure 18:** Prices of cotton futures and their first log differences

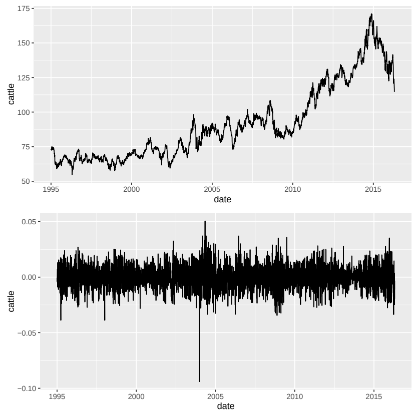


**Figure 19:** Prices of cattle futures and their first log differences

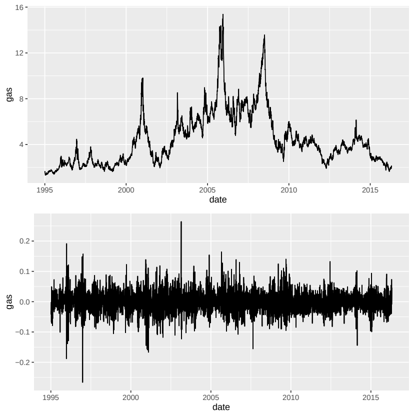


**Figure 20:** Prices of natural gas futures and their first log differences

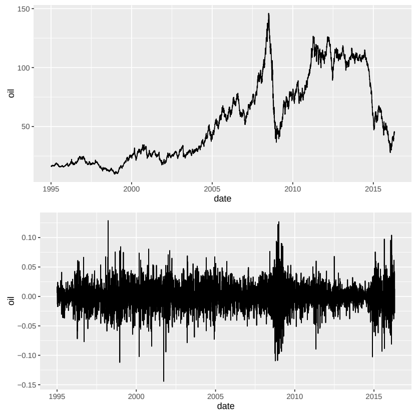


**Figure 21:** Prices of Brent crude oil futures and their first log differences

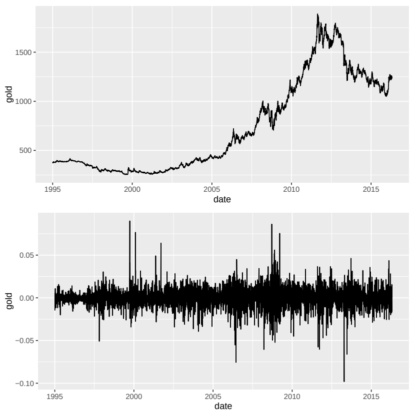


**Figure 22:** Prices of gold futures and their first log differences

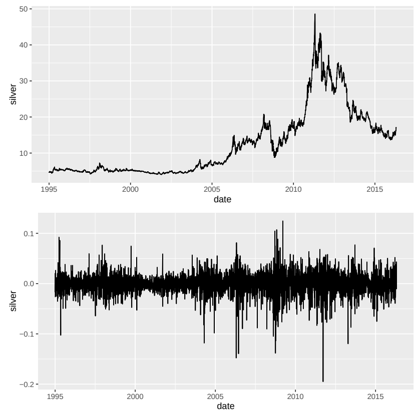


**Figure 23:** Prices of silver futures and their first log differences

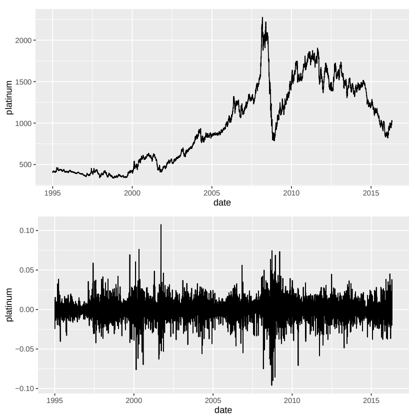


**Figure 24:** Prices of platinum futures and their first log differences

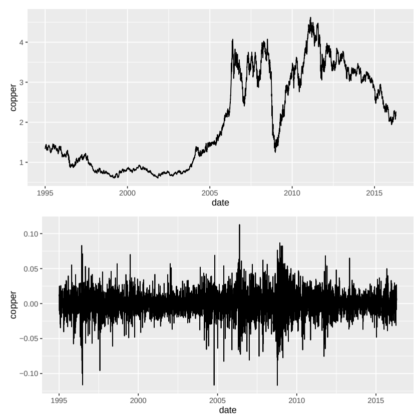


**Figure 25:** Prices of copper futures and their first log differences

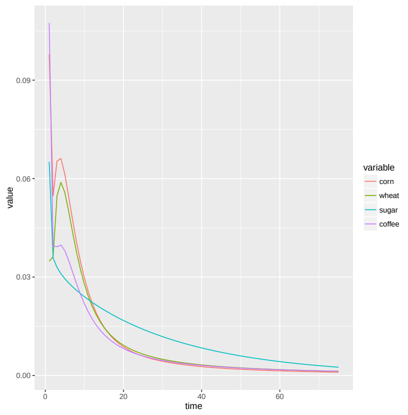


**Figure 26:** 1st-75th estimates of FIGARCH $\psi_i$

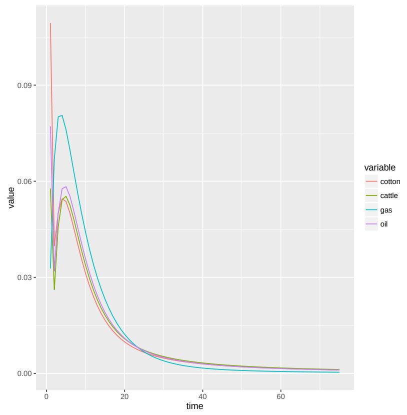


**Figure 27:** 1st-75th estimates of FIGARCH $\psi_i$

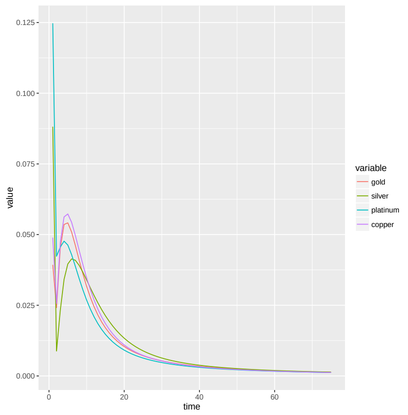


**Figure 28:** 1st-75th estimates of FIGARCH $\psi_i$

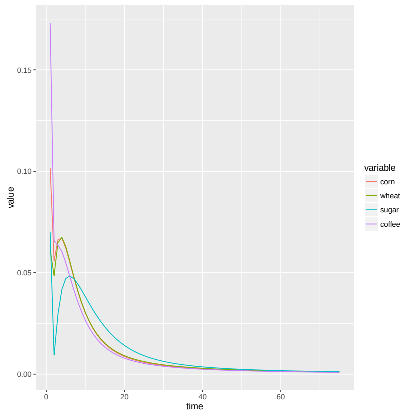


**Figure 29:** 1st-75th estimates of HYGARCH $\psi_i$

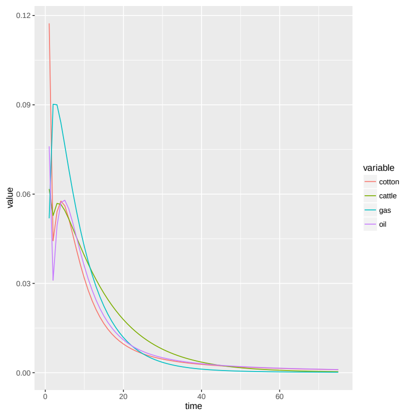


**Figure 30:** 1st-75th estimates of HYGARCH $\psi_i$

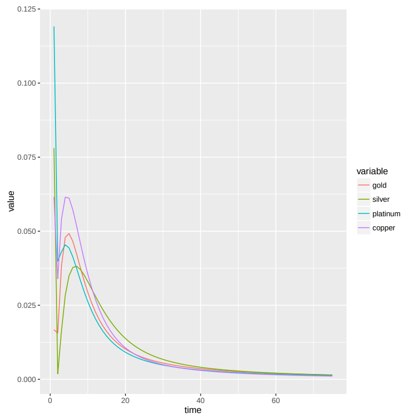


**Figure 31:** 1st-75th estimates of HYGARCH $\psi_i$

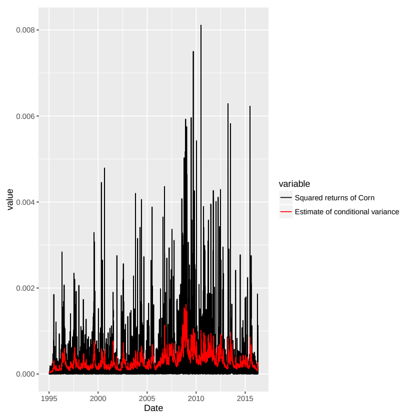


**Figure 32:** Estimated conditional variance of corn returns and squared returns

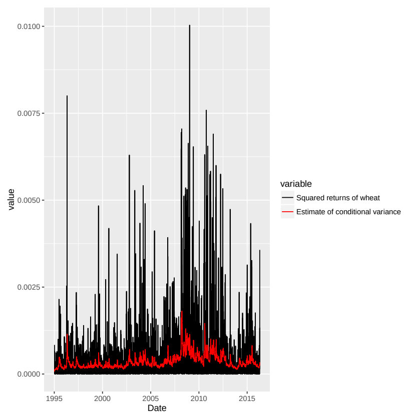


**Figure 33:** Estimated conditional variance of wheat returns and squared returns

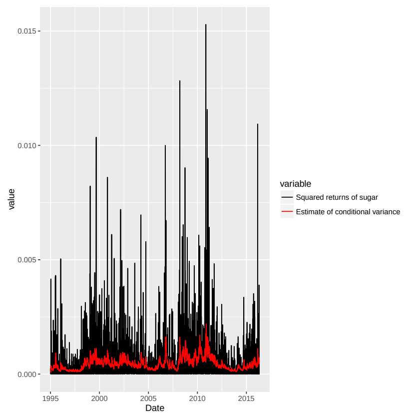


**Figure 34:** Estimated conditional variance of sugar returns and squared returns

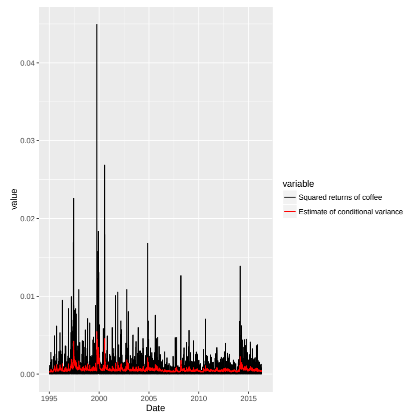


**Figure 35:** Estimated conditional variance of coffee returns and squared returns

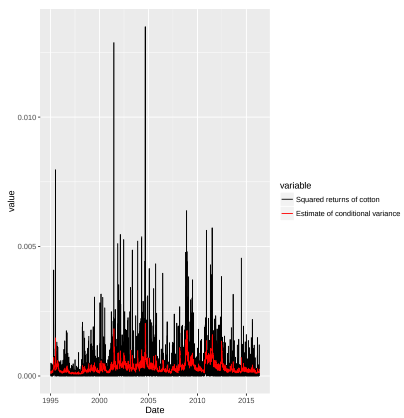


**Figure 36:** Estimated conditional variance of cotton returns and squared returns

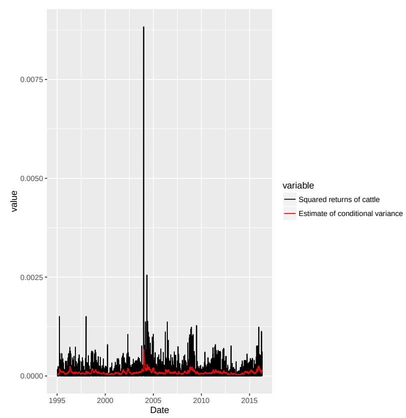


**Figure 37:** Estimated conditional variance of cattle returns and squared returns

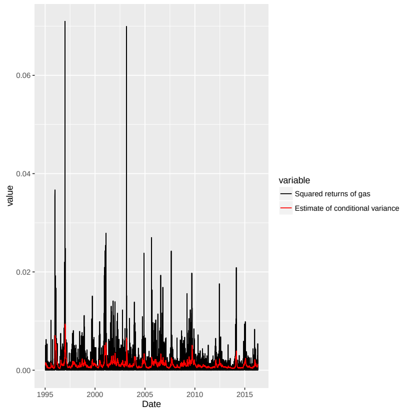


**Figure 38:** Estimated conditional variance of gas returns and squared returns

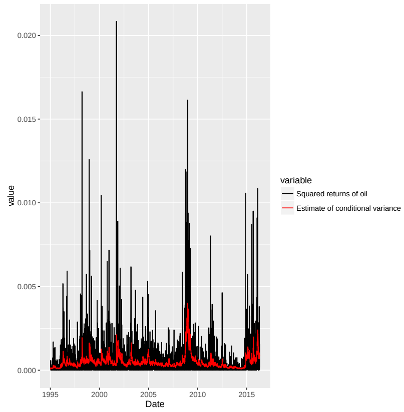


**Figure 39:** Estimated conditional variance of oil returns and squared returns

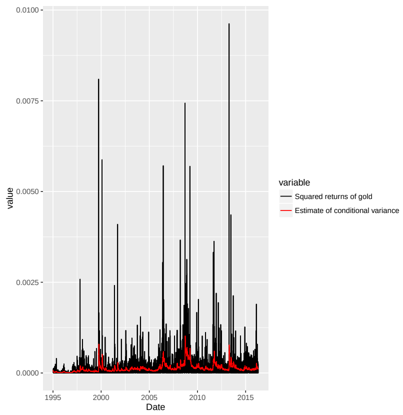


**Figure 40:** Estimated conditional variance of gold returns and squared returns

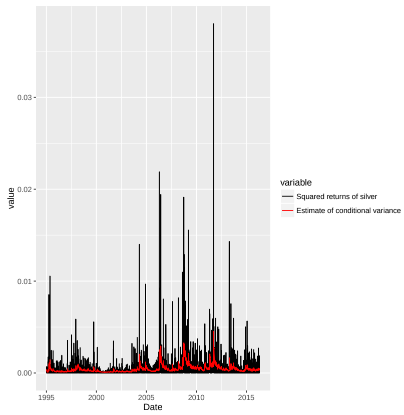


**Figure 41:** Estimated conditional variance of silver returns and squared returns

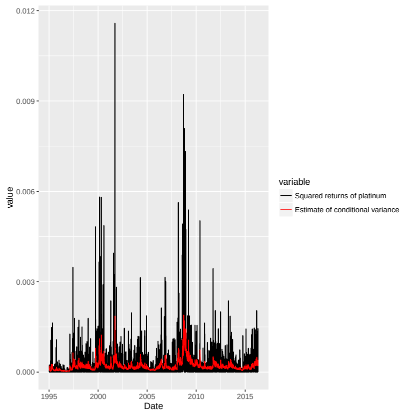


**Figure 42:** Estimated conditional variance of platinum returns and squared returns

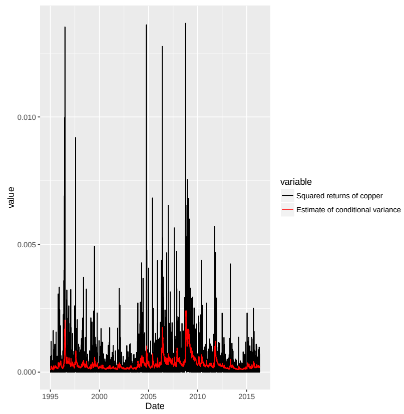


**Figure 43:** Estimated conditional variance of copper returns and squared returns