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Apibendrintųjų z-skirstinių Cornish–Fisher skleidiniai

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Nagrinėsime vienamačius apibendrintus z-skirstinius [1], kurių charkteringoji funkcija yra

$$f_{2\delta}(t) = \left(\frac{B\left(\beta_1 + \frac{it\alpha}{2\pi}, \beta_2 - \frac{it\alpha}{2\pi}\right)}{B(\beta_1, \beta_2)}\right)^{2\delta} \mathrm{e}^{it\mu},$$

kur α, β_1, β_2 ir $\delta > 0$, $\mu \in R^1$, $B(\beta_1, \beta_2)$ – β-funkcija.

Tokių skirstinių klasę žymėsime

$$P\{\xi_{G(2\delta)} < x\} \sim GZD(\alpha, \beta_1, \beta_2, \delta, \mu).$$

Atsitiktinis dydis $\xi_{G(2\delta)}$ yra be galo dalus ir jo semiinvariantai yra

$$\kappa_1 = \frac{\alpha\delta}{\pi}\nu_1(\beta_1, \beta_2) + \mu,$$
$$\kappa_m = \frac{2\alpha^m\delta}{(2\pi)^m}\nu_m(\beta_1, \beta_2), \quad m = 2, 3, \ldots .$$

Čia

$$\nu_m(\beta_1, \beta_2) = \int_0^\infty x^{m-1}\frac{\mathrm{e}^{-\beta_2 x} + (-1)^m \mathrm{e}^{-\beta_1 x}}{1 - \mathrm{e}^{-x}}\,\mathrm{d}x, \quad m = 1, 2, \ldots .$$

Yra žinoma [1,2], kad atsitiktinis dydis $\xi_{G(2\delta)}$ turi tankį $p(u)$ „su sunkia uodega“:

$$p(x) \sim C_\pm |x|^{\varrho_\pm} \mathrm{e}^{-\sigma_\pm |x|},$$

kai $x \to \pm\infty$, kur $\varrho_+, \varrho_- \in R^1$, C_+, C_-, σ_+ ir $\sigma_- > 0$.

Aišku, kad Gauso dėsnis klasei $GZD(\alpha, \beta_1, \beta_2, \delta, \mu)$ nepriklauso.

Mus domina lygties

$$P\{\xi_{G(2\delta)} < x\} = p$$

sprendinys $x = x_p$, $0 < p < 1$.

Šią lygtį lengva išspręsti, kai $2\delta = 1$, nes

$$P\{\xi_{G(1)} < x\} = P\left\{\frac{\alpha}{2\pi}\ln\frac{1 - Y}{Y} + \mu < x\right\},$$

kur

$$P\{Y < x\} = \frac{1}{B(\beta_1, \beta_2)} \int_0^x t^{\beta_1 - 1}(1-t)^{\beta_2 - 1}\mathrm{d}t.$$

Mes atsitiktinį dydį

$$\xi_{G(2\delta)} \sim GZD(\alpha, \beta_1, \beta_2, \delta, \mu),$$

kai $2\delta \neq 1$ aproksimuosime atsiktiniais dydžiais

$$\xi_{G_n(1)} \sim GZD\big(\alpha(n), \beta_1(n), \beta_2(n), 1/2, \mu(n)\big),$$

kur $\alpha(n)$, $\beta_1(n)$, $\beta_2(n)$, $\mu(n)$ priklauso nuo 2δ, α, $n = 1, 2, \ldots$, t.y.

$$\xi_{G(2\delta)} = \xi_{G_n(1)} + \ldots$$

Kadangi atsitiktiniai dydžiai $\xi_{G(2\delta)}$ ir $\xi_{G_n(1)}$ yra be galo dalūs, tai

$$P(x) = P\left\{\xi_{G(2\delta)} < x\right\} = P_n^{*n}(x)$$

ir

$$G(x) = P\{\xi_{G_n(1)} < x\} = G_n^{*n}(x),$$

kai $n = 1, 2, \ldots$.

Pasinaudodami H. Bergström [3] tapatybe gauname

$$\begin{aligned} P(x) = G(x) &+ \sum_{\nu=1}^{s} \sum_{m=0}^{\infty} \left(\frac{1}{n}\right)^m \frac{(-\nu)^m}{m!\nu!} \frac{d^m}{d\tau^m} \Big[G^{*\tau} * \big(n(P_n - G_n)\big)^{*\nu}(x)\Big]_{\tau=1} \\ &+ \sum_{j=1}^{s-1} \sum_{m=0}^{\infty} \Big(-\frac{1}{n}\Big)^{j+m} \sum_{k=0}^{(j-1)\wedge(s-j-1)} q_{jk} \sum_{l=0}^{s-j-k-1} \frac{1}{l!} \frac{(j+k+l+1)^m}{m!} \\ &\times \frac{\mathrm{d}^m}{\mathrm{d}\tau^m} \Big[G^{*\tau} * \big(n(P_n - G_n)\big)^{*(j+k+l+1)(x)}\Big]_{\tau=1} + r_n^{(s+1)}(x), \end{aligned} \tag{1}$$

kur $s = 1, 2, \ldots$, $n > s$, $x \in R^1$,

$$r_n^{(s+1)}(x) = \sum_{\mu=s+1}^{n} C_{\mu-1}^{s} P_n^{*(n-\mu)} * (P_n - G_n)^{*(s+1)} * G_n^{*(\mu-s-1)}(x).$$

Čia $G^{*\tau}(x)$ yra tikimybinis skirstinys, kurio charakteringoji funkcija yra

$$g^{\tau}(t) = \left(\int_{-\infty}^{\infty} \mathrm{e}^{itx}\mathrm{d}G(x)\right)^{\tau},$$

kai $\tau > 0$ (žiūr. B. Grigelionis [1]).

Formalioje tapatybėje (1) vietoje x galime įstatyti $y(x_p)$ tokią, kad

$$P\left\{\xi_{G(2\delta)} < y(x_p)\right\} = p + r_n^{(s+1)}(y(x_p)).$$

Čia

$$G(x_p) = p, \quad 0 < p < 1.$$

$y(x_p)$ parinkimui panaudosime lygybę

$$G(x_p) = G(y(x_p)) + \sum_{\nu=1}^{s}\sum_{m=0}^{s}\Big(\frac{1}{n}\Big)^m (-\nu)^m \frac{1}{m!\nu!}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\Big[G^{*\tau} * \big(n(P_n - G_n)\big)^{*\nu}(y(x_p))\Big]_{\tau=1}$$

$$+ \sum_{j=1}^{s}\sum_{m=0}^{s}\Big(-\frac{1}{n}\Big)^{j+m}\sum_{k=0}^{(j-1)\wedge(s-j-1)} q_{jk}$$

$$\times \sum_{l=0}^{s-j-k-1}\frac{1}{l!}\frac{(j+k+l+1)^m}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\Big[G^{*\tau} * \big(n(P_n - G_n)\big)^{*(j+k+l+1)}(y(x_p))\Big]_{\tau=1}.$$

Toliau žymėsime

$$G(x_p) = G(y(x_p)) + A_1^s(y(x_p)).$$

Dešiniąją lygybės

$$G(x_p) = G\big(x_p + y(x_p) - x_p\big) + A_1^s\big(x_p + y(x_p) - x_p\big)$$

pusę skleidžiame Teiloro eilute $y(x_p) - x_p$ laipsniais ir gauname

$$-A_1^s(x_p) = \sum_{l=1}^{\infty} b_l\big(y(x_p) - x_p\big)^l, \tag{2}$$

kur

$$b_l = \frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}u^l}\Big[G(u) + A_1^{(s)}(u)\Big]_{u=x_p}.$$

Apvertus (2) gauname

$$y(x_p) = x_p + \sum_{k=1}^{\infty} a_k\big(-A_1^s(x_p)\big)^k, \tag{3}$$

kur

$$a_k = \sum_{\substack{\nu_1+2\nu_2+\dots+(k-1)\nu_{k-1}=k-1\\ \nu_1+\dots+\nu_{k-1}=s}} \frac{(-1)^s k(k+1)\dots(k+s-1)}{\nu_1!\nu_2!\dots\nu_{k-1}!}\Big(\frac{\mathrm{d}}{\mathrm{d}u}\big[G(u) + A_1^s(u)\big]_{u=x_p}\Big)^{-k-s}$$

$$\times \prod_{i=1}^{k-1}\Big(\frac{1}{i+1}\frac{\mathrm{d}^{i+1}}{\mathrm{d}u^{i+1}}\big[G(u) + A_1^s(u)\big]\Big)_{u=x_p}.$$

Iš (3) seka, kad

$$y(x_p) = x_p + \sum_{m=0}^{\infty} \Big(\frac{1}{n}\Big)^m J_m(x_p).$$

Koeficientus $J_m(x_p)$ prie $\big(\frac{1}{n}\big)^m$ randame tradiciniu būdu:

$$J_m(x_p) = \sum_{k=1}^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}\varepsilon^m}\Big[a_k(\varepsilon)\big(-A_1^s(x_p,\varepsilon)\big)^k\Big]_{\varepsilon=0}.$$

Čia

$$A_1^s(x_p,\varepsilon) = \sum_{\nu=1}^{s}\sum_{m=0}^{s} \varepsilon^m \frac{(-\nu)^m}{m!\nu!}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\Big[G^{*\tau} * \big(n(P_n - G_n)\big)^{*\nu}(x_p)\Big]_{\tau=1}$$

$$+ \sum_{j=1}^{s-1}\sum_{m=0}^{s}(-\varepsilon)^{j+m} \sum_{k=0}^{(j-1)\wedge(s-j-1)} q_{jkj}$$

$$\times \sum_{l=0}^{s-j-k-1} \frac{1}{l!}\frac{(j+k+l+1)^m}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\Big[G^{*\tau} * \big(n(P_n - G_n)\big)^{*(j+k+l+1)}(x_p)\Big]_{\tau=1}$$

$$a_k(\varepsilon) = \sum_{\substack{\nu_1+2\nu_2+\dots+(k-1)\nu_{k-1}=k-1\\ \nu_1+\dots+\nu_{k-1}=s}} \frac{(-1)^s k(k+1)\dots(k+s-1)}{\nu_1!\nu_2!\dots\nu_{k-1}!}\Big[\frac{\mathrm{d}}{\mathrm{d}u}\big[G(u)+A_1^s(u,\varepsilon)\big]_{u=x_p}\Big]^{-k-s}$$

$$\times \prod_{i=1}^{k-1}\Big(\frac{1}{i+1}\frac{\mathrm{d}^{i+1}}{\mathrm{d}u^{i+1}}\big[G(u) + A_1^s(u,\varepsilon)\big]_{u=x_p}\Big)^{\nu_i}.$$

Tikimybinio skirstinio

$$P_n^{*n} \in GZD(\alpha, \beta_1, \beta_2, \delta, \mu)$$

momentai yra žinomi, o tikimybinio skirstinio

$$G_n^{*n} \in GZD\big(\alpha(n), \beta_1(n), \beta_2(n), 1/2, \mu(n)\big)$$

parametrus parenkame taip, kad galiotų lygybės

$$\int_{-\infty}^{\infty} x^l \mathrm{d}P_n(x) = \int_{-\infty}^{\infty} x^l \mathrm{d}G_n(x),$$

kai $l = 1, 2, 3, 4$. Tuomet

$$y(x_p) = x_p - \frac{n}{5!}\frac{\mathrm{d}^5 G(y)}{\mathrm{d}y^5}_{|y=x_p} \kappa_{5,n} + \dots,$$

kur

$$\kappa_{5,n} = \int_{-\infty}^{\infty} u^5 \mathrm{d}(P_n - G_n)(u).$$

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SUMMARY

J. Turkuvienė, A. Bikelis. Cornish–Fisher expansions of generalized z-distribution

The article presents Cornish–Fisher expansions of generalized z-distribution.

Keywords: generalized z-distribution, Cornish–Fisher expansion.