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FACULTY OF INFORMATICS AND MATHEMATICS

Master thesis

Alternative Functional Restrictions for MIDAS Regression

Alternatyvūs Funkciniai Apribojimai MIDAS regresijai

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Introduction

MIDAS regression model was introduced by Ghysels et al. [2005]. It allows to deal with data sampled at different time frequencies. Data of different time frequencies are of general interest when amount of data generated is increasing constantly, followed also by a greater accessibility level of it.

One of the key features of MIDAS regression is that parametric restrictions can be imposed on original parameters. This is of huge help when higher lag orders with more variables prevail in the model, making it possibly infeasible. Probably one of the most common restrictions used is normalized exponential Almon polynomial. It is quite popular in distributed lag literature and is used due to its flexibility for shapes and low number of parameters needed to estimate.

In Andrade et al. [2013], Almon polynomial is one of the restrictions used to forecast future inflation using different frequency variables. Also, MIDAS can be used in a mixture with other models as well. In Engle et al. [2009] and Colacito et al. [2009], MIDAS model is used together with GARCH or DCC models, extending the standard ARCH model family.

In this work we apply new restriction functions that are be used to forecast realized volatility. MIDAS regression with Almon restriction and HAR model as introduced by Corsi [2009] are used as benchmark models. According to Müller et al. [1997] we present restriction functions for market components that can be complementary structures comparing to HAR model. Another approach is to use a combination of two Gamma probability density functions as a restriction function. It is capable to reflect more continuous separation of two market components than HAR model.

In section 1 general MIDAS model is presented. Further in section 2 HAR model is described, which is used to estimate realized volatility. In the same section HAR model is linked with MIDAS. In section 3 new restriction functions are presented which are used as alternatives for commonly used restriction functions to estimate realized volatility. Later in section 4 it is shown that MIDAS model is capable to recognize Gamma combination restriction function. Section 5 introduces the data used in this thesis. Section 6 contains results and comparisons of the estimations made with chosen restriction functions for S&P 500, Russel 2000 and Nikkei 225 indexes. Additional results for Kospi, Euro and FTSE 100 indexes can be found in section 6.3.

1 MIDAS

MIDAS model was introduced by Ghysels et al. [2005] to deal with data sampled at different freqencies. Prior to MIDAS, aggregation was one of the main solutions, which alligns low and high frequency variable to be at the same frequencies. Data aggregation often leads to some information being lost. MIDAS represents a simple time series model that allows to use data sampled at different frequencies for left-hand and right-hand side variables. It is especially important currently when information gathering is improving dramatically. Important thing to mention is that MIDAS can be both an autoregressive and non-autoregressive model, which depends if the data in the past is of the same frequency or not.

1.1 MIDAS regression

In a most compact form, MIDAS regression can be expressed as:

$$\alpha(B)y_t = \beta(L)^T \boldsymbol{x}_{t,0} + \epsilon_t, \tag{1}$$

where

$$\alpha(z) = 1 - \sum_{j=1}^{p} \alpha_j z^j,$$

$$\boldsymbol{x}_{t,0} := \left(x_{tm_0}^{(0)}, \ldots, x_{tm_i}^{(i)}, \ldots, x_{tm_l}^{(l)}\right)^T,$$

$$\beta(z) = \sum_{j=0}^{l} \beta_j z^j, \quad \beta_j = \left(\beta_j^{(0)}, \ldots, \beta_j^{(i)}, \ldots, \beta_j^{(l)}\right)^T,$$

$$L^j \boldsymbol{x}_{t,0} := \boldsymbol{x}_{t,j} = \left(L^j x_{tm_0}^{(0)}, \ldots, L^j x_{tm_i}^{(i)}, \ldots, L^j x_{tm_l}^{(l)}\right)^T.$$

Here, $\{y_t, t \in \mathbb{Z}\}$ is a univariate process observed at low frequency with B being low frequency lag operator. $\{x_\tau^{(i)}, \tau \in \mathbb{Z}\}$, $i = 0, \ldots, k$ observed at higher frequency with L being higher frequency lag operator. l denote a single order of the lag polynomials. If some components of $\beta(z)$ would be of lower order, it is easy to set the rest of coefficients of the polynomial to zero. ith high frequency period τ can be represent in terms of low frequency period t as $\tau = (t-1)m_i + j, \quad j = 1, \ldots, m_i$.

Such notation of MIDAS allows to allign frequencies of left hand and right hand side variables. Model in equation 1 can be also expressed in matrix notation. Explicit matrix notation with several simple examples how frequencies are alligned can be found in Ghysels et al. [2016].

1.2 Functional Restrictions

In some cases, the number of lags used for MIDAS regression can be quite high. If there would be observations y_t affected by daily values $x_\tau^{(i)}$ of six months, 120 high-frequency lags (assuming month has 20 days and week has 5 days) would be needed. In such case, model can easily become unfeasible, meaning that estimation of all parameters on the right-hand side of equation 1 of high-frequency variables would take a lot of computational power. To bypass this issue, a sufficiently flexible parametric estimation function with so-called hyper-parameters can be imposed on the original parameters.

$$\beta_j^i = f_i(j, \theta_i), \quad j = 0, \dots, l_i, \quad \theta_i = \left(\theta_1^{(i)}, \dots, \theta_{q_i}^{(i)}\right), \quad q_i \in \mathbb{N}.$$

Usually, number of hyper-parameters is quite low and makes it easy to estimate. It is convenient to take β_j^i in the following form:

$$\beta_j^i := \frac{f(j; \theta_i)}{\sum_{k=0}^{l_i} f(k; \theta_i)}. \tag{2}$$

The advantage of such formulation is that it satisfies normalization constraint (sum adds up to 1) and given that $f(j; \theta)$ is non-negative, equation 2 will also produce non-negative values. Probably one of the most popular functional forms is exponential Almon polynom. Naming is related to the "Almon lags" that are popular in the distributed lag literature. Under Almon restriction approach, function $f(j; \theta)$ in equation 2 is of the following form:

$$f(j; \theta) := \exp\left\{\sum_{i=1}^{q} \theta_i j^i\right\}, \tag{3}$$

where q denotes the number of parameters to be estimated and is freely chosen by a user. Normalized exponential Almon restriction is one of the most frequent restrictions chosen when using MIDAS. In papers Engle et al. [2009], Colacito et al. [2009] and Andrade et al. [2013] Almon restriction is one of the main restrictions used for estimations. It usually takes only few parameters to estimate and is capable to reflect various possible functional shapes for parameters. In this work Almon restriction function will be used as one of the benchmark restrictions. Examples of different shapes of Almon restriction can be seen in figure 1.

There are other predefined functional restrictions for MIDAS regression in R package **midasr**, like Beta, Gompertz, Log-Cauchy, Nakagami (all are analogues of probability density functions). Definitions of these functional restriction forms can be found in Ghysels et al. [2016].

2 Realized volatility

In this section, a clasical HAR model for realized volatility is described. Then it is shown that it can be linked with MIDAS regression model. Since HAR model is also one of the benchmark models in this work, it is good to get familiar with HAR model fairly closely.

2.1 HAR model for realized volatility

HAR-RV (Heterogeneous Autoregressive model of Realized Volatility) model was presented in Corsi [2009]. The idea of the model was to introduce a simple com-

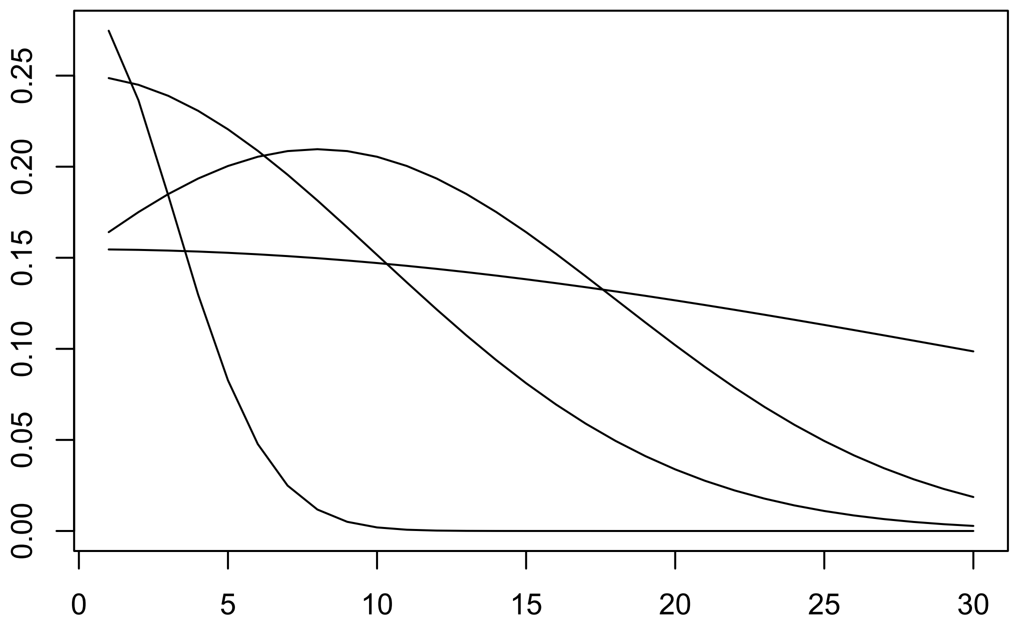

Figure 1: Possible shapes of Almon restriction function. Shapes in the figure are normalized exponential Almon polynomials built with R function **nealmon** with parameters: θ_1, θ_2, θ_3 equal to (1; 0; -0.05), (3; 0; -0.005), (4; 0; -0.0005) and (4; 0.08; -0.005). Parameter θ_1 is the scaling parameter, while θ_2 and θ_3 are Almon polynomial parameters.

ponent model for conditional volatility which would be able to reproduce main empirical features in the data, such as a very strong persistence of autocorrelations of the square and absolute returns that last for long time periods or leptokurtic probability density functions of return with shapes depending on the time scale, presenting a very slow convergence to the normal distribution as time scale increase. As it is stated in Corsi [2009], popular GARCH and stochastic volatility models are not able to reproduce all of these features.

Standard definition of realized volatility is:

$$RV_t^{(d)} = \sqrt{\sum_{j=0}^{M-1} r^2_{t-j\cdot\Delta}}, \tag{4}$$

where $\Delta = 1d/M$ and $r_{t-j\cdot\Delta} = p(t - j\cdot\Delta) - p(t - (j+1)\cdot\Delta)$ defines continuously compounded Δ-frequency returns, i.e. intraday returns sampled at time interval Δ. $1d$ denotes full trading day with natural number M dividing it into equally spaced intraday return series. t denotes day index and j denotes time within the day. In this case, quantities over various time horizons are designed as normalized sums of one-period realized volatilities. For example, weekly realized volatility would be a simple average of daily realized volatilities:

$$RV_t^{(w)} = \frac{1}{5}\left(RV_t^{(d)} + RV_{t-1d}^{(d)} + \cdots + RV_{t-4d}^{(d)}\right). \tag{5}$$

$RV_t^{(m)}$ would define monthly realized volatility as average of 20 daily realized volatilities.

HAR-RV model itself can be written as:

$$RV_{t+1d}^{(d)} = c + \beta^{(d)} RV_t^{(d)} + \beta^{(w)} RV_t^{(w)} + \beta^{(m)} RV_t^{(m)} + \omega_{t+1d}, \tag{6}$$

where ω_{t+1d} are measurement errors of realized volatility.

2.2 Realized volatility in MIDAS

From the first paper about MIDAS, one of primary applications that MIDAS was used for was to forecast realized volatility. For instance, Ghysels et al. [2005] estimated realized volatility from within-month daily returns as:

$$\sigma_{t+1}^2 = \sum_{d=0}^{22} r_{t+1-d}^2.$$

Another example to forecast next month's variance was proposed by French et al. [1987]. It is called rolling window approach which also uses within-month daily squared returns for forecasting:

$$V_t^{RW} = 22 \sum_{d=0}^{D} \frac{1}{D} r_{t-d}^2,$$

where D is the number of days used in the estimation and multiplier 22 is for measuring variance in monthly units.

2.3 HAR with MIDAS

It is known that MIDAS is primarily used for different frequency sampled data and that realized volatility is one of the main subjects MIDAS is used to estimate. Since HAR model in section 2.1 considers realized volatilities over different interval sizes it is quite natural to link it with MIDAS regression. Recalling, that in HAR model quantities over various time horizons are designed as normalized sums of one-period realized volatilities and assuming that a week has 5 days and a month has 20 days, HAR model can be rewritten as a special case of MIDAS regression:

$$RV_{t+1}^{(d)} = c + \sum_{j=0}^{19} \beta_j RV_{t-j}^{(d)} + \omega_{t+1}, \tag{7}$$

where

$$\beta_j = \begin{cases} \beta^{(d)} + \frac{1}{5}\beta^{(w)} + \frac{1}{20}\beta^{(m)}, & \text{for } j = 0, \\ \frac{1}{5}\beta^{(w)} + \frac{1}{20}\beta^{(m)}, & \text{for } j = 1, \ldots, 4, \\ \frac{1}{20}\beta^{(m)}, & \text{for } j = 5, \ldots, 19. \end{cases} \tag{8}$$

The latter equation is an expression of functional restriction to use in MIDAS regression. The result is a three-step restriction function.

3 Selection of new functional restrictions

The main purpose of this work is to find alternative functional restrictions that could outperform HAR and Almon in realized volatility forecasting.

3.1 Gamma restriction

As a first approach, probability density function of Gamma distribution was taken. Gamma probability density functions was chosen due to flexibility of shapes it can reflect. We denote Gamma density restriction function as:

$$f(x; \gamma, \alpha, \beta) := \gamma \frac{\alpha^\beta x^{\beta-1} e^{-x\alpha}}{\Gamma(\beta)}. \tag{9}$$

Here x is a lag series, while α, β and γ are three parameters that need to be estimated. γ parameter works as a scaling factor here. Few instances of Gamma restriction function can be seen in figure 2.

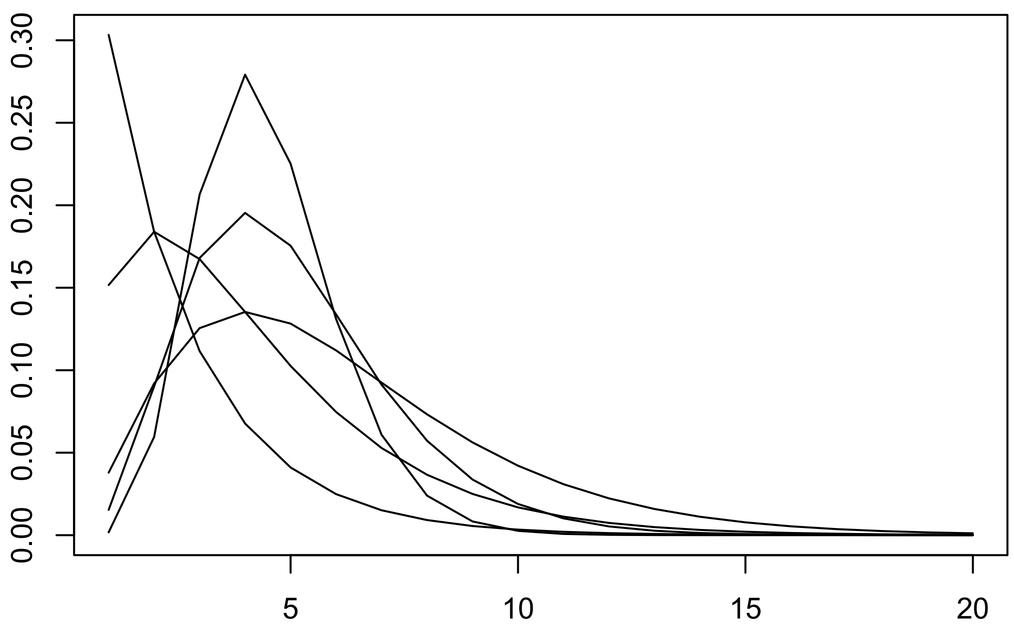

Figure 2: Various shapes of Gamma restriction function. Shapes in the figure are built using different sets of α, β and γ parameters in equation 9.

3.2 HAR alternatives

As proposed in HAR model in Corsi [2009], different time intervals reflect time frames of one day, one week and one month. But these components are based mostly on calendar periods. There could be other breakdowns of time intervals that would reflect market components possibly better.

Another possible approach was proposed by Müller et al. [1997], which is also mentioned in Corsi [2009] as an inspiration for building HAR model. Paper was written for HARCH (heterogeneous ARCH) model due to asymmetric propagation of volatility between long and short time horizons. Originally, HARCH model was not used for realized volatility estimation, but since Corsi [2009] adjusted the model for realized volatility estimation, in this work we use the original idea of market components for HARCH model and apply them for MIDAS regression. Idea in the paper is that market components could be determined by power of two natural numbers: p^m, where m determines market components. Thresholds of each market component then are determined by p^i, $i = 0, \ldots, m-1$. In the original paper, p and m were chosen to be 4 and 7, respectively. That lead to 4096 lags of time series. Depending on the frequency of time series, range of 4096 in time series can lead up to 12 weeks for 30 minutes lagged data up to 17 years for daily data. It is important to choose numbers p and m reasonably. In this work we consider two cases: $p = 3$ with $m = 5$ and $p = 2$ with $m = 7$. The aim is to take the number of market components close to the one in the original HARCH model. But 8 market components on base as 4 would lead to too many lags. One of the decisions was to take smaller amount of market components based on smaller p number. Other structures could be taken as well, for example, one could take only few market components, which would be based on fairly high number p.

In figure 3 there is presented a comparison of estimated parameters for first 20 lagged variables using HAR, HAR 3^5 and HAR 2^7 as market component structures.

3.3 Gamma combination

In figure 3 interesting behaviour of estimated parameters can be seen for HAR 2^7 structure. It suggests a possible existence of more expressed market component in the area between 10 - 20 lagged periods. It gave a birth to an idea to have a smoother transition between different market components than alternatives of HAR model. Recall Gamma restriction function 9. Main idea of this work is to show how a combination of two Gamma restriction functions would perform against standard HAR or Almon restrictions. Restriction function using a combination of two Gamma density functions is defined according to formula:

$$f(x;\theta,\alpha_1,\beta_1,\alpha_2,\beta_2) := \theta_1 \left(\theta_2 \frac{\alpha_1^{\beta_1} x^{\beta_1-1} e^{-x\alpha_1}}{\Gamma(\beta_1)} + (1-\theta_2)\frac{\alpha_2^{\beta_2} x^{\beta_2-1} e^{-x\alpha_2}}{\Gamma(\beta_2)}\right), \qquad (10)$$

where θ_1 is a scaling parameter and θ_2 is a weight distribution between two Gamma density functions. An example of combination of two Gamma distribution functions

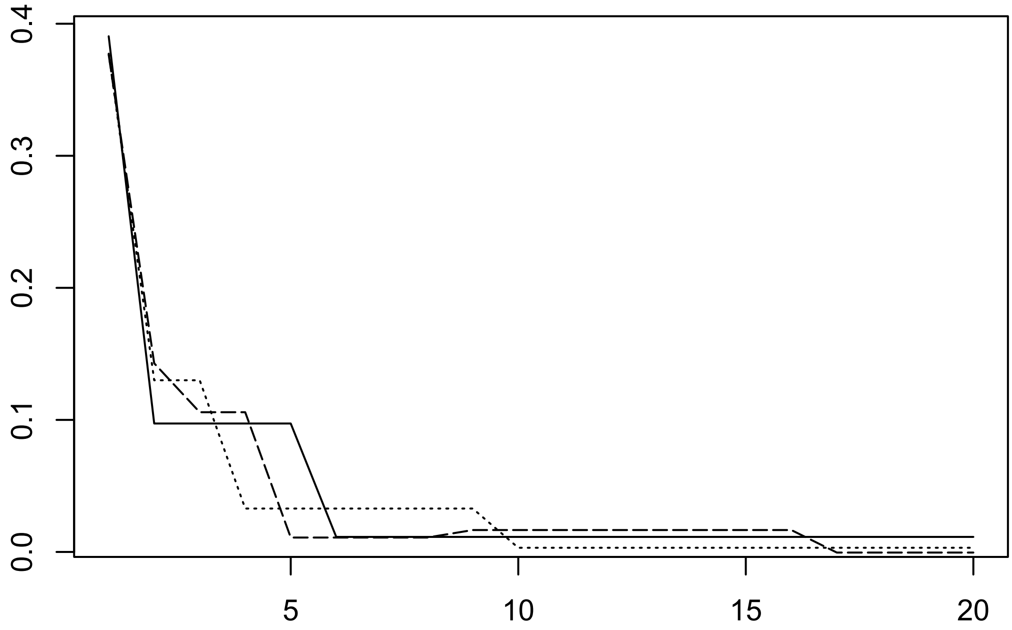

Figure 3: Market component structure for different HAR approaches for first 20 lags. Solid line stands for classical HAR, dotted line stands for HAR 3^5 and dashed line stands for HAR 2^7.

can be seen in figure 4.

4 Simulation

In this section estimation results on simulated data regarding Gamma combination restriction function are presented. Before using Gamma combination restriction function for real data estimation, it must be assured that MIDAS can identify the process.

1000 simulations were made on $N(0,1)$ data with $N(0,0.1)$ residuals. Two estimations were done, with correct starting values for optimization process and false starting values for optimisation process. Restriction function to simulate the process is exactly the same as in figure 4. As can be seen in figure 5, correct starting values lead to good estimation results. Reading histograms from left to right and from top to bottom represent parameters θ_1, θ_2, α_1, β_1, α_2, β_2 in the formula 10, respectively. On the other hand, if false starting values for optimisation algorithm are taken, it depends which Gamma density function of two starting values are closer to. For example, figure 6 shows how parameters α_1, β_1 changed places with parameters α_2, β_2.

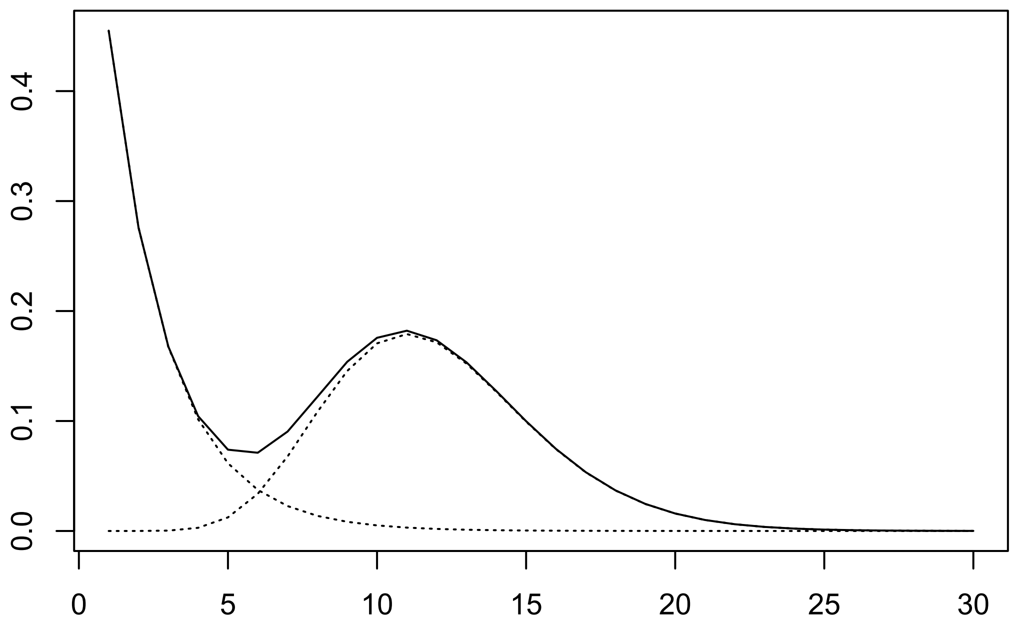

Figure 4: Restriction function using combination of two Gamma density functions.

But this is reasonable, since restriction function used to simulate the process took θ_2 weight as 0.5. That means that switching place of Gamma density functions inside the restriction function doesn't change the output. But if weight θ_2 would be something else than 0.5, histograms of estimation results can be somewhat misleading. For simplicity, θ_2 was taken 0.5 just for demonstrative purposes.

5 Data

All the estimations and tests made in this work are based on realized volatility data provided by Oxford-Man Institute of Quantitative Finance[1]. In this paper data used from the source is daily reaziled volatility which is built on 5-minutes index data. At the time of estimations were done, data source contained time series from January 3^{rd}, 2001, up to September 22^{nd}, 2016. Sample contained more than 4000 obervations. Realized volatilities from the source were transformed to reflect annualized realized volatities using transformation $rv_t = \log(100\sqrt{252RV_t})$, where RV_t is the initial series of realized volatilities.

[1]Data source can be accessed via link http://realized.oxford-man.ox.ac.uk/data/download. See Gerd et al. [2009] for full reference.

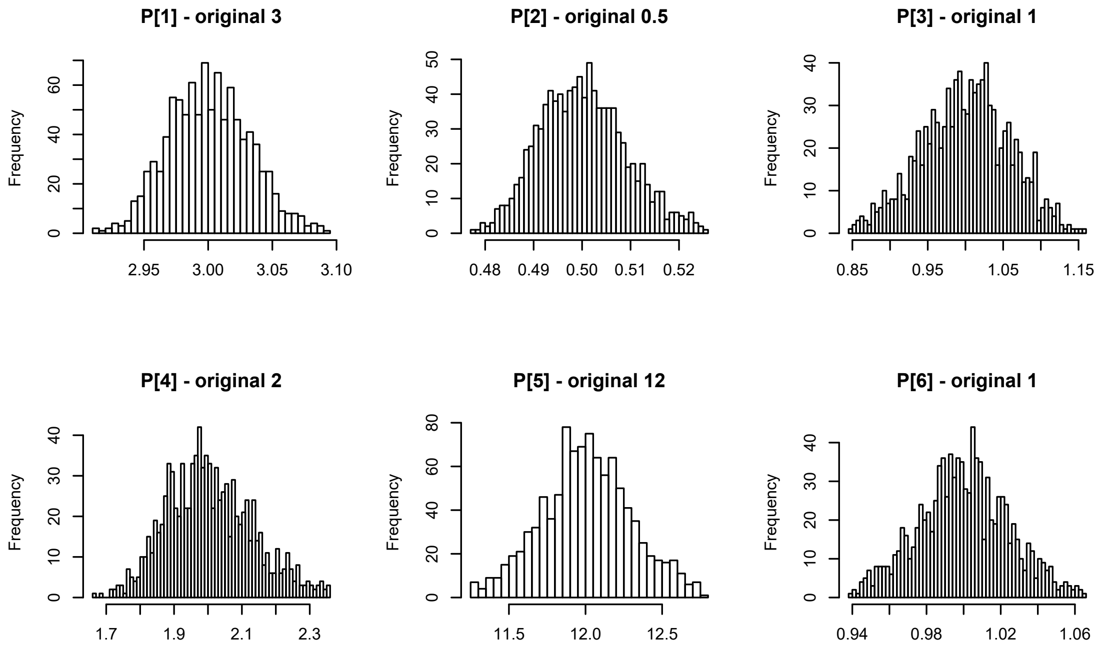

Figure 5: Histograms of estimated parameters. Correct starting values used for optimisation algorithm. Original parameter values under the name of each histogram

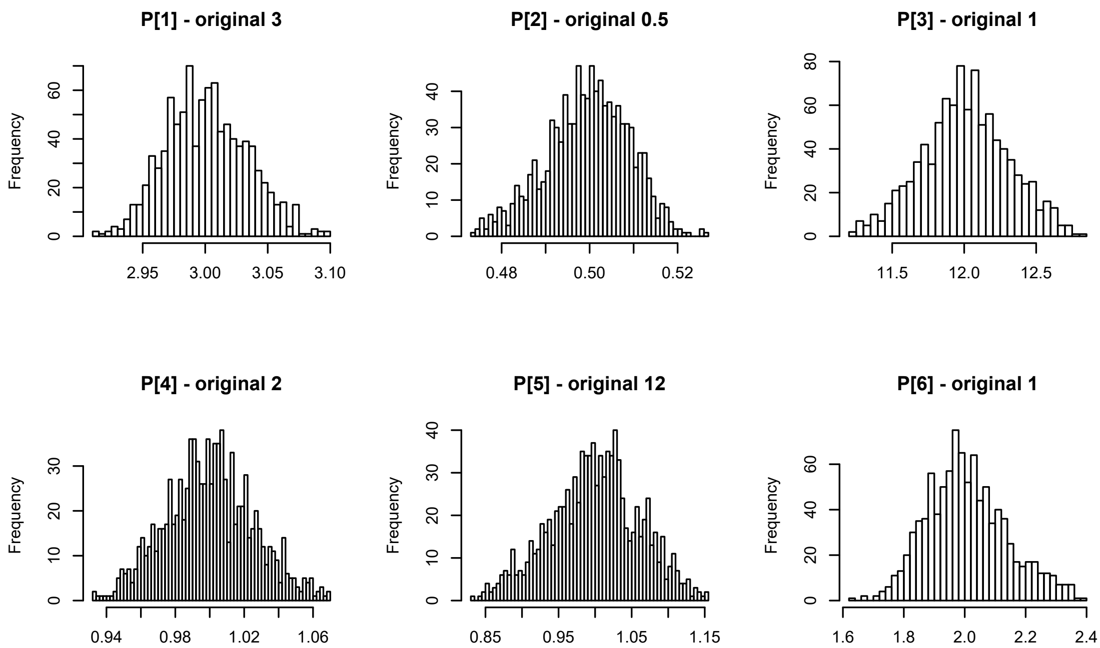

Figure 6: Histograms of estimated parameters. False starting values used for optimisation algorithm. Original parameter values under the name of each histogram

There were six indexes of realized volatilities selected on which all the estimations were made: S&P 500, Russel 2000, Nikkei 225, Cospi, Euro and FTSE 100.

Indexes were chosen to reflect MIDAS performance on various markets in different geographical locations in the world. Results for first three of them can be found in section 6 while results for the rest can be found in section 6.3. Graphs of time series for S&P 500, Russel 2000 and Nikkei 225 can be seen in figures 7, 8 and 9, respectively.

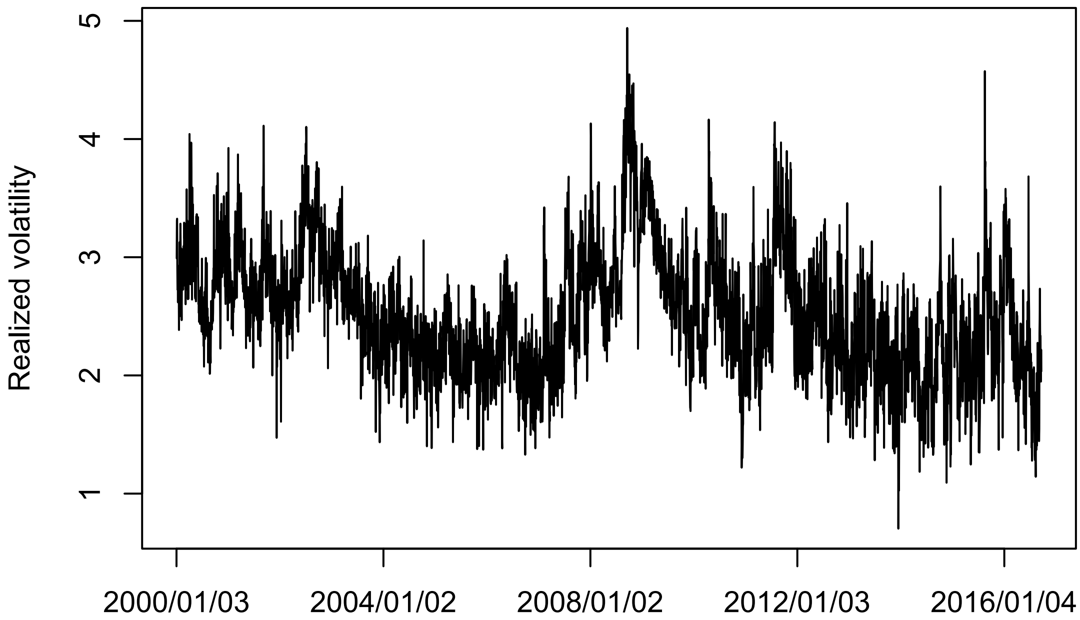

Figure 7: Realized volatility of S&P 500 index

6 Results

In this section we present some results for the functional restrictions mentioned in sections 2.3 and 3. Just few more additional comments on the size of lags chosen for some restrictions are mentioned.

Size of lags can be split into short and long regions. Short region consists basically of two options, choices of 12 and 20 lags. Decision to take such lag sizes is taken according to paper Ishida and Kvedaras [2015]. Since the source of the data used in referenced paper is the same, there is no contradiction not to use same specifications for restriction functions, which are 12 and 20 lags. For lag size of 12, Almon, Gamma (as in equation 9) and Gamma combination (as in equation 10) restriction functions are used. For lag size of 20, HAR restriction function is additionally used. For so-called long region of lags, size of lags is determined mainly by the predefined length of

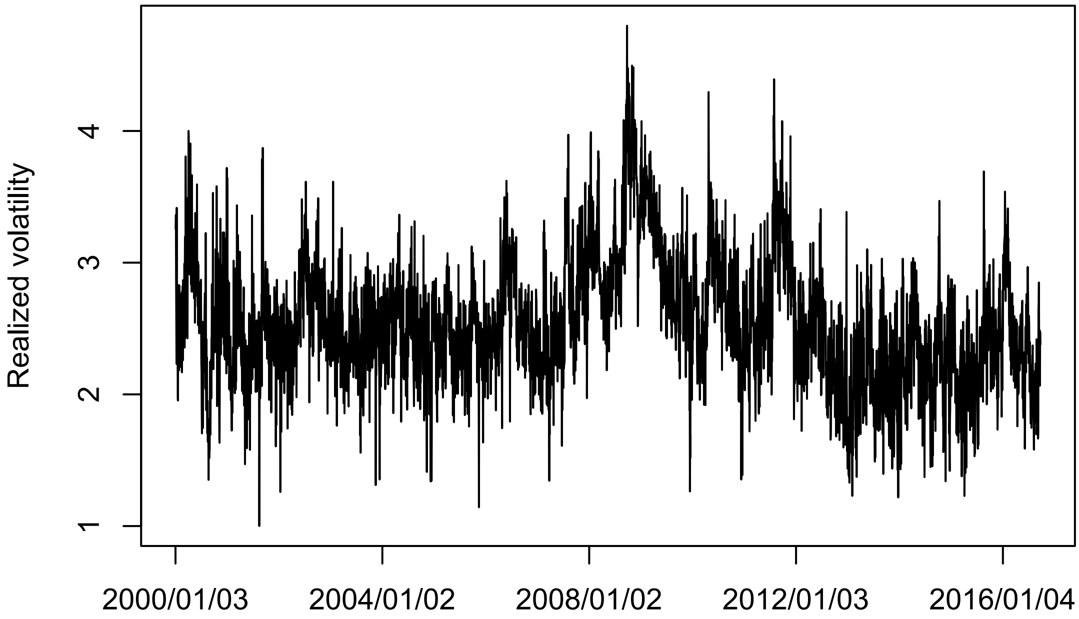

Figure 8: Realized volatility of Russel 2000 index

HAR 2^7 and HAR 3^5, which is 128 and 243, respectively. Also, another extension of HAR was used, to reflect daily, weekly, monthly, 3-monthly and 6-monthly market components. Extension is straight forward from original HAR model, except it takes 120 lags and is called HAR-HALFYEAR. For simplicity, Almon, Gamma and Gamma combination restrictions are based on size of 128 lags to avoid too many models being presented. There will be seen two Gamma combination models, named GAMMA-COMB and GAMMA-COMB-2. Difference between them is the method of the estimation of parameters. GAMMA-COMB model uses BFGS optimization method for *optim* function, but has a bypass in the restriction function. It is known, that Gamma function with negative values can produce infinite values, which lead to exceptions during optimization algorithm with no outcome. To avoid this issue, restriction function returns function $f(x) = 1/x$, in case it gets negative candidates for parameters (x is a lag order). GAMMA-COMB-2 uses L-BFGS-B optimization algorithm with explicitly predefined restrictions for estimated parameters, where $\theta_1 > 0$, $\theta_2 \in [0, 1]$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$.

For each index there are three tables presented. One table with p-values of hAh test (hAh test checks the adequacy of the MIDAS regression coefficients. Regarding hAh test, more can be found in Kvedaras and Zemlys [2012]). Second table presents AIC/BIC values for each model on different sampling sizes. While third table presents mean squared errors of rolling forecast.

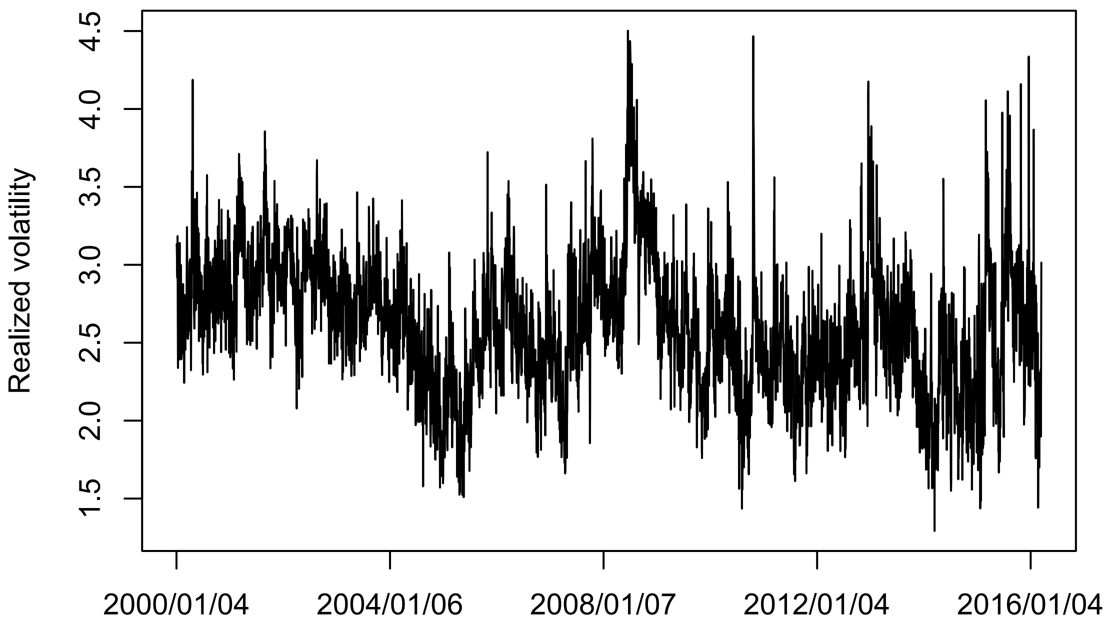

Figure 9: Realized volatility of Nikkei 225 index

6.1 S&P 500

hAh test p-values can be found in Table 1. The majority of the models show that MIDAS coefficients are adequate, except few models under the biggest sampling size. Table 2 shows that both AIC/BIC criterias are more favourable for ALMON model with lower lag order. Table 3 shows that at least one of the newly introduced restriction functions for MIDAS regression can be better at forecasting realized volatility regarding mean squared error (MSE). For sample sizes of 1 - 3000, in section of high order of lags, ALMON restriction shows the worst performance of all models.

Sample:	1 - 1000	1 - 2000	1 - 3000
Criterion:	**hAh - p-value**	**hAh - p-value**	**hAh - p-value**
ALMON(12)	0.38	0.35	0.29
GAMMA(12)	0.25	0.07	**0.04**
GAMMA-COMB(12)	0.18	0.47	0.15
GAMMA-COMB-2(12)	0.14	0.36	0.09
HAR	0.32	0.11	**0.00**
ALMON(20)	0.41	0.30	0.32
GAMMA(20)	0.60	0.34	**0.00**
GAMMA-COMB(20)	0.39	0.31	0.22
GAMMA-COMB-2(20)	0.40	0.21	0.22
ALMON(128)	0.37	0.09	**0.03**
GAMMA(128)	0.47	0.24	0.09
GAMMA-COMB(128)	0.40	0.20	0.11
GAMMA-COMB-2(128)	0.40	0.21	0.12
HAR-2to7(128)	0.46	0.40	0.15
HAR-3to5(243)	0.85	0.31	0.08
HAR-HALFYEAR	0.23	**0.04**	**0.01**

Table 1: hAh test p-values for different sample sizes of S&P 500 index. Values in bold denote functional constraints that can be rejected as inadequate.

Sample:	1 - 1000		1 - 2000		1 - 3000	
Criterion:	**AIC**	**BIC**	**AIC**	**BIC**	**AIC**	**BIC**
ALMON(12)	**158.81**	**183.29**	**415.05**	**443.03**	**894.68**	**924.69**
GAMMA(12)	162.65	187.13	424.97	452.94	903.57	933.58
GAMMA-COMB(12)	165.64	204.81	422.73	467.49	900.37	948.39
GAMMA-COMB-2(12)	166.91	206.07	427.94	472.69	905.78	953.80
HAR	160.99	185.43	422.47	450.42	917.05	947.04
ALMON(20)	159.46	183.89	417.48	445.44	**891.88**	**921.88**
GAMMA(20)	**158.05**	**182.49**	**417.05**	**445.01**	896.40	926.40
GAMMA-COMB(20)	162.72	201.82	420.30	465.03	896.44	944.44
GAMMA-COMB-2(20)	163.57	202.67	421.45	466.17	897.45	945.44
ALMON(128)	133.35	157.21	397.07	424.75	875.24	905.05
GAMMA(128)	129.97	**153.83**	388.50	**416.17**	867.30	**897.11**
GAMMA-COMB(128)	135.36	173.53	392.55	436.83	866.51	914.21
GAMMA-COMB-2(128)	136.31	174.47	393.02	437.30	866.97	914.68
HAR-2to7(128)	134.55	182.26	383.96	439.31	**864.75**	924.38
HAR-3to5(243)	**124.88**	161.92	**379.72**	423.49	868.79	916.16
HAR-HALFYEAR	134.35	167.81	387.10	425.87	884.70	926.46

Table 2: AIC/BIC values using different samples of S&P 500 index. Values in bold represent best models regarding AIC/BIC for each set of different lag models.

Out - sample:	1001 - 2000	2001 - 3000	3001 - 4000
In - sample:	**1 - 1000**	**1 - 2000**	**1 - 3000**
Criterion:	**MSE**	**MSE**	**MSE**
ALMON(12)	1.00000	1.00000	1.00000
GAMMA(12)	1.00384	**0.99972**	1.00658
GAMMA-COMB(12)	1.00230	1.00001	**0.99957**
GAMMA-COMB-2(12)	1.00281	**0.99952**	1.00009
HAR	1.00783	1.01740	1.01896
ALMON(20)	1.00000	1.00000	1.00000
GAMMA(20)	1.00320	1.00232	1.01364
GAMMA-COMB(20)	1.00444	1.00080	1.00008
GAMMA-COMB-2(20)	**0.99960**	1.00227	**0.99927**
ALMON(128)	1.00000	1.00000	1.00000
GAMMA(128)	**0.99691**	**0.99884**	**0.99526**
GAMMA-COMB(128)	1.00189	**0.99840**	**0.98448**
GAMMA-COMB-2(128)	**0.99601**	**0.99868**	**0.98451**
HAR-2to7(128)	1.00346	1.00321	**0.98159**
HAR-3to5(243)	1.00468	1.01063	**0.98285**
HAR-HALFYEAR	1.00052	1.01795	**0.99818**

Table 3: Relative MSE values of different samples for S&P 500 index. Values in bold represent better MSE results comparing to the best of HAR or ALMON model for particular set of models with same order of lags.

6.2 Russel 2000

If S&P 500 index consists mainly of large capitalization stocks, Russel 2000 index is combined from small to medium market capitalization stocks traded at New York Stock Exchange and is used as a most common benchmark for mutual funds.

In this work we present Russel 2000 index as an example that MIDAS regression is not always a good choice. Table 4 shows that MIDAS regression coefficients are adequate mainly for 1-1000 sample size only. Most of the models for higher sample sizes show that functional constraints tend not to hold. Table 5 shows same pattern as for S&P 500 index, AIC/BIC criterias are more favourable for new restriction functions using higher amount of information. Table 6 with MSE values shows somewhat controversial results. If GAMMA-COMB(20) shows better forecasting prediction for sample size 1-1000, better forecasting errors for sample size 1-2000 for the same model could be questioned because hAh test is on the very limit to pass adequacy of the MIDAS coefficients not to be rejected. All other models with sample sizes 1-2000 or 1-3000 that have MSE values lower than 1 have very low p-values for hAh test and adequacy of MIDAS coefficients doesn't hold either.

Sample:	1 - 1000	1 - 2000	1 - 3000
Criterion:	**hAh - p-value**	**hAh - p-value**	**hAh - p-value**
ALMON(12)	0.20	0.07	**0.04**
GAMMA(12)	0.19	0.06	**0.01**
GAMMA-COMB(12)	0.25	**0.01**	**0.01**
GAMMA-COMB-2(12)	0.17	**0.03**	**0.01**
HAR	0.57	0.07	**0.00**
ALMON(20)	0.43	0.12	**0.04**
GAMMA(20)	0.44	0.12	**0.03**
GAMMA-COMB(20)	0.07	0.05	**0.02**
GAMMA-COMB-2(20)	0.25	**0.05**	**0.02**
ALMON(128)	0.12	**0.00**	**0.00**
GAMMA(128)	0.12	**0.00**	**0.02**
GAMMA-COMB(128)	0.05	**0.00**	0.05
GAMMA-COMB-2(128)	0.09	**0.00**	0.06
HAR-2to7(128)	0.06	**0.00**	**0.04**
HAR-3to5(243)	0.92	**0.01**	**0.03**
HAR-HALFYEAR	0.16	**0.01**	**0.00**

Table 4: hAh test p-values for different sample sizes of Russel 2000 index. Values in bold denote functional constraints that can be rejected as inadequate.

Sample:	1 - 1000		1 - 2000		1 - 3000	
Criterion:	**AIC**	**BIC**	**AIC**	**BIC**	**AIC**	**BIC**
ALMON(12)	**548.98**	**573.46**	**862.53**	**890.50**	**1231.86**	**1261.88**
GAMMA(12)	549.21	573.69	863.03	891.00	1237.00	1267.01
GAMMA-COMB(12)	554.97	594.13	869.16	913.92	1237.77	1285.79
GAMMA-COMB-2(12)	555.45	594.62	869.16	913.92	1240.00	1288.02
HAR	**543.94**	**568.38**	860.94	888.89	1246.10	1276.10
ALMON(20)	891.88	921.88	858.82	886.77	**1225.44**	**1255.44**
GAMMA(20)	545.83	570.27	**858.73**	**886.69**	1228.88	1258.88
GAMMA-COMB(20)	551.78	590.88	864.75	909.47	1232.31	1280.30
GAMMA-COMB-2(20)	551.79	590.89	864.92	909.65	1231.21	1279.20
ALMON(128)	465.75	489.60	777.47	805.15	1159.31	1189.12
GAMMA(128)	465.76	489.62	777.20	804.88	1149.71	1179.52
GAMMA-COMB(128)	470.88	509.05	783.20	827.48	1145.18	1192.88
GAMMA-COMB-2(128)	471.50	509.66	783.11	827.39	1150.07	1197.78
HAR-2to7(128)	476.58	524.29	784.48	839.83	1146.64	1206.27
HAR-3to5(243)	**361.40**	**398.43**	**673.93**	**717.70**	**1051.27**	**1098.64**
HAR-HALFYEAR	471.41	504.87	785.19	823.97	1169.64	1211.40

Table 5: AIC/BIC values using different samples of Russel 2000 index. Values in bold represent best models regarding AIC/BIC for each set of different lag models.

Out - sample:	**1001 - 2000**	**2001 - 3000**	**3001 - 4000**
In - sample:	**1 - 1000**	**1 - 2000**	**1 - 3000**
Criterion:	**MSE**	**MSE**	**MSE**
ALMON(12)	1.00000	1.00000	1.00000
GAMMA(12)	1.00014	1.00521	1.00661
GAMMA-COMB(12)	1.00200	1.06614	1.00077
GAMMA-COMB-2(12)	**0.99875**	**0.99846**	1.00073
HAR	1.00895	1.01935	1.01232
ALMON(20)	1.00000	1.00000	1.00000
GAMMA(20)	1.00039	1.00497	1.00971
GAMMA-COMB(20)	**0.99943**	**0.99975**	**0.99920**
GAMMA-COMB-2(20)	1.00022	1.00018	**0.99915**
ALMON(128)	1.00000	1.00000	1.00000
GAMMA(128)	1.00031	**0.99138**	**0.98904**
GAMMA-COMB(128)	1.00609	**0.98444**	**0.97693**
GAMMA-COMB-2(128)	1.00082	**0.98441**	**0.97612**
HAR-2to7(128)	1.00586	**0.98303**	**0.97763**
HAR-3to5(243)	1.01104	**0.99471**	**0.97148**
HAR-HALFYEAR	1.00494	1.00541	**0.98911**

Table 6: Relative MSE values of different samples for Russel 2000 index. Values in bold represent better MSE results comparing to the best of HAR or ALMON model for particular set of models with same order of lags.

6.3 Nikkei 225

Nikkei 225 index is a stock market index of Tokyo Stock Exchange and reflects the performance of Japanese equities. Table 7 shows that only few models didn't pass hAh test of MIDAS coefficients adequacy. All except one model that failed hAh adequacy test used sample size of 1-2000 and almost half of them are ALMON models. There is slightly different behaviour for AIC/BIC values for Nikkei 225 index comparing to S&P 500 or Russel 2000 indexes. Table 8 shows that HAR model is best regarding AIC/BIC at the middle section of the table. Again as for S&P 500 and Russel 2000 indexes, AIC/BIC values for higher lag models goes for new restriction MIDAS models. Nikkei 225 index is also interesting in sense that better forecasting performance regarding MSE values is achieved for all high order of lags GAMMA-COMBO and GAMMA-COMBO-2 models using all possible sample sizes (see Table 9).

Sample:	**1 - 1000**	**1 - 2000**	**1 - 3000**
Criterion:	**hAh - p-value**	**hAh - p-value**	**hAh - p-value**
ALMON(12)	0.76	**0.04**	0.86
GAMMA(12)	0.98	**0.00**	0.60
GAMMA-COMB(12)	0.93	0.21	0.80
GAMMA-COMB-2(12)	0.93	0.80	0.90
HAR	0.40	0.35	**0.01**
ALMON(20)	0.46	0.32	0.08
GAMMA(20)	0.85	**0.00**	0.50
GAMMA-COMB(20)	0.79	0.82	0.72
GAMMA-COMB-2(20)	0.79	0.85	0.83
ALMON(128)	0.73	**0.00**	0.07
GAMMA(128)	0.80	0.65	0.66
GAMMA-COMB(128)	0.82	0.77	0.68
GAMMA-COMB-2(128)	0.83	0.77	0.73
HAR-2to7(128)	0.85	0.92	0.83
HAR-3to5(243)	0.97	0.68	0.42
HAR-HALFYEAR	0.72	0.74	0.24

Table 7: hAh test p-values for different sample sizes of Nikkei 225 index. Values in bold denote functional constraints that can be rejected as inadequate.

Sample:	**1 - 1000**		**1 - 2000**		**1 - 3000**	
Criterion:	**AIC**	**BIC**	**AIC**	**BIC**	**AIC**	**BIC**
ALMON(12)	-190.48	-166.00	**-120.04**	**-92.06**	**-46.54**	**-16.52**
GAMMA(12)	**-191.01**	**-166.53**	286.36	314.33	-29.50	0.52
GAMMA-COMB(12)	-188.37	-149.20	-115.63	-70.87	-40.64	7.38
GAMMA-COMB-2(12)	-184.22	-145.05	-114.36	-69.60	-34.04	13.98
HAR	-189.61	-165.17	**-132.75**	**-104.80**	-38.73	-8.73
ALMON(20)	-190.65	-166.21	-123.39	-95.43	-48.06	**-18.06**
GAMMA(20)	**-193.64**	**-169.20**	403.95	431.90	-37.44	-7.45
GAMMA-COMB(20)	-191.76	-152.66	-128.82	-84.09	**-56.21**	-8.22
GAMMA-COMB-2(20)	-190.97	-151.87	-122.04	-77.31	-43.27	4.73
ALMON(128)	-206.00	-182.15	207.18	234.85	-33.68	-3.87
GAMMA(128)	**-208.96**	**-185.11**	-129.86	-102.19	-47.76	-17.95
GAMMA-COMB(128)	-207.29	-169.12	-138.11	-93.83	-62.10	-14.39
GAMMA-COMB-2(128)	-204.90	-166.73	-133.17	-88.89	-55.73	-8.03
HAR-2to7(128)	-207.03	-159.33	**-152.71**	-97.36	**-67.50**	-7.87
HAR-3to5(243)	-208.09	-171.06	-142.23	-98.46	-65.66	**-18.28**
HAR-HALFYEAR	-205.67	-172.21	-148.43	**-109.65**	-53.29	-11.54

Table 8: AIC/BIC values using different samples of Nikkei 225 index. Values in bold represent best models regarding AIC/BIC for each set of different lag models.

Out - sample:	1001 - 2000	2001 - 3000	3001 - 4000
In - sample:	1 - 1000	1 - 2000	1 - 3000
Criterion:	MSE	MSE	MSE
ALMON(12)	1.00000	1.00000	1.00000
GAMMA(12)	1.00336	1.33194	1.00939
GAMMA-COMB(12)	1.00515	1.00327	**0.99550**
GAMMA-COMB-2(12)	**0.99940**	1.00155	**0.99522**
HAR	1.00000	1.01776	1.00000
ALMON(20)	1.00937	1.00000	1.00138
GAMMA(20)	1.01063	1.48834	1.00426
GAMMA-COMB(20)	1.00935	**0.99872**	**0.99313**
GAMMA-COMB-2(20)	1.00574	**0.99453**	**0.99319**
ALMON(128)	1.00000	1.00000	1.00000
GAMMA(128)	**0.98747**	**0.85013**	**0.98373**
GAMMA-COMB(128)	**0.98765**	**0.84829**	**0.97269**
GAMMA-COMB-2(128)	**0.98037**	**0.84578**	**0.97237**
HAR-2to7(128)	**0.97320**	**0.85748**	**0.97377**
HAR-3to5(243)	**0.98046**	**0.85235**	**0.97815**
HAR-HALFYEAR	**0.97709**	**0.86449**	**0.97873**

Table 9: Relative MSE values of different samples for Nikkei 225 index. Values in bold represent better MSE results comparing to the best of HAR or ALMON model for particular set of models with same order of lags.

Summary

In this work, MIDAS regression model was used as a main tool to forecast realized volatility. Classical HAR model and normalized exponential Almon polynomial were used as benchmark functional restriction to compare other MIDAS models. New restriction functions were introduced as alternatives to HAR and Almon. Alternatives for HAR model were chosen according to Müller et al. [1997], where different market participants with different time horizons can be expressed as a power of two natural numbers. Some empirical results lead to an inclusion of restriction function as a combination of two Gamma probability density functions.

Estimation results have shown that newly proposed functional constraints quite often show better forecasting performance regarding mean squared errors. There are also indexes for which new restriction functions are worse or don't fit at all, what means that one must not choose new restriction functions blindly against classical ones all the time.

If combination of two Gamma probability density functions is fairly easy to understand, there are some more questions regarding decision what parameters one need to choose for restriction function proposed by Müller et al. [1997]. More tests could be done to clarify the decision.

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Appendix

In this section results are presented for indexes Kospi, Euro and FTSE 100. Same output is provided as for indexes S&P 500, Russel 2000 and Nikkei 225.

Kospi index

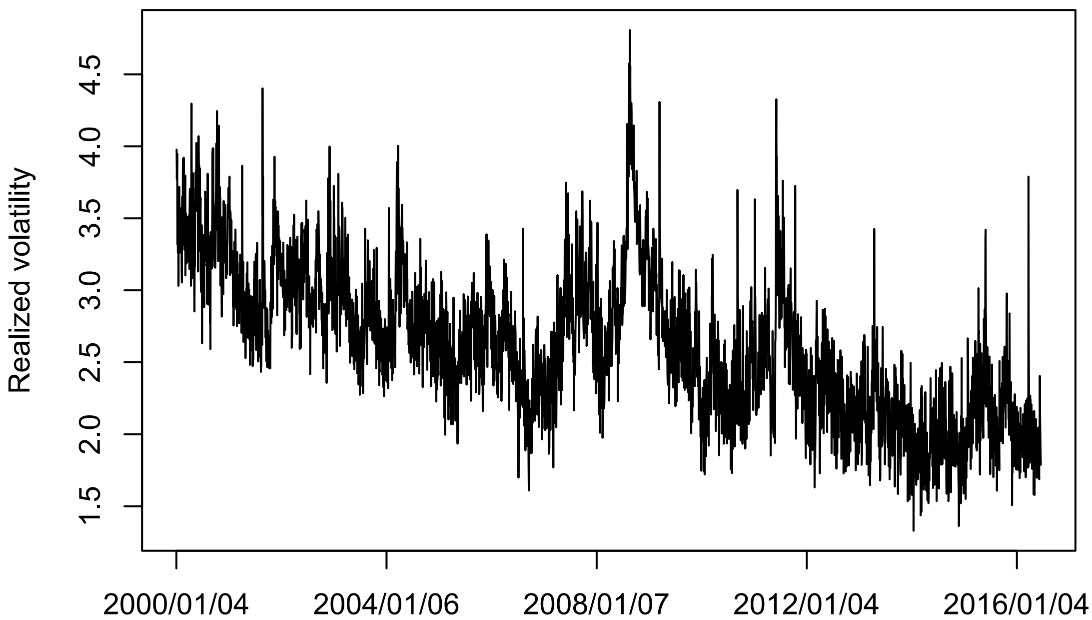

Figure 10: Realized volatility of Kospi index

Sample:	1 - 1000	1 - 2000	1 - 3000
Criterion:	**hAh - p-value**	**hAh - p-value**	**hAh - p-value**
ALMON(12)	0.26	0.33	0.28
GAMMA(12)	0.47	0.25	0.09
GAMMA-COMB(12)	0.28	0.09	0.13
GAMMA-COMB-2(12)	0.27	0.08	0.12
HAR	0.89	0.08	**0.00**
ALMON(20)	0.43	0.32	0.19
GAMMA(20)	0.59	0.09	**0.04**
GAMMA-COMB(20)	0.55	0.30	0.26
GAMMA-COMB-2(20)	0.41	0.41	0.21
ALMON(128)	0.92	0.54	**0.01**
GAMMA(128)	0.93	0.71	0.13
GAMMA-COMB(128)	0.91	0.85	0.24
GAMMA-COMB-2(128)	0.90	0.82	0.26
HAR-2to7(128)	0.94	0.84	0.17
HAR-3to5(243)	0.98	0.66	0.27
HAR-HALFYEAR	0.94	0.79	**0.01**

Table 10: hAh test p-values for different sample size of Kospi index. Values in bold denote functional constraints that can be rejected.

Sample:	1 - 1000		1 - 2000		1 - 3000	
Criterion:	**AIC**	**BIC**	**AIC**	**BIC**	**AIC**	**BIC**
ALMON(12)	**-135.24**	**-110.76**	**-287.02**	**-259.05**	**-247.91**	**-217.90**
GAMMA(12)	-135.11	-110.63	-280.13	-252.16	-230.97	-200.96
GAMMA-COMB(12)	-131.95	-92.79	-280.31	-229.85	-242.58	-194.56
GAMMA-COMB-2(12)	-129.06	-89.89	-279.11	-234.35	-240.93	-192.91
HAR	**-140.16**	**-115.72**	-286.73	-258.78	-229.69	-199.69
ALMON(20)	-132.89	-108.45	**-293.02**	**-265.06**	**-254.04**	**-224.04**
GAMMA(20)	-132.00	-107.56	-277.71	-183.81	-230.39	-200.39
GAMMA-COMB(20)	-129.37	-90.27	-286.83	-225.56	-251.23	-203.23
GAMMA-COMB-2(20)	-128.95	-89.85	-284.29	-239.56	-250.86	-202.86
ALMON(128)	-153.27	**-129.41**	-300.08	-272.41	-239.13	-209.32
GAMMA(128)	-153.03	-129.18	-301.08	**-273.41**	-250.78	-220.97
GAMMA-COMB(128)	-148.84	-110.68	-310.49	-228.74	-268.58	-220.88
GAMMA-COMB-2(128)	-147.68	-109.52	-306.11	-261.83	-268.96	-221.26
HAR-2to7(128)	-150.46	-102.75	-308.14	-252.79	-262.90	-203.28
HAR-3to5(243)	**-159.73**	-122.69	**-311.05**	-267.28	**-271.74**	**-224.36**
HAR-HALFYEAR	-153.34	-119.88	-301.36	-262.59	-238.82	-197.06

Table 11: AIC/BIC values of different samples for Kospi index. Values in bold represent best models regarding AIC/BIC for each set of different lag models.

Out - sample:	1001 - 2000	2001 - 3000	3001 - 4000
In - sample:	**1 - 1000**	**1 - 2000**	**1 - 3000**
Criterion:	**MSE**	**MSE**	**MSE**
ALMON(12)	1.00000	1.00000	1.00000
GAMMA(12)	1.00311	1.00702	1.00525
GAMMA-COMB(12)	1.00458	**0.99981**	**0.99780**
GAMMA-COMB-2(12)	1.00145	**0.99891**	**0.99771**
HAR	1.01261	1.01635	1.00000
ALMON(20)	1.00000	1.00000	1.00594
GAMMA(20)	1.00316	1.00734	1.01141
GAMMA-COMB(20)	1.02094	**0.99873**	1.00196
GAMMA-COMB-2(20)	1.00302	**0.99891**	1.00184
ALMON(128)	1.00000	1.00000	1.00000
GAMMA(128)	**0.99631**	**0.99089**	**0.98031**
GAMMA-COMB(128)	**0.99652**	**0.98334**	**0.97026**
GAMMA-COMB-2(128)	**0.99502**	**0.98166**	**0.96977**
HAR-2to7(128)	1.00547	**0.99074**	**0.96294**
HAR-3to5(243)	1.00327	**0.98379**	**0.96763**
HAR-HALFYEAR	1.00729	1.00272	**0.96388**

Table 12: Relative MSE values of different samples for Kospi index. Values in bold represent better MSE results comparing to the best of HAR or ALMON model for particular set of models with same order of lags.

Euro index

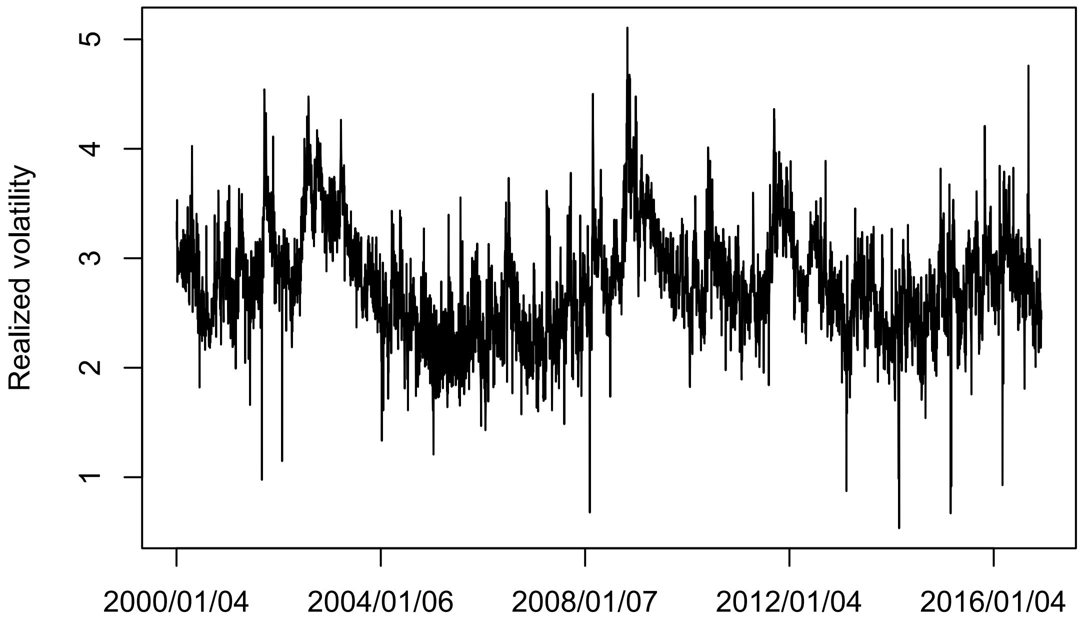

Figure 11: Realized volatility of Euro index

Sample:	1 - 1000	1 - 2000	1 - 3000
Criterion:	hAh - p-value	hAh - p-value	hAh - p-value
ALMON(12)	**0.01**	**0.00**	**0.00**
GAMMA(12)	0.07	**0.01**	**0.02**
GAMMA-COMB(12)	0.10	**0.00**	**0.00**
GAMMA-COMB-2(12)	0.11	**0.00**	**0.00**
HAR	0.19	**0.03**	**0.00**
ALMON(20)	0.09	**0.00**	**0.00**
GAMMA(20)	0.23	0.05	**0.01**
GAMMA-COMB(20)	0.28	**0.02**	**0.00**
GAMMA-COMB-2(20)	0.32	**0.02**	**0.00**
ALMON(128)	0.15	**0.00**	**0.00**
GAMMA(128)	0.24	**0.02**	**0.00**
GAMMA-COMB(128)	0.32	**0.02**	**0.00**
GAMMA-COMB-2(128)	0.29	**0.02**	**0.00**
HAR-2to7(128)	0.17	**0.02**	**0.00**
HAR-3to5(243)	0.96	0.06	**0.04**
HAR-HALFYEAR	0.27	**0.03**	**0.00**

Table 13: hAh test p-values for different sample size of Euro index. Values in bold denote functional constraints that can be rejected.

Sample:	1 - 1000		1 - 2000		1 - 3000	
Criterion:	AIC	BIC	AIC	BIC	AIC	BIC
ALMON(12)	138.61	163.09	601.94	629.91	801.38	**831.39**
GAMMA(12)	**135.31**	**159.79**	599.83	**627.80**	804.90	834.90
GAMMA-COMB(12)	135.49	174.65	598.15	642.90	**795.69**	843.71
GAMMA-COMB-2(12)	136.83	176.00	**597.60**	642.36	796.87	844.89
HAR	141.02	165.46	**596.58**	**624.54**	797.21	**827.21**
ALMON(20)	144.34	168.78	603.79	631.74	803.66	833.66
GAMMA(20)	**140.74**	**165.17**	600.64	628.59	806.47	836.47
GAMMA-COMB(20)	144.18	183.28	610.64	655.37	798.58	846.58
GAMMA-COMB-2(20)	142.06	181.16	599.79	644.52	**797.18**	845.18
ALMON(128)	119.32	143.18	586.71	614.38	798.09	827.91
GAMMA(128)	114.40	**138.25**	572.41	**600.08**	777.60	**807.42**
GAMMA-COMB(128)	**113.01**	151.17	572.05	616.33	770.52	818.22
GAMMA-COMB-2(128)	116.17	154.34	573.68	617.96	771.26	818.96
HAR-2to7(128)	122.78	170.49	576.03	631.37	768.03	827.65
HAR-3to5(243)	122.21	159.24	575.18	618.95	775.62	822.99
HAR-HALFYEAR	117.68	151.14	**569.32**	608.09	**767.56**	809.31

Table 14: AIC/BIC values of different samples for Euro index. Values in bold represent best models regarding AIC/BIC for each set of different lag models.

Out - sample:	**1001 - 2000**	**2001 - 3000**	**3001 - 4000**
In - sample:	**1 - 1000**	**1 - 2000**	**1 - 3000**
Criterion:	**MSE**	**MSE**	**MSE**
ALMON(12)	1.00000	1.00000	1.00000
GAMMA(12)	**0.99745**	1.00251	**0.99743**
GAMMA-COMB(12)	**0.99924**	**0.99781**	1.00345
GAMMA-COMB-2(12)	**0.99760**	**0.99857**	1.00337
HAR	1.00000	1.00142	1.00832
ALMON(20)	1.00266	1.00000	1.00000
GAMMA(20)	**0.99968**	1.00380	1.00601
GAMMA-COMB(20)	**0.99862**	1.02256	1.01041
GAMMA-COMB-2(20)	1.00112	**0.99988**	1.00929
ALMON(128)	1.00000	1.00000	1.00000
GAMMA(128)	**0.98902**	**0.99033**	**0.99713**
GAMMA-COMB(128)	**0.99158**	**0.98573**	**0.99980**
GAMMA-COMB-2(128)	**0.99031**	**0.98693**	**0.99872**
HAR-2to7(128)	**0.99842**	**0.98416**	1.00329
HAR-3to5(243)	1.00002	**0.99005**	**0.99425**
HAR-HALFYEAR	**0.99370**	**0.98861**	**0.99862**

Table 15: Relative MSE values of different samples for Euro index. Values in bold represent better MSE results comparing to the best of HAR or ALMON model for particular set of models with same order of lags.

FTSE 100 index

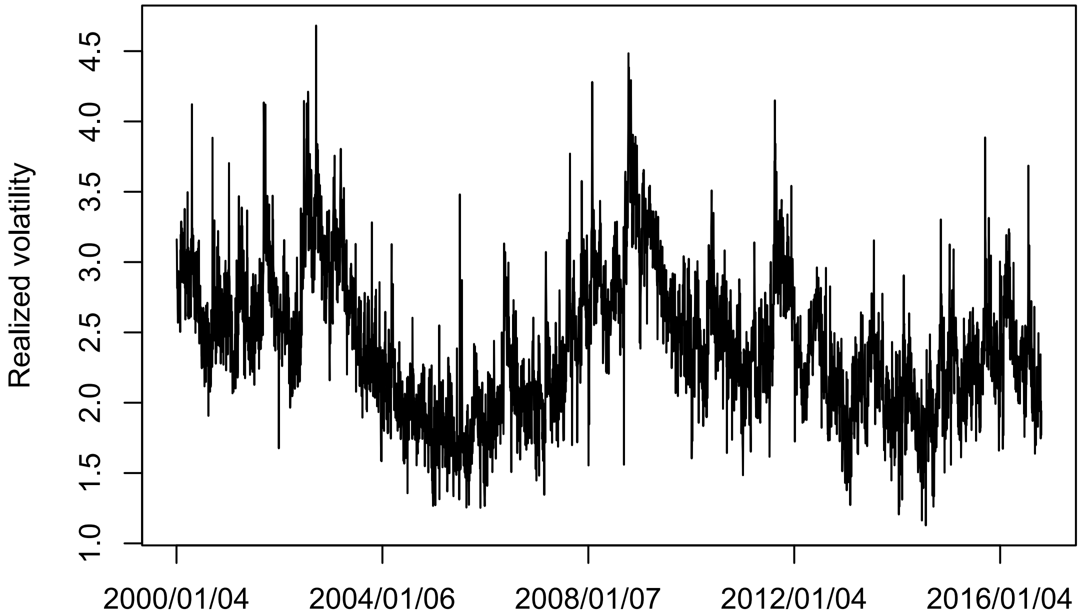

Figure 12: Realized volatility of FTSE100 index

Sample:	1 - 1000	1 - 2000	1 - 3000
Criterion:	**hAh - p-value**	**hAh - p-value**	**hAh - p-value**
ALMON(12)	0.64	**0.02**	**0.00**
GAMMA(12)	0.73	**0.02**	**0.00**
GAMMA-COMB(12)	0.53	**0.02**	**0.00**
GAMMA-COMB-2(12)	0.54	**0.02**	**0.00**
HAR	**0.02**	**0.02**	**0.00**
ALMON(20)	0.19	**0.02**	**0.00**
GAMMA(20)	0.24	0.11	**0.00**
GAMMA-COMB(20)	0.12	0.05	**0.00**
GAMMA-COMB-2(20)	0.12	0.05	**0.00**
ALMON(128)	0.54	0.21	**0.00**
GAMMA(128)	0.64	0.08	**0.01**
GAMMA-COMB(128)	0.57	0.22	**0.01**
GAMMA-COMB-2(128)	0.57	0.16	**0.01**
HAR-2to7(128)	0.48	0.15	**0.01**
HAR-3to5(243)	0.99	0.13	0.05
HAR-HALFYEAR	0.50	0.14	**0.01**

Table 16: hAh test p-values for different sample size of FTSE100 index. Values in bold denote functional constraints that can be rejected.

Sample:	1 - 1000		1 - 2000		1 - 3000	
Criterion:	**AIC**	**BIC**	**AIC**	**BIC**	**AIC**	**BIC**
ALMON(12)	129.11	153.59	**124.41**	**152.38**	2571.23	2601.24
GAMMA(12)	**128.48**	**152.96**	125.12	153.10	208.66	**238.67**
GAMMA-COMB(12)	133.21	172.38	125.64	170.40	217.79	265.81
GAMMA-COMB-2(12)	134.13	173.29	126.26	171.02	**201.59**	249.61
HAR	140.07	164.51	124.88	152.83	**200.29**	**230.29**
ALMON(20)	130.96	155.40	125.36	153.31	205.18	235.18
GAMMA(20)	**130.16**	**154.59**	**123.45**	**151.41**	207.02	237.02
GAMMA-COMB(20)	135.42	174.52	124.86	169.59	213.00	261.00
GAMMA-COMB-2(20)	135.71	174.81	126.12	170.85	200.91	248.91
ALMON(128)	141.85	165.71	144.66	172.33	235.56	265.38
GAMMA(128)	138.30	**162.16**	130.31	157.99	213.60	243.41
GAMMA-COMB(128)	143.74	181.91	129.91	174.19	206.42	254.13
GAMMA-COMB-2(128)	143.76	181.92	132.50	176.78	206.87	254.57
HAR-2to7(128)	149.43	197.14	132.39	187.73	200.72	260.35
HAR-3to5(243)	**125.34**	162.37	**113.14**	**156.91**	**181.24**	**228.62**
HAR-HALFYEAR	143.28	176.74	121.29	160.06	190.92	232.68

Table 17: AIC/BIC values of different samples for FTSE100 index. Values in bold represent best models regarding AIC/BIC for each set of different lag models.

Out - sample:	**1001 - 2000**	**2001 - 3000**	**3001 - 4000**
In - sample:	**1 - 1000**	**1 - 2000**	**1 - 3000**
Criterion:	**MSE**	**MSE**	**MSE**
ALMON(12)	1.00000	1.00000	1.00000
GAMMA(12)	1.00012	1.00076	**0.49997**
GAMMA-COMB(12)	1.00012	**0.99474**	**0.50022**
GAMMA-COMB-2(12)	**0.99762**	**0.99603**	**0.50174**
HAR	1.00000	1.00000	1.00000
ALMON(20)	1.01092	1.00454	1.00008
GAMMA(20)	1.00974	1.00569	1.00435
GAMMA-COMB(20)	1.00915	1.00013	1.00531
GAMMA-COMB-2(20)	1.00765	1.00098	1.00809
ALMON(128)	1.00000	1.00000	1.00000
GAMMA(128)	**0.98485**	**0.98957**	**0.99613**
GAMMA-COMB(128)	**0.98265**	**0.98092**	**0.99549**
GAMMA-COMB-2(128)	**0.98639**	**0.98621**	**0.99754**
HAR-2to7(128)	**0.98575**	**0.98092**	**0.99629**
HAR-3to5(243)	**0.99212**	**0.97964**	1.00280
HAR-HALFYEAR	**0.97745**	**0.97988**	**0.99222**

Table 18: Relative MSE values of different samples for FTSE100 index. Values in bold represent better MSE results comparing to the best of HAR or ALMON model for particular set of models with same order of lags.