

## On the argument of zeta-functions of certain cusp forms

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Let  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  denote the sets of all positive integers, integers, real and complex numbers, respectively. The set

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is called the full modular group. Suppose that the function  $F(z)$  is holomorphic in the upper half – plane  $\text{Im } z > 0$ , with some  $\kappa$ , for all elements  $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$  of  $SL(2, \mathbb{Z})$  satisfies the functional equation

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^\kappa F(z)$$

and  $\lim_{\text{Im } z \rightarrow \infty} F(z) = 0$ . Then  $F(z)$  is called a cusp form of weight  $\kappa$  for the full modular group. We also suppose that  $F(z)$  is a normalized eigenform for all Hecke operators. In this case,  $F(z)$  has the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}, \quad c(1) = 1.$$

Let  $s = \sigma + it$  be a complex variable. Then the function

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

is called the zeta-function attached to the cusp form  $F(z)$ . The arithmetical function  $c(m)$  is multiplicative. Therefore,  $\varphi(s, F)$  has the Euler product expansion over primes

$$\varphi(s, F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where  $c(p) = \alpha(p) + \beta(p)$ . In view of Deligne's estimates [1]

$$|\alpha(p)| \leq p^{\frac{\kappa-1}{2}}, \quad |\beta(p)| \leq p^{\frac{\kappa-1}{2}}$$

it is easily seen that both the Euler product and the Dirichlet series for  $\varphi(s, F)$  converge absolutely for  $\sigma > (\kappa + 1)/2$ , and in this region  $\varphi(s, F)$  is a non-vanishing holomorphic function. Moreover,  $\varphi(s, F)$  is analytically continuable to an entire function and satisfies the following functional equation

$$(2\pi)^{-s} \Gamma(s) \varphi(s, F) = (-1)^{\kappa/2} (2\pi)^{s-\kappa} \Gamma(\kappa - s) \varphi(\kappa - s, F),$$

where, as usual,  $\Gamma(s)$  denotes the Euler gamma-function. The nontrivial zeros of  $\varphi(s, F)$  lie in the critical strip  $\{s \in \mathbb{C}: (\kappa - 1)/2 < \sigma < (\kappa + 1)/2\}$ . The analogue of the Riemann hypothesis says that all nontrivial zeros of  $\varphi(s, F)$  lie on the critical line  $\sigma = \kappa/2$ .

The aim of this note is to obtain a formula for  $\arg(\varphi(s, F))$  near the critical line. We will use the notation

$$\sigma_T = \frac{\kappa}{2} + \frac{1}{l_T},$$

where  $l_T \rightarrow \infty$  as  $T \rightarrow \infty$ .

Taking logarithms and differentiating both sides of the Euler product with respect to  $s$ , we find that, for  $\sigma > (\kappa + 1)/2$ ,

$$-\frac{\varphi'}{\varphi}(s, F) = \sum_p \sum_{m=1}^{\infty} (\alpha^m(p) + \beta^m(p)) p^{-ms} \log p. \quad (1)$$

Now we define

$$\Lambda_F(n) = \begin{cases} (\alpha^m(p) + \beta^m(p)) \log p, & \text{if } n = p^m, \\ 0, & \text{otherwise.} \end{cases}$$

Then in view of (1)

$$-\frac{\varphi'}{\varphi}(s, F) = \sum_{n=2}^{\infty} \Lambda_F(n) n^{-s}, \quad \sigma > \frac{\kappa + 1}{2}.$$

Let for  $x > 1$ ,

$$\Lambda_{x,F}(n) = \begin{cases} \Lambda_F(n), & \text{if } 1 \leq n \leq x, \\ \Lambda_F(n) \frac{(\log(x^3/n))^2 - 2(\log(x^2/n))^2}{2(\log x)^2}, & \text{if } x \leq n \leq x^2, \\ \Lambda_F(n) \frac{(\log(x^3/n))^2}{2(\log x)^2}, & \text{if } x^2 \leq n \leq x^3. \end{cases} \quad (2)$$

**THEOREM.** For  $t \geq 2$ ,  $2 \leq x \leq t^2$ , we have

$$\begin{aligned} \arg \varphi(\sigma_T + it, F) &= - \sum_{n < x^3} \frac{\Lambda_{x,F}(n) \sin(t \log n)}{n^{\sigma_T, x, t} \log n} \\ &+ O\left( (\sigma_{T, x, t} - \sigma_T) \left| \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T, x, t}}} \right| \right) + O((\sigma_{T, x, t} - \sigma_T) \log t), \end{aligned}$$

where

$$\sigma_{T,x,t} = \sigma_T + 2 \max(\beta - \kappa/2, 2/\log x) \quad (3)$$

with  $\rho = \beta + i\gamma$  running over those zeros  $\varphi(s, F)$  for which

$$|t - \gamma| \leq x^{3|\beta - \kappa/2|} (\log x)^{-1}, \quad (4)$$

and  $\Lambda_{x,F}(n)$  is as in (2).

For the proof of the theorem we will apply two following lemmas.

LEMMA 1. *If  $s \neq \rho$ ,  $t \geq 2$ , then*

$$\frac{\varphi'}{\varphi}(s, F) = \sum_{\rho} ((s - \rho)^{-1} + \rho^{-1}) + O(\log t)$$

uniformly for  $\kappa/2 \leq \sigma \leq \kappa/2 + 10$ .

Proof of this lemma can be found in [3].

LEMMA 2. *Let  $t \geq 2$  and  $2 \leq x \leq t^2$ . Then we have that*

$$\sum_{\rho} \frac{\sigma_{T,x,t} - \frac{\kappa}{2}}{(\sigma_{T,x,t} - \beta)^2 + (t - \beta)^2} = O \left| \sum_{m < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t} + it}} \right| + O(\log t)$$

and

$$\begin{aligned} \frac{\varphi'}{\varphi}(s, F) &= - \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^s} \\ &+ O \left( (x^{\sigma_T/2 - \sigma/2}) \left| \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t} + it}} \right| \right) + O(x^{\sigma_T/2 - \sigma/2} \log t) \end{aligned} \quad (5)$$

for  $\sigma > \sigma_{T,x,t}$ .

Proof of this lemma is similar to the proof of Theorem 3.5.2 from [2].

*Proof of the Theorem.* Clearly,

$$\begin{aligned} \arg \varphi(\sigma_T + it) &= - \int_{\sigma_T}^{\infty} \operatorname{Im} \frac{\varphi'}{\varphi}(\sigma + it) d\sigma \\ &= \int_{\sigma_{T,x,t}}^{\infty} \operatorname{Im} \frac{\varphi'}{\varphi}(\sigma + it) d\sigma + (\sigma_{T,x,t} - \sigma_T) \operatorname{Im} \frac{\varphi'}{\varphi}(\sigma_{T,x,t} + it) \\ &\quad + \int_{\sigma_T}^{\sigma_{T,x,t}} \operatorname{Im} \left\{ \frac{\varphi'}{\varphi}(\sigma_{T,x,t} + it) - \frac{\varphi'}{\varphi}(\sigma + it) \right\} d\sigma \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (6)$$

In view of estimate (5)

$$I_1 = \operatorname{Im} \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t} + it} \log n} + O\left(\frac{1}{\log x} \left| \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t} + it}} \right|\right) + O\left(\frac{\log t}{\log x}\right). \quad (7)$$

Similarly as in the proof of Lemma 3.6.3 from [2], we find that

$$I_2 = O\left((\sigma_{T,x,t} - \sigma_T) \left| \sum_{m < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t} + it}} \right|\right) + O((\sigma_{T,x,t} - \sigma_T) \ln t). \quad (8)$$

From Lemma 1, taking the imaginary part of both sides, it follows that (we suppose for simplicity that  $t \neq \gamma$ )

$$\begin{aligned} & \operatorname{Im} \left( \frac{\varphi'}{\varphi}(\sigma_{T,x,t} + it) - \frac{\varphi'}{\varphi}(\sigma + it) \right) \\ &= \sum_{\rho} \operatorname{Im} \left( \frac{1}{\sigma_{T,x,t} - \beta + i(t - \gamma)} - \frac{1}{\sigma - \beta + i(t - \gamma)} \right) + O(\log t) \\ &= \sum_{\rho} \frac{(\sigma_{T,x,t} - \sigma)(\sigma + \sigma_{T,x,t} - 2\beta)(t - \gamma)}{((\sigma_{T,x,t} - \beta)^2 + (t - \gamma)^2)((\sigma - \beta)^2 + (t - \gamma)^2)} + O(\log t) \end{aligned}$$

for  $\kappa/2 \leq \sigma \leq \sigma_{T,x,t}$ . Therefore,

$$\begin{aligned} |I_3| &\leq \sum_{\rho} \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{(\sigma_{T,x,t} - \sigma)|\sigma + \sigma_{T,x,t} - 2\beta||t - \gamma|}{((\sigma_{T,x,t} - \beta)^2 + (t - \gamma)^2)((\sigma - \beta)^2 + (t - \gamma)^2)} d\sigma \\ &\quad + O((\sigma_{T,x,t} - \sigma_T) \log t) \\ &\leq \sum_{\rho} \frac{\sigma_{T,x,t} - \sigma_T}{(\sigma_{T,x,t} - \beta)^2 + (t - \gamma)^2} \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma + \sigma_{T,x,t} - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma \\ &\quad + O((\sigma_{T,x,t} - \sigma_T) \log t). \end{aligned} \quad (9)$$

First, let

$$|\beta - \kappa/2| \leq \frac{1}{2}(\sigma_{T,x,t} - \kappa/2).$$

Then

$$|\sigma + \sigma_{T,x,t} - 2\beta| \leq (\sigma - \kappa/2) + (\sigma_{T,x,t} - \kappa/2) + 2|\beta - \kappa/2| \leq 3(\sigma_{T,x,t} - \kappa/2)$$

for  $\sigma_T \leq \sigma \leq \sigma_{T,x,t}$ . Thus

$$\begin{aligned} & \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma + \sigma_{T,x,t} - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma \leq 3(\sigma_{T,x,t} - \kappa/2) \int_{-\infty}^{\infty} \frac{|t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma \\ &= 3(\sigma_{T,x,t} - \kappa/2) \int_{-\infty}^{\infty} \frac{|t - \gamma|}{y^2 + (t - \gamma)^2} dy = 3\pi(\sigma_{T,x,t} - \kappa/2). \end{aligned} \quad (10)$$

If

$$|\beta - \kappa/2| > \frac{\sigma_{T,x,t} - \kappa/2}{2} \geq \frac{\sigma_{T,x,t} - \sigma_T}{2},$$

then it follows from (3) and (4) that

$$|t - \gamma| > \frac{x^{3|\beta - \kappa/2|}}{\log x} > 3|\beta - \kappa/2|,$$

and

$$|\sigma + \sigma_{T,x,t} - 2\beta| \leq (\sigma - \kappa/2) + (\sigma_{T,x,t} - \kappa/2) + 2|\beta - \kappa| < 6|\beta - \kappa/2|$$

for  $\sigma_T \leq \sigma \leq \sigma_{T,x,t}$ . Therefore, in this case, too

$$\begin{aligned} \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma + \sigma_{T,x,t} - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma &< \int_{\sigma_T}^{\sigma_{T,x,t}} \frac{|\sigma + \sigma_{T,x,t} - 2\beta| d\sigma}{|t - \gamma|} \\ &< 2 \frac{|\beta - \kappa/2|}{|\beta - \kappa/2|} \int_{\sigma_T}^{\sigma_{T,x,t}} d\sigma < 10(\sigma_{T,x,t} - \sigma_T) < 10(\sigma_{T,x,t} - \kappa/2). \end{aligned}$$

This, (7), (8) and Lemma 2 yield

$$\begin{aligned} |I_3| &< 10(\sigma_{T,x,t} - \sigma) \sum_{\rho} \frac{\sigma_{T,x,t} - \kappa/2}{(\sigma_{T,x,t} - \beta)^2 + (t - \gamma)^2} + O((\sigma_{T,x,t} - \sigma_T) \log t) \\ &= O\left( (\sigma_{T,x,t} - \sigma_T) \left| \sum_{n < x^3} \frac{\Lambda_{x,F}(n)}{n^{\sigma_{T,x,t} + it}} \right| \right) + O((\sigma_{T,x,t} - \sigma_T) \log t). \end{aligned} \quad (11)$$

Now, substituting the estimates (7), (8) and (11), we obtain the assertion of the theorem.

## References

1. P. Deligne, La conjecture de Weil I, *Pub. I. H. E. S.*, **43**, 273–307 (1974).
2. A. Laurinćikas, *Limit Theorems for the Riemann Zeta-function*, Kluwer, Dordrecht, Boston, London (1996).
3. A. Sankaranarayanan, On Hecke  $L$ -functions associated with cusp forms, *Acta Arith.*, **108**(3), 217–234 (2003).

REZIUMĖ

**R. Ivanauskaitė. Parabolinių formų dzeta funkcijos argumentas**

Darbe įrodyta parabolinių formų dzeta funkcijos teorema argumentui.