VILNIAUS UNIVERSITETAS
MATEMATIKOS IR INFORMATIKOS FAKULTETAS
TIKIMYBIŲ TEORIJOS IR SKAIČIŲ TEORIJOS KATEDRA

Laima Kaziulytė

Asymptotic Distribution of Beurling Integers

Magistro baigiamasis darbas

Leidžiu ginti ....

Darbo vadovas prof. Ramūnas Garunkštis

Vilnius 2016

Contents

Abstract

We study generalised prime (g-prime) systems $\mathcal{P}$ and g-integer systems $\mathcal{N}$ obtained from them. The asymptotic distribution of g-integers is given with the assumption that g-prime counting function $\pi_{\mathcal{P}}(x)$ behaves as

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + \mathrm{O}(x^{\alpha}) \qquad (x \to +\infty)$$

for some $b > 0, \alpha \in (0, 1)$.

Chapter 1

Introduction

1.1 Generalised Primes and Beurling Zeta Functions

As usually, let s denote a complex variable with σ and t it's real and imaginary parts respectively. We always assume that $x \to \infty$.

A *generalised prime system* $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, \ldots$ satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_n \leq \ldots$$

and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of *generalised integers* or *Beurling integers*; that is, the numbers of the form

$$p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

where $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{N}_0$.[1] This system generalises the notion of prime numbers and the natural numbers obtained from them. Such systems (along with the attached zeta functions) were first introduced by Beurling [1] and have been studied by numerous authors since then (see, for instance, the papers by Fainleib [6], Hall [7], Hilberdink and Lapidus [9], Stankus [15] and Zhang [18–20]).

First define the *counting functions* $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$ by

$$\pi_{\mathcal{P}}(x) = \sum_{p \leq x,\, p \in \mathcal{P}} 1, \tag{1.1}$$

$$N_{\mathcal{P}}(x) = \sum_{n \leq x,\, n \in \mathcal{N}} 1. \tag{1.2}$$

[1] Here and henceforth $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Here, as elsewhere in the paper, we write $\sum_{p\in\mathcal{P}}$ to mean a sum over all the generalised primes, counting multiplicities. Similarly for $\sum_{n\in\mathcal{N}}$. Much of the research on this subject has been about connecting the asymptotic behaviour of the *generalised prime counting function* (1.1) and of the *generalised integer counting function* (1.2) as $x \to \infty$. Specifically, given the asymptotic behaviour of $\pi_{\mathcal{P}}(x)$, what can be said about the bahaviour of $N_{\mathcal{P}}(x)$, and vice versa.

Many of the known results involve the associated *zeta function*, often referred to as a *Beurling zeta function* in the literature, which we define formally by the *Euler product*

$$\zeta_{\mathcal{P}}(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}.$$

This infinite product may be formally multiplied out to give the *Dirichlet series*

$$\zeta_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{1}{n^s}.$$

Note that when $\mathcal{P}$ is the set of (rational) primes, and hence $\mathcal{N}$ is the set of natural numbers, $\zeta_{\mathcal{P}}$ coincides with the classical Riemann zeta function. Further, $\pi_{\mathcal{P}}(x)$ (resp. $N_{\mathcal{P}}(x)$) is just the standard prime (resp. integer) counting function.

All the classical functions (when $\mathcal{N} = \mathbb{N}$) are written without any index: $\zeta(s), \Lambda(n)$.

1.2 Overview of Relevant Known Results

In this section we give a summary of the known results relating the asymptotic behaviour of $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$.

The research has concentrated on finding conditions for which results of the form

$$N_{\mathcal{P}}(x) = ax + E_1(x) \qquad \Longleftrightarrow \qquad \pi_{\mathcal{P}}(x) = \mathrm{li}(x) + E_2(x)$$

hold. Here a is a positive constant, $\mathrm{li}(x)$ is the logarithmic integral given by

$$\mathrm{li}(x) = \lim_{\varepsilon\to 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{\mathrm{d}t}{\log t},$$

and $E_1(x)$ and $E_2(x)$ are error terms of smaller order than x and $\mathrm{li}(x)$, respectively. The error terms which have been studied (and seem to occur naturally) are of three types; namely, those of the form

$$\mathrm{O}\left(\frac{x}{(\log x)^{\gamma}}\right), \mathrm{O}\left(x\mathrm{e}^{-c(\log x)^{\alpha}}\right) \quad \text{and} \quad \mathrm{O}(x^{\theta}),$$

where $\gamma > 1$, $c > 0$ and α, $\theta \in (0,1)$.

- Beurling ([1], 1937) showed that

$$N_{\mathcal{P}}(x) = ax + \mathrm{O}\left(\frac{x}{(\log x)^{\gamma}}\right) \quad \text{for some } \gamma > 3/2 \quad \text{implies} \quad \pi_{\mathcal{P}}(x) \sim \frac{x}{\log x},^{2}$$

  which is an analogue (in this more general context) of the Prime Number Theorem. Furthermore, he showed by example that this is false in general for $\gamma = 3/2$. Conversely, it follows from Diamond's work ([4], Theorem 2) that

$$\pi_{\mathcal{P}}(x) = \frac{x}{\log x} + \mathrm{O}\left(\frac{x}{(\log x)^{1+\delta}}\right) \quad \text{for some } \delta > 0 \text{ implies } N_{\mathcal{P}}(x) \sim ax.$$

- Nyman ([13], 1949) showed that

$$N_{\mathcal{P}}(x) = ax + \mathrm{O}\left(\frac{x}{(\log x)^{A}}\right) \ (\forall A) \iff \pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \mathrm{O}\left(\frac{x}{(\log x)^{A}}\right) \ (\forall A).$$

- Malliavin ([12], 1961) showed that

$$N_{\mathcal{P}}(x) = ax + \mathrm{O}\left(x\mathrm{e}^{-c_1(\log x)^{\alpha}}\right) \tag{1.3}$$

  for some $\alpha \in (0,1)$ and $c_1 > 0$, implies

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \mathrm{O}\left(x\mathrm{e}^{-c_2(\log x)^{\beta}}\right) \tag{1.4}$$

  for some $c_2 > 0$, where $\beta = \alpha/10$. Hall ([8], 1971) improved this to $\beta = \alpha/7.91$. Conversely, Malliavin ([12], 1961) showed that if (1.4) holds for some $\beta \in (0,1)$ and $c_2 > 0$, then (1.3) holds for some $a, c_1 > 0$ and $\alpha = \frac{\beta}{2+\beta}$. Diamond ([3], 1970) improved this to $\alpha = \frac{\beta}{1+\beta}$, and furthermore, Diamond's result contains $\log x \log\log x$ instead of $\log x$ in the exponent.

- Landau ([10], 1903) proved that

$$N_{\mathcal{P}}(x) = ax + \mathrm{O}(x^{\theta}) \qquad \text{for some } \theta < 1 \tag{1.5}$$

  implies

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \mathrm{O}(x\mathrm{e}^{-c\sqrt{\log x}}) \qquad \text{for some } c > 0.$$

  Diamond, Montgomery and Vorhauer have recently shown (see [5]) that this is essentially best possible by exhibiting a (discrete) system for which (1.5) holds but

$$\pi_{\mathcal{P}}(x) - \mathrm{li}(x) = \Omega(x\mathrm{e}^{-c'\sqrt{\log x}}) \qquad \text{for some } c' > 0.^{3}$$

[2] Here and henceforth, all such statements are implicitly assumed to be asymptotic as $x \to \infty$. Moreover, by $f(x) \sim g(x)$, we mean $f(x)/g(x) \to 1$ as $x \to \infty$.

- Bredikhin ([2], 1960) proved that if

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + O\left(\frac{x}{\log^{1+\varepsilon} x}\right)$$

for some $b > 0$ and $\varepsilon > 0$, then

$$N_{\mathcal{P}}(x) = Cx \log^{b-1} x + O\left(\frac{x \log^{b-1} x}{(\log \log x)^{\varepsilon_1}}\right), \tag{1.6}$$

where $\varepsilon_1 = \min(1, \varepsilon), C > 0$ is a constant.

- Hilberdink and Lapidus ([9], 2008) showed that if

$$\pi_{\mathcal{P}}(x) = \frac{x}{\log x} + O(x^{\alpha})$$

for some $\alpha \in (0, 1)$, then there exist positive constants C and δ such that

$$N_{\mathcal{P}}(x) = Cx + O\left(xe^{-\delta\sqrt{\log x \log \log x}}\right). \tag{1.7}$$

1.3 The Main Result

In order to state our result we need several notations and results.

We shall see below that the following definition plays a special role.

Definition 1. *Let $b > 0$ and set the function*

$$Z(s) = s^{-1}((s-1)\zeta(s))^b,$$

where $Z(s)$ is defined on any simply connected domain which does not contain a zero of $\zeta(s)$ and does not contain the point $s = 0$. We shall always suppose that this domain includes the real half-line $[1, +\infty)$. We can then choose the principal value of the complex logarithm, so that $Z(1) = 1$.

Lemma 2. *The function $Z(s)$ is holomorphic in the disc $|s - 1| < 1$, and can be represented there by the Taylor series*

$$Z(s) = \sum_{j=0}^{\infty} \frac{1}{j!}\gamma_j(b)(s-1)^j$$

where the coefficients $\gamma_j(b)$ for all $\varepsilon > 0$ satisfy the upper bound

$$\frac{1}{j!}\gamma_j(b) \ll_{b,\varepsilon} (1+\varepsilon)^j.$$

[3] Here, $f(x) = \Omega(g(x))$ as $x \to \infty$ means that there exists $c > 0$ such that $|f(x)| \geq cg(x)$ for some arbitrary large x.

The proof can be found in [16], pp. 182.

Throughout this paper, we shall use the weighted counting function

$$\psi_{\mathcal{P}}(x) = \sum_{p^k \le x, k \in \mathbb{N}} \log p = \sum_{n \le x, n \in \mathcal{N}} \Lambda_{\mathcal{P}}(n).$$

Here $\Lambda_{\mathcal{P}}$ denotes the (generalised) von Mangoldt function, defined for n in the multiset $\mathcal{N}$ by $\Lambda_{\mathcal{P}}(n) = \log p$ if $n = p^m$ for some $p \in \mathcal{P}$ and $m \in \mathbb{N}$, and $\Lambda_{\mathcal{P}}(n) = 0$ otherwise. We shall also write

$$\phi_{\mathcal{P}}(s) = -\frac{\zeta'_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(s)} = \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s}.$$

The counting functions $N_{\mathcal{P}}(x)$ and $\psi_{\mathcal{P}}(x)$ are related to $\zeta_{\mathcal{P}}(s)$ and $\phi_{\mathcal{P}}(s)$ via

$$\zeta_{\mathcal{P}}(s) = s\int_1^\infty \frac{N_{\mathcal{P}}(x)}{x^{s+1}}\,\mathrm{d}x \quad \text{and} \quad \phi_{\mathcal{P}}(s) = s\int_1^\infty \frac{\psi_{\mathcal{P}}(x)}{x^{s+1}}\,\mathrm{d}x.$$

As a result, it is often more convenient to work with $\psi_{\mathcal{P}}(x)$, rather than $\pi_{\mathcal{P}}(x)$. Note that for $\alpha \in [\frac{1}{2}, 1)$, $b > 0$, the statements

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + \mathrm{O}\left(x^{\alpha+\varepsilon}\right) \quad (\forall \varepsilon > 0) \quad \text{and} \quad \psi_{\mathcal{P}}(x) = bx + \mathrm{O}\left(x^{\alpha+\varepsilon}\right) \quad (\forall \varepsilon > 0),$$

are equivalent. For $\mathcal{N} = \mathbb{N}$, it is well-known that the above are equivalent to the absence of zeros of the Riemann zeta function in the region $\{s \in \mathbb{C} : \Re s > \alpha\}$.

It could be that $\zeta_{\mathcal{P}}(s)$ has no meromorphic continuation to some region to the left of 1. Because of that we will use an auxiliary function

$$G(s) = \zeta_{\mathcal{P}}\, \zeta^{-b}(s) \qquad (\sigma > 1).$$

Denote

$$R = \begin{cases} s \in \mathbb{C} : \sigma \ge \max\left(1 - \frac{A}{(\log|t|)^{-\frac{2}{3}}(\log\log|t|)^{-\frac{1}{3}}}, \alpha\right) & \text{for } |t| \ge 3, \\ s \in \mathbb{C} : \sigma \ge \max\left(1 - \frac{A}{(\log 3)^{-\frac{2}{3}}(\log\log 3)^{-\frac{1}{3}}}, \alpha\right) & \text{for } |t| \le 3, \end{cases}$$

where $\alpha \in (0,1)$, A is a positive constant from Lemma 11. We can assume that $A < 1$.

Lemma 3. *Suppose that for some $\alpha \in (0,1)$ and $b > 0$, we have*

$$\psi_{\mathcal{P}}(x) = bx + \mathrm{O}(x^\alpha).$$

Then $G(s)$ has an analytic continuation to the region R, which is defined above.

The proof of this lemma is given in section 2.1.

In the region R, where $G(s)$ is analytic, we set

$$\lambda_k(b) = \frac{1}{\Gamma(b-k)} \sum_{h+j=k} \frac{1}{h!j!} G^{(h)}(1)\gamma_j(b) \tag{1.8}$$

where the $\gamma_j(b)$ are the coefficients appearing in Lemma 2.

Our aim is to prove the following theorem.

Theorem 4. *Let N be a positive integer, $b > 0$ and $\alpha \in (0,1)$. If*

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + O\left(x^{\alpha}\right)$$

then we have, uniformly in N,

$$N_{\mathcal{P}}(x) = x(\log x)^{b-1} \left\{ \sum_{k=0}^{N} \frac{\lambda_k(b)}{(\log x)^k} + \mathrm{O}\left(R_N(x)\right) \right\} \tag{1.9}$$

with

$$R_N(x) = \mathrm{e}^{-c_6\sqrt{\log x}} + \left(\frac{c_4 N + 1}{\log x}\right)^{N+1}.$$

The positive constants c_4, c_6 and the implicit constant in the Landau symbol depend at most on b and α. The coefficients $\lambda_k(b)$ are defined by formula (1.8).

We prove this theorem in section 2.2. Theorem 4 can be compared to the results of Bredikhin (formula (1.6)) and Hilberdink and Lapidus (formula (1.7)).

Corollary 5. *Let $b \in \mathbb{N}$ and $\alpha \in (0,1)$. If*

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + O\left(x^{\alpha}\right),$$

then we obtain

$$N_{\mathcal{P}}(x) = x(\log x)^{b-1} \left\{ P\left(\frac{1}{\log x}\right) + \mathrm{O}\left(\mathrm{e}^{-c_6\sqrt{\log x}}\right) \right\},$$

where P is a polynomial of degree at most $b-1$, $c_6 > 0$ is a constant which depends at most on b and α.

Proof. By formula (1.8) we have $\lambda_k(b) = 0$ whenever $k \geq b$. We can hence choose N so as to minimise the error term in (1.9). By choosing $N = [(\log x)/\mathrm{e}c_4]$ [4] in Theorem 4, we get the desired result. □

[4] Here we use square brackets to denote the integer part of a number.

Chapter 2

Proofs

2.1 Auxiliary Statements

Definition 6. *In the half-plane $\sigma > 0$ Gamma function $\Gamma(s)$ is defined by the integral*

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1}\, \mathrm{d}u.$$

Definition 7. *Given a positive parameter r, we designate by Hankel contour the path in the complex plane continuing from $-\infty$ along the real line (arbitrary close, but below it) to $-r$, counterclockwise around a circle of radius r at 0, back to $-r$ on the real line, and back to $-\infty$ along the real line (arbitrary close, but above it). (See Figure 2.1 below).*

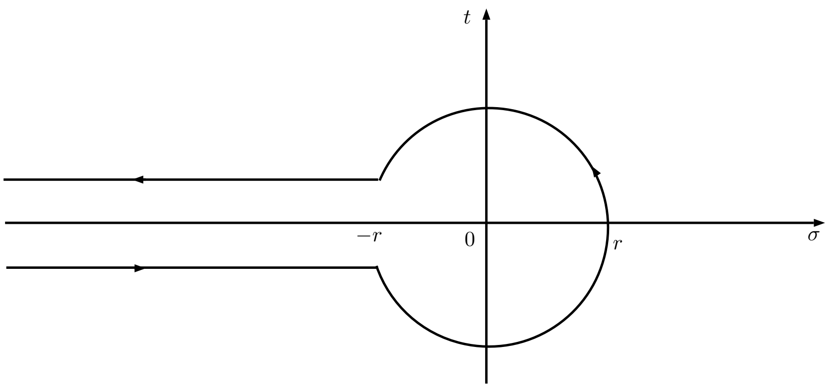

Figure 2.1: Hankel contour

Lemma 8. *For each $x > 1$, let $\mathcal{H}(x)$ denote the part of the Hankel contour situated* *in the half-plane $\sigma > -x$. Then we have uniformly for $z \in \mathbb{C}$*

$$\frac{1}{2\pi i} \int_{\mathcal{H}(x)} s^{-z} \mathrm{e}^{s} \ \mathrm{d}s = \frac{1}{\Gamma(z)} + \mathrm{O}\left(47^{|z|}\, \Gamma(1+|z|)\, \mathrm{e}^{-\frac{1}{2}x}\right).$$

The proof can be found in [16], pp. 184.

Lemma 9. ***(Stirling's formula)*** *For $\delta > 0 \ \exists\, c = c(\delta)$, such that for $s \in \{s \in \mathbb{C} : -\pi + \delta \le \arg s \le \pi - \delta,\ s \ne 0\}$,*

$$\left|\log \Gamma(s) - \left((s - 1/2)\log s - s + \log\sqrt{2\pi}\right)\right| < \frac{c}{|s|},$$

where we take the principal part of the logarithm.

For a proof see, for example [11], pp. 30.

Lemma 10. ***(Perron's formula)*** *Let*

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be a Dirichlet series with abscissa of convergence σ_c and

$$A(x) = \sum_{n \le x} a_n \qquad\qquad (x \ge 0)$$

be the summatory function of its coefficients. Then for $\kappa > \max(0, \sigma_c)$ and $x \ge 1$, we have

$$\int_0^x A(t)\ \mathrm{d}t = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} F(s) x^{s+1} \frac{\mathrm{d}s}{s(s+1)}.$$

For the proof see [16], pp. 134.

Lemma 11. *Let A be a positive constant. The region*

$$\sigma \ge 1 - A(\log t)^{-\frac{2}{3}} (\log\log t)^{-\frac{1}{3}} \qquad (t \ge 3)$$

is free of zeros of the function $\zeta(s)$ and in this region

$$\frac{1}{\zeta(s)} \ll (\log t)^{\frac{2}{3}} (\log\log t)^{\frac{1}{3}} \qquad (t \to \infty).$$

The proof can be found in [17], pp. 135.

Proof of Lemma 3. Denote

$$f(s) = \left(\log \frac{\zeta_{\mathcal{P}}(s)}{\zeta^{b}(s)}\right)'.$$

Then we can write $f(s)$ as

$$f(s) = \frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) - b\,\frac{\zeta'}{\zeta}(s) = \frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s) - \left(b\,\frac{\zeta'}{\zeta}(s) + b\,\zeta(s)\right). \tag{2.1}$$

The sum in the parentheses of (2.1) is analytic in the region R. It follows because of the following facts. The function $\zeta(s)$ is analytic in the whole plane, except for a simple pole at $s = 1$, with residue 1 and $\zeta(s)$ does not have any zeros for $\sigma \geq 1 - A(\log|t|)^{-\frac{2}{3}}(\log\log|t|)^{-\frac{1}{3}}$, $|t| \geq 3$ (see Lemma 11). We also know (see [17], pp. 389) that $\zeta(s)$ does not have any zeros in the half-plane $\sigma > 0$ with $|t| \leq 14$. Hence $\frac{\zeta'}{\zeta}(s)$ is analytic in the region R, except for a simple pole at $s = 1$ with residue equal to -1. In the sum $b\,\frac{\zeta'}{\zeta}(s) + b\,\zeta(s)$ the first terms of the Laurent series of the functions $\zeta(s)$ and $\frac{\zeta'}{\zeta}(s)$ cancell out, making this sum analytic in R.

Further, we write $\frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s)$ as a Dirichlet series

$$\frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s) = -\sum_{n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s} + \sum_{n\in\mathbb{N}} \frac{b}{n^s} = \sum_{n\in\mathbb{N}\cup\mathcal{N}} \frac{a_n}{n^s} \qquad (\sigma > 1),$$

where

$$a_n = \begin{cases} -\Lambda_p(n) & , \quad \text{if } n \in \mathcal{N}, n \neq \mathbb{N} \\ b & , \quad \text{if } n \in \mathbb{N}, n \neq \mathcal{N} \\ b - \Lambda_p(n) & , \quad \text{if } n \in \mathcal{N} \cap \mathbb{N}. \end{cases}$$

Thus, from the lemma's assumption $\psi_{\mathcal{P}}(x) = \sum_{\substack{n\leq x\\ n\in\mathcal{N}}} \Lambda_{\mathcal{P}}(n) = bx + \mathrm{O}(x^{\alpha})$ and the fact that $\sum_{\substack{n\leq x\\ n\in\mathbb{N}}} 1 = x + \mathrm{O}(1)$, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{N}\cup\mathbb{N}}} a_n = \sum_{\substack{n\leq x\\ n\in\mathbb{N}}} b - \sum_{\substack{n\leq x\\ n\in\mathcal{N}}} \Lambda_{\mathcal{P}}(n) = \mathrm{O}(x^{\alpha}).$$

Using this result and applying Abel summmation

$$\sum_{n\in\mathbb{N}\cup\mathcal{N}} \frac{a_n}{n^s} = -s\ \mathrm{O}\left(\int_1^{\infty} \frac{x^{\alpha}}{x^{\sigma+1}}\,\mathrm{d}x\right).$$

Since the latter integral converges in the half-plane $\sigma > \alpha$, the function $\frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s)$ is analytic in this region. Hence,

$$f(s) = \frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s) - \left(b\,\frac{\zeta'}{\zeta}(s) + b\,\zeta(s)\right)$$

is analytic in the region R.

Therefore, the function

$$G(s) = \frac{\zeta_{\mathcal{P}}(s)}{\zeta^b(s)} = \exp\left(\int_2^s f(u)\,\mathrm{d}u + \log\frac{\zeta_{\mathcal{P}}(2)}{\zeta^b(2)}\right)$$

is also analytic in the region R, which was our claim. □

2.2 Proof of the Main Theorem

For the proof we use the Selberg-Delange method, see Chapter II.5 in Tenenbaum [16].

For a beginning of the proof of Theorem 4 define the domain D by deleting the real segment $(\alpha, 1]$ from the region R.

Set

$$N_{\mathcal{P}}(x) = \sum_{n \le x,\, n \in \mathcal{N}} 1.$$

Then Perron formula (Lemma 10) allows us to write

$$\int_1^x N_{\mathcal{P}}(t)\ \mathrm{d}t = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \zeta_{\mathcal{P}}(s) x^{s+1} \frac{\mathrm{d}s}{s(s+1)} \tag{2.2}$$

with $\kappa = 1 + \frac{1}{\log x}$. Using this version of Perron's formula we get the function $N_{\mathcal{P}}$, that we are searching for, under the integral sign. Fortunately, this does not cause much trouble, since we can take it out and approximate using relations we will get in the sequel (see Lemma 13 in the end of the proof).

Let $T > \exp(\mathrm{e}^{2b})$ be a parameter whose value will be determined later. Put $\varepsilon(t) = A(1-\alpha)\frac{\log\log|t|}{\log|t|}$ for $|t| \ge 3$, where A is a constant from the definition of the region R. It is worth noting that $\alpha < 1 - \varepsilon(t) < 1$. The residue theorem allows us to deform the segment of integration $[\kappa - iT, \kappa + iT]$ into some path joining the endpoints $\kappa - iT, \kappa + iT$ and contained entirely in D. We choose the path symmetrically with respect to the real axis (see Figure 2.2 below). Its upper part is made up of: the truncated Hankel countour Γ, surrounding the point $s = 1$, with radius $r = \frac{1}{2\log x}$, and linear part joining $1 - r$ to $1 - \varepsilon(3)$; the vertical segment $[1 - \varepsilon(3), 1 - \varepsilon(3) + 3i]$; the curve

$$\sigma(t) = 1 - \varepsilon(t) = 1 - A(1-\alpha)\frac{\log\log|t|}{\log|t|}$$

for $3 \le t \le T$; and the horizontal segment $[\sigma(T) + iT, \kappa + iT]$.

The contour is entirely contained in D since

$$A(1-\alpha)\frac{\log\log|t|}{\log|t|} < \frac{A}{\log^{\frac{2}{3}}(|t|)(\log\log|t|)^{\frac{1}{3}}}$$

for $|t| \ge 3$, which implies that $1 - \varepsilon(t)$ is in the region R for all $|t| \le T$.

We shall see that the main contribution arises from the integral over the truncated Hankel contour Γ. We denote this integral by I_1:

$$I_1 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)}\ \mathrm{d}s.$$

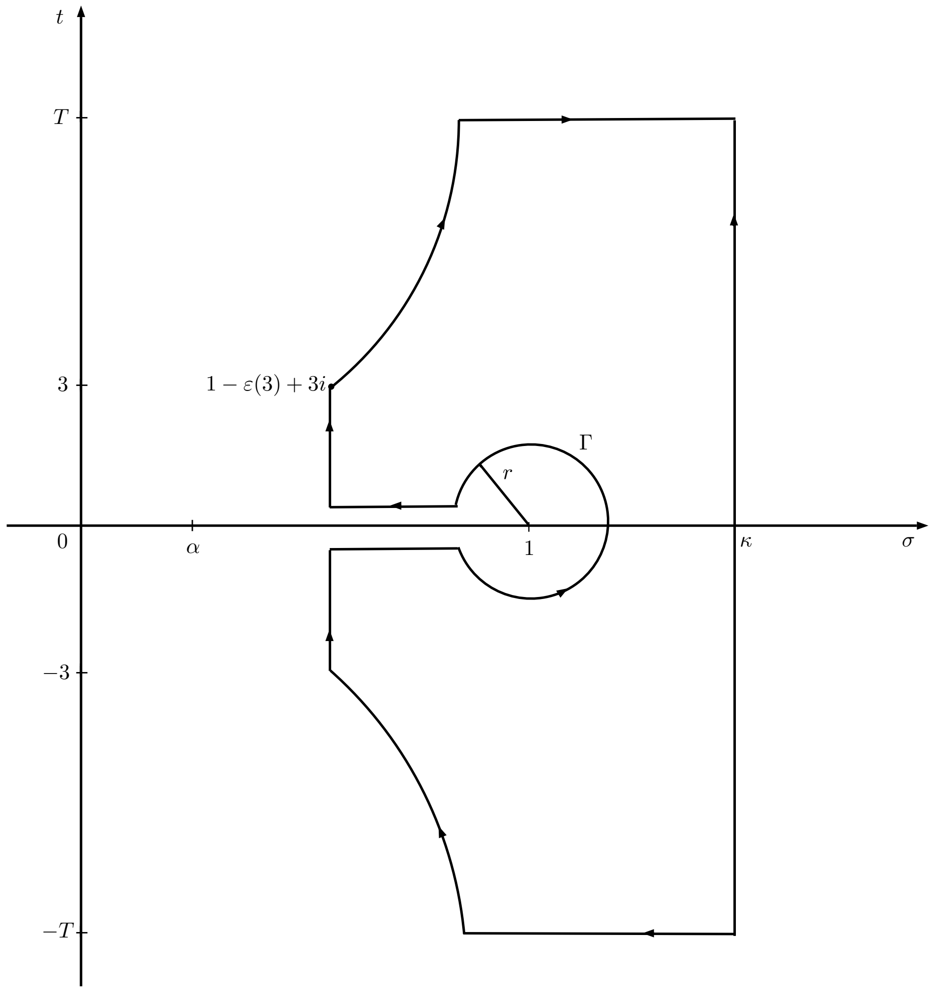

Figure 2.2: Contour of integration

The other parts of the path are denoted as follows.

$$I_2 = \frac{1}{2\pi i} \int_{[1-\varepsilon(3), 1-\varepsilon(3)+3i]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{[1-\varepsilon(3), 1-\varepsilon(3)-3i]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s,$$

$$I_3 = \frac{1}{2\pi i} \int_{1-\varepsilon(t)} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s \qquad \text{for } t \in [-T, -3] \cup [3, T],$$

$$I_4 = \frac{1}{2\pi i} \int_{[\sigma(T)+iT, \kappa+iT]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{[\sigma(-T)-iT, \kappa-iT]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s,$$

$$I_5 = \frac{1}{2\pi i} \int_{[\kappa+iT, \kappa+i\infty]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{[\kappa-iT, \kappa-i\infty]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s,$$

$$I_0 = \frac{1}{2\pi i} \int_{[\kappa-iT, \kappa+iT]} \frac{\zeta_{\mathcal{P}}(x) x^{s+1}}{s(s+1)} \, \mathrm{d}s,$$

where $\int_{[s_1, s_2]}$ denotes an integral over the interval starting at s_1 and ending at s_2. Using these notations (see Figure 2.2) we have

$$I_0 = I_1 + I_2 + I_3 + I_4.$$

Notice that all the integrals I_j for $j = 0, 1, \ldots, 5$ are functions of x, i.e. $I_1 = I_1(x)$, $I_2 = I_2(x)$ and so on. To evaluate some of these integrals, we need a bound of $\zeta_{\mathcal{P}}$.

Lemma 12. *For sufficiently large $|t|$ we have*

$$|\zeta_{\mathcal{P}}(1-\varepsilon(t)+it)| \leq |t|^{\frac{2b}{\log\log|t|}}.$$

Proof. To get this bound we follow part of the proof of Theorem 2.2 of Hilberdink and Lapidus paper [9]. Let $\psi_{\mathcal{P}}(x) = bx + r(x)$, so that $r(x) = \mathrm{O}(x^{\alpha})$. It follows that

$$\phi_{\mathcal{P}}(s) = s\int_1^{\infty} \frac{bx + r(x)}{x^{s+1}}\,\mathrm{d}x = \frac{bs}{s-1} + \int_1^{\infty} \frac{r(x)}{x^{s+1}}\,\mathrm{d}x.$$

The latter integral converges for $\Re s > \alpha$ and represents an analytic function in this half-plane. This provides the analytic continuation of $\phi(s)$ to $\{s \in \mathbb{C} : \Re s > \alpha\}$ except for a simple pole at $s = 1$ with residue b. Moreover, $\zeta_{\mathcal{P}}(s)$ has no zeros in this region, for if it did, then $\phi_{\mathcal{P}}(s) = -\frac{\zeta_{\mathcal{P}}(s)'}{\zeta_{\mathcal{P}}(s)}$ would have a singularity.

Now consider the sum $\sum_{n\leq x} \frac{\Lambda_{\mathcal{P}}(n)}{n^s}$ for $\Re s > \alpha$, where n ranges over elements of $\mathcal{N}$. We have

$$\begin{aligned}\sum_{n\leq x, n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s} &= \frac{\psi_{\mathcal{P}}(x)}{x^s} + s\int_1^x \frac{\psi_{\mathcal{P}}(y)}{y^{s+1}}\,\mathrm{d}y \\ &= bx^{1-s} + \frac{r(x)}{x^s} + bs\int_1^x \frac{1}{y^s}\,\mathrm{d}y + s\int_1^x \frac{r(y)}{y^{s+1}}\,\mathrm{d}y \qquad (2.3)\\ &= \frac{bx^{1-s}}{1-s} + \phi_{\mathcal{P}}(s) + \frac{r(x)}{x^s} - s\int_x^{\infty} \frac{r(y)}{y^{s+1}}\,\mathrm{d}y.\end{aligned}$$

Thus

$$\phi_{\mathcal{P}}(s) = \sum_{n\leq x, n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s} - \frac{bx^{1-s}}{1-s} - \frac{r(x)}{x^s} + s\int_x^{\infty} \frac{r(y)}{y^{s+1}}\,\mathrm{d}y.$$

Writing $s = \sigma + it$, and using $r(x) = \mathrm{O}(x^{\alpha})$, we obtain

$$|\phi_{\mathcal{P}}(\sigma+it)| \leq \sum_{n\leq x, n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^{\sigma}} + \mathrm{O}\left(\frac{x^{1-\sigma}}{|t|}\right) + \mathrm{O}(|t|x^{\alpha-\sigma}). \qquad (2.4)$$

Here and further in this lemma we assume that $|t|$ under the big O symbol approaches infinity. To estimate the first term on the right of (2.4), put $t = 0$ in (2.3) to give

$$\begin{aligned}\sum_{n\leq x, n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^{\sigma}} &= \frac{bx + r(x)}{x^{\sigma}} + b\sigma\int_1^x y^{-\sigma}\,\mathrm{d}y + \sigma\int_1^x \frac{r(y)}{y^{\sigma+1}}\,\mathrm{d}y \\ &= bx^{1-\sigma} + \frac{b\sigma}{1-\sigma}\left(x^{1-\sigma} - 1\right) + \mathrm{O}(x^{\alpha-\sigma}) + \mathrm{O}(1) \\ &= b\,\frac{x^{1-\sigma} - 1}{1-\sigma} + \mathrm{O}(1). \qquad (2.5)\end{aligned}$$

This also holds for $\sigma = 1$, if we interpret the first term on the right of (2.5) as $b\log x$. Moreover, with this interpretation, the above estimate (2.5) is uniform for $\sigma \in [\alpha+\delta, c_0]$ for any $c_0 > 1$ and $\delta > 0$. Combining these gives

$$|\phi_{\mathcal{P}}(\sigma+it)| \leq b\,\frac{x^{1-\sigma} - 1}{1-\sigma} + \mathrm{O}(1) + \mathrm{O}\left(\frac{x^{1-\sigma}}{|t|}\right) + \mathrm{O}(|t|x^{\alpha-\sigma}).$$

The optimal choice for x occurs when $x^{1-\sigma}$ and $|t|x^{\alpha-\sigma}$ are of the same order. So putting $x=|t|^{\frac{1}{1-\alpha}}$, we obtain

$$|\phi_{\mathcal{P}}(\sigma+it)| \leq b\,\frac{|t|^{\frac{1-\sigma}{1-\alpha}}-1}{1-\sigma}+\mathrm{O}(1)+\mathrm{O}(|t|^{\frac{1-\sigma}{1-\alpha}}). \tag{2.6}$$

Note that for $\sigma=1$ this is

$$|\phi_{\mathcal{P}}(1+it)| \leq \frac{b}{1-\alpha}\log|t|+\mathrm{O}(1).$$

We use these inequalities to obtain bounds for $|\zeta_{\mathcal{P}}(s)|$. For $\sigma\in(\alpha,1)$,

$$\begin{aligned}\log\zeta_{\mathcal{P}}(\sigma+it) &= \int_{[\sigma+it,2+it]}\phi_{\mathcal{P}}(z)\ \mathrm{d}z+\log\zeta_{\mathcal{P}}(2+it)\\ &= \int_{\sigma}^{2}\phi_{\mathcal{P}}(y+it)\ \mathrm{d}y+\mathrm{O}(1).\end{aligned}$$

Taking real parts, we obtain

$$\log|\zeta_{\mathcal{P}}(\sigma+it)| \leq \int_{\sigma}^{2}|\phi_{\mathcal{P}}(y+it)|\ \mathrm{d}y+\mathrm{O}(1).$$

Letting $\sigma=1-\varepsilon(t), |t|\geq 3$, we deduce from (2.6) that

$$\log|\zeta_{\mathcal{P}}(1-\varepsilon(t)+it)| \leq \int_{1-\varepsilon(t)}^{2} b\,\frac{|t|^{\frac{1-y}{1-\alpha}}-1}{1-y}\ \mathrm{d}y+\mathrm{O}(1)+\mathrm{O}\left(\int_{1-\varepsilon(t)}^{2}|t|^{\frac{1-y}{1-\alpha}}\ \mathrm{d}y\right).$$

The latter integral equals

$$\frac{1-\alpha}{\log|t|}\left(|t|^{\frac{\varepsilon(t)}{1-\alpha}}-|t|^{-\frac{1}{1-\alpha}}\right)<\frac{1-\alpha}{\log|t|}|t|^{\frac{\varepsilon(t)}{1-\alpha}}=1-\alpha.$$

Hence

$$\begin{aligned}\log|\zeta_{\mathcal{P}}(1-\varepsilon(t)+it)| &\leq b\int_{1-\varepsilon(t)}^{2}\frac{|t|^{\frac{1-y}{1-\alpha}}-1}{1-y}\ \mathrm{d}y+\mathrm{O}(1)\\ &= b\int_{0}^{\varepsilon(t)}\frac{|t|^{\frac{u}{1-\alpha}}-1}{u}\ \mathrm{d}u+b\int_{0}^{1}\frac{1-|t|^{-\frac{v}{1-\alpha}}}{v}\ \mathrm{d}v+\mathrm{O}(1)\\ &= b\int_{0}^{\frac{\varepsilon(t)\log|t|}{1-\alpha}}\frac{\mathrm{e}^{y}-1}{y}\ \mathrm{d}y+b\int_{0}^{\frac{\log|t|}{1-\alpha}}\frac{1-\mathrm{e}^{-x}}{x}\ \mathrm{d}x+\mathrm{O}(1)\\ &= b\int_{1}^{\log\log|t|}\frac{\mathrm{e}^{y}}{y}\ \mathrm{d}y+\mathrm{O}(\log\log|t|)\\ &\sim b\,\frac{\log|t|}{\log\log|t|}.\end{aligned}$$

Thus,

$$|\zeta_{\mathcal{P}}(1-\varepsilon(t)+it)| \leq \exp\left(\frac{2b\log|t|}{\log\log|t|}\right),$$

for all $|t|$ sufficiently large, which is our claim. □

We continue the proof of Theorem 4. Appealing to Lemma 12, we see immediately that

$$I_5 \ll \int_T^\infty \frac{t^{\frac{2b}{\log\log t}} x^2}{t^2}\ \mathrm{d}t \ll x^2 \int_T^\infty t^{\frac{2b}{\log\log T}-2}\ \mathrm{d}t \ll_b x^2 T^{\frac{2b}{\log\log T}-1}.$$

This upper bound is equally valid for the integral I_4 since

$$I_4 \ll \int_{\sigma(T)}^{1+\frac{1}{\log x}} \frac{T^{\frac{2b}{\log\log T}} x^2}{T^2}\ \mathrm{d}\sigma \ll x^2 T^{\frac{2b}{\log\log T}-1}.$$

The integral I_2 is

$$I_2 \ll x^{2-\varepsilon(3)}.$$

Finally, to get the upper bound for the arc $\sigma(t)$ we choose a number k_0 such that Lemma 12 is valid for all $|t| \geq k_0$. Splitting the integral, we have

$$I_3 \ll \int_3^{k_0} \frac{|\zeta_{\mathcal{P}}(s)| x^{1+\sigma(k_0)}}{t^2}\ \mathrm{d}t + x^{1+\sigma(T)} \int_{k_0}^T t^{\frac{2b}{\log\log t}-2}\ \mathrm{d}t \ll_b x^{1+\sigma(T)}.$$

Selecting $T = \mathrm{e}^{\sqrt{\log x}}$ for $x > \exp(\mathrm{e}^{4b})$, leads us to the main formula

$$\int_1^x N_{\mathcal{P}}(t)\ \mathrm{d}t = I_1(x) + \mathrm{O}(x^2 \mathrm{e}^{-c_1\sqrt{\log x}}), \tag{2.7}$$

with

$$I_1(x) = \frac{1}{2\pi i}\int_\Gamma \zeta_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)}\ \mathrm{d}s,$$

where Γ is the truncated Hankel contour.

Here and for the rest of the proof we make the convention that all constants, explicit ($c_1, c_2, c_3, c_4, \ldots$) or implicit, depend at most on b and α.

It remains to study the main term $I_1(x)$ of (2.7). Clearly, $I_1(x)$ is an infinitely differentiable function of x on $\mathbb{R}^+$, and in particular we have

$$I_1'(x) = \frac{1}{2\pi i}\int_\Gamma \zeta_{\mathcal{P}}(s) x^s \frac{\mathrm{d}s}{s}, \qquad I_1''(x) = \frac{1}{2\pi i}\int_\Gamma \zeta_{\mathcal{P}}(s) x^{s-1}\ \mathrm{d}s.$$

Recall that $Z(s) = s^{-1}((s-1)\zeta(s))^b$. For $s \in D$, we then can write

$$\zeta_{\mathcal{P}}(s) = s\, G(s) Z(s)(s-1)^{-b}.$$

From this and the result of Lemma 3, for $s \in \Gamma$,

$$\zeta_{\mathcal{P}}(s) \ll |s-1|^{-b}, \qquad \text{as } x \to +\infty.$$

Since $r = 1/(2\log x)$, it follows that

$$I_1''(x) \ll \int_\Gamma \left(\frac{1}{2\log x}\right)^{-b} x^{s-1}\ \mathrm{d}s \ll (\log x)^b. \tag{2.8}$$

As both $G(s)$ and $Z(s)$ are holomorphic in the open set containing the disk $|s-1| < 1-\varepsilon(3)$, so is their product, which can be represented there by the Taylor series

$$G(s)Z(s) = \sum_{k=0}^{\infty} g_k(b)(s-1)^k$$

with

$$g_k(b) = \frac{1}{k!} \sum_{h+j=k} \binom{k}{j} G^{(h)}(1)\gamma_j(b) = \Gamma(b-k)\lambda_k(b).$$

In addition, since $G(s)Z(s)$ is $\mathrm{O}(1)$ in the disk $|s-1| < 1-\varepsilon(3)$, the Cauchy formulae imply that

$$g_k(b) \ll (1-\varepsilon(3))^{-k}.$$

Observing that Γ is contained in the disk $|s-1| \leq 1-\varepsilon(3)$, we can write for $s \in \Gamma$, $x \to +\infty$, $N \geq 0$,

$$G(s)Z(s) = \sum_{k=0}^{N} g_k(b)(s-1)^k + \mathrm{O}\left(\left(\frac{|s-1|}{1-\varepsilon(3)}\right)^{N+1}\right).$$

Therefore

$$I_1'(x) = \sum_{k=0}^{N} g_k(b) \frac{1}{2\pi i} \int_{\Gamma} x^s (s-1)^{k-b} \, \mathrm{d}s + \mathrm{O}((1-\varepsilon(3))^{-N-1} B(x)) \tag{2.9}$$

with

$$\begin{aligned} B(x) &= \int_{\Gamma} |x^s (s-1)^{N+1-b}| \, |\mathrm{d}s| \\ &\ll \int_{1-\varepsilon(3)}^{1-r} (1-\sigma)^{N+1-b} x^{\sigma} \, \mathrm{d}\sigma + x^{r+1} r^{N+2-b}. \end{aligned}$$

Using the change of variable $t = (1-\sigma)\log x$, we obtain

$$\begin{aligned} B(x) &\ll x(\log x)^{b-2-N} \left(\int_{\frac{1}{2}}^{\infty} t^{N+1-b} e^{-t} \, \mathrm{d}t + 2^{-N} \right) \\ &\ll x(\log x)^{b-2-N} \left(\int_{\frac{1}{2}}^{1} \left(\frac{1}{2}\right)^{1-b} e^{-1/2} \, \mathrm{d}t + \int_{1}^{\infty} t^{N+1+b} \mathrm{e}^{-t} \, \mathrm{d}t + 2^{-N} \right) \\ &\ll x(\log x)^{b-2-N} \Gamma(N+b+2) \ll x(\log x)^{b-1} \left(\frac{N+1}{\log x}\right)^{N+1}, \end{aligned}$$

To estimate the integral which appears in (2.9), we change the variable $w = (s-1)\log x$ and with the notation of Lemma 8 and the use of Stirling's formula (Lemma 9), we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} x^s (s-1)^{k-b} \, \mathrm{d}s &= \frac{x}{2\pi i} (\log x)^{b-1-k} \int_{\mathcal{H}(\varepsilon(3)\log x)} w^{k-b} e^{w} \, \mathrm{d}w \\ &= x(\log x)^{b-1-k} \left\{ \frac{1}{\Gamma(b-k)} + \mathrm{O}\left(47^{b-k} \Gamma(|b-k|+1) \, \mathrm{e}^{-\frac{\varepsilon(3)}{2}\log x}\right) \right\} \\ &= x(\log x)^{b-1-k} \left\{ \frac{1}{\Gamma(b-k)} + \mathrm{O}\left((c_2 k+1)^k x^{-\varepsilon(3)/2}\right) \right\}. \end{aligned}$$

Thus for the main term of (2.9) we have

$$\sum_{k=0}^{N} g_k(b)\frac{1}{2\pi i}\int_\Gamma x^s(s-1)^{k-b}\ \mathrm{d}s = x(\log x)^{b-1}\left\{\sum_{k=0}^{N}\frac{\lambda_k(b)}{(\log x)^k}+\mathrm{O}(E_N)\right\}$$

with

$$\begin{aligned}E_N &= x^{-\frac{\varepsilon(3)}{2}}\sum_{k=0}^{N}|g_k(b)|\left(\frac{c_2k+1}{\log x}\right)^k\\ &\ll x^{-\frac{\varepsilon(3)}{2}}\sum_{k=0}^{N}(1-\varepsilon(3))^{-k}\left(\frac{c_2k+1}{\log x}\right)^k\\ &\ll x^{-\frac{\varepsilon(3)}{2}}\left(\frac{c_3N+1}{\log x}\right)^N \ll \left(\frac{c_3N+1}{\log x}\right)^{N+1}.\end{aligned}$$

Substituting in (2.9), it follows that

$$I_1'(x) = x(\log x)^{b-1}\left\{\sum_{k=0}^{N}\frac{\lambda_k(b)}{(\log x)^k}+\mathrm{O}\left(\frac{c_4N+1}{\log x}\right)^{N+1}\right\}. \tag{2.10}$$

Next we show that $I_1'(x)$ is a suitable approximation for $N_{\mathcal{P}}(x)$.

Lemma 13. $N_{\mathcal{P}}(x) = I_1'(x) + \mathrm{O}\left(x\mathrm{e}^{-c_5\sqrt{\log x}}\right)$.

Proof. To this end, let us take a parameter h, $0 < h < \frac{x}{2}$, and apply (2.7) for both x and $x+h$. Subtracting these estimates, we obtain

$$\int_x^{x+h} N_{\mathcal{P}}(t)\ \mathrm{d}t = I_1(x+h) - I_1(x) + \mathrm{O}\left(x^2\mathrm{e}^{-c_1\sqrt{\log x}}\right), \tag{2.11}$$

while (2.8) implies that

$$\begin{aligned}I_1(x+h)-I_1(x) &= hI_1'(x) + h^2\int_0^1(1-t)I_1''(x+th)\ \mathrm{d}t\\ &= hI_1'(x)+\mathrm{O}\left(h^2(\log x)^b\right).\end{aligned} \tag{2.12}$$

Further,

$$\int_x^{x+h}\left(N_{\mathcal{P}}(t)-N_{\mathcal{P}}(x)\right)\ \mathrm{d}t = \int_x^{x+h} N_{\mathcal{P}}(t)\ \mathrm{d}t - N_{\mathcal{P}}(x)h.$$

From here,

$$N_{\mathcal{P}}(x) = h^{-1}\int_x^{x+h}N_{\mathcal{P}}(t)\ \mathrm{d}t - h^{-1}\int_x^{x+h}\left(N_{\mathcal{P}}(t)-N_{\mathcal{P}}(x)\right)\ \mathrm{d}t.$$

Making use of equalities (2.11) and (2.12) we can write

$$N_{\mathcal{P}}(x) = I_1'(x) + \mathrm{O}\left(x^2h^{-1}\mathrm{e}^{-c_1\sqrt{\log x}} + h(\log x)^b + h^{-1}L\right),$$

with

$$L = \int_x^{x+h} (N_{\mathcal{P}}(t) - N_{\mathcal{P}}(x))\ \mathrm{d}t.$$

By (2.11) and (2.12)

$$\begin{aligned} L &\le \int_x^{x+h} (N_{\mathcal{P}}(t) - N_{\mathcal{P}}(x))\ \mathrm{d}t \le \int_x^{x+h} N_{\mathcal{P}}(t)\ \mathrm{d}t - \int_x^{x+h} N_{\mathcal{P}}(x) dt \\ &\le \int_x^{x+h} N_{\mathcal{P}}(t)\ \mathrm{d}t - \int_{x-h}^{x} N_{\mathcal{P}}(t)\ \mathrm{d}t \\ &\ll x^2 \mathrm{e}^{-c_1\sqrt{\log x}} + h^2 (\log x)^b. \end{aligned}$$

Thus, choosing

$$h = x\mathrm{e}^{-\frac{1}{2}c_1\sqrt{\log x}},$$

it follows that

$$N_{\mathcal{P}}(x) = I_1'(x) + \mathrm{O}\left(x\mathrm{e}^{-c_5\sqrt{\log x}}\right), \tag{2.13}$$

as required. □

Theorem 4 follows by combining (2.10)and (2.13).

Beurling'o natūraliųjų skaičių asimptotinis pasiskirstymas

Santrauka

Nagrinėjame apibendrintų pirminių skaičių sistemas $\mathcal{P}$, iš jų gautas apibendrintų natūraliųjų (Beurlingo) skaičių sistemas $\mathcal{N}$ ir apibendrintas dzeta funkcijas $\zeta_{\mathcal{P}}(s)$. Tardami, kad funkcija $\pi_{\mathcal{P}}(x)$, skaičiuojanti apibendrintus pirminius neviršijančius x, yra lygi

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + \mathrm{O}(x^{\alpha}) \qquad (x \to +\infty),$$

čia $b > 0, \alpha \in (0,1)$, gauname asimptotinį apibendrintų natūraliųjų skaičių pasiskirstymą. Šis rezultatas yra gaunamas pritaikius Perron'o formulę specialiai sukonstruotai Dirichlet eilutei ir panaudojus kontūrinio integravimo metodą.

Bibliography

[1] A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés*, I. Acta Math., **68** (1937), 255–291.

[2] B.M. Bredikhin, *The remainder term in the asymptotic formula for $\nu_G(x)$*, Izv. Vyssh. Uchebn. Zaved. Mat., **6** (1960), 40–49.

[3] H.G. Diamond, *Asymptotic distribution of Beurling generalised integers*, Illinois J. Math., **14** (1970), 12–28.

[4] H.G. Diamond, *When do Beurling generalised integers have a density?*, J. Reine Angew. Math., **295** (1977), 22–39.

[5] H. Diamond, H. Montgomery, U. Vorhauer, *Beurling primes with large oscillation* , Math. Ann., **334** (2006), 1–36.

[6] A. S. Fainleib, *On the distribution of Beurling integers*, J. Number Theory, **111** (2005), 227–247.

[7] R. S. Hall, *Beurling generalized prime number systems in which the Chebyshev inequalities fail*, Proc. Amer. Math. Soc., **40** (1) (1973), 79–82.

[8] R. S. Hall, *Theorems about Beurling's generalised primes and the associated zeta function*, Ph.D. Thesis, University of Illinois, Urbana, USA (1967).

[9] T. W. Hilberdink and M. L. Lapidus, *Beurling zeta functions, generalised primes, and fractal membranes*, Acta Appl. Math., **94** (1) (2008), 21–48.

[10] E. Landau, *Neuer beweis des primzahlsatzes und beweis des primidealsatzes* , Math. Ann., **56** (1903), 645–670.

[11] A. Laurinčikas, *Rymano dzeta funkcijos teorijos pagrindai*, Vilniaus universiteto leidykla, 1992.

[12] P. Malliavin, *Sur le reste de la loi asymptotique de répartition des nombres premiers généralisés de Beurling*, Acta Math., **106** (1961), 281–298.

[13] B. Nyman, *A general prime number theorem*, Acta Math., **81** (1949), 299–307.

[14] A. G. Postnikov, *Introduction to analytic number theory*, Translation of Mathematical Monographs, 68. Providence, RI: American Mathematical Society (AMS). VI, 1988

[15] E. Stankus, *On some generalized integers*, Lithuanian Mathematical Journal, **36** (1) (1996), 115–123.

[16] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press 1995

[17] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., revised by D.R. Heath-Brown, Oxford University Press 1986

[18] W.-B Zhang, *Extensions of Beurling's prime number theorem*, J. Number Theory, **11** (5) (2015), 1589–1616.

[19] W.-B Zhang, *Normalization of Beurling generalized primes with Riemann hypothesis*, Annales Univ. Sci. Budapest., Sect. Comp., **39** (2013), 459–469.

[20] W.-B Zhang, *Beurling primes with RH and Beurling primes with large oscillation*, Math. Ann., **337** (2007), 671–704.