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The Boolean zeta function

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Abstract. This paper provides analysis on Dirichlet series with a_n coefficients obtained from $\mathrm{MAJ}_m(x_1, \dots, x_m)$ function known in theoretical computer science.

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An *ordinary fractal string L* is a bounded open subset Ω of $\mathbb{R}$. Such a set consists of countably many open intervals, the lengths of which will be denoted by $l_1, l_2, l_3, \dots$, called the *lengths* of the string [4]. Let us define binary number by $[n]_2 = x_1 x_2 \dots x_m$, $x_j \in \{0, 1\}$, $j = 1, \dots, m$.

Let

$$a_n = \begin{cases} 1, & \text{if } \sum_{1 \leqslant i < m} x_i \geqslant m/2, \\ 0, & \text{otherwise.} \end{cases}$$

It is well known in the computer science the majority function $\mathrm{MAJ}_m(x_1, \dots, x_m) = a_n$ [1].

Now we can define a zeta function

$$\zeta_{\mathrm{BM}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is holomorphic for $\sigma > 1$.

Let us define strings' lengths of the zeta function by a_n/n, $n \in \mathbb{N}$. As we see string lengths can be divided into two types: with length 0 and other with lengths $1/n$, $n \in \mathbb{N}$.

Let us divide a set $\mathbb{N}$ into subsets (intervals), where $a_n = 0$ and $a_n = 1$. Let's number these intervals. In the next step let's number the elements from every interval. Then we can denote by c_{kl}, $k \in \mathbb{N}$, $l = 1, \dots, m$, the l-th element from the k-th interval with a given value n. As an example, we can write the first eight elements: $c_{11} = 1$, $c_{12} = 2$, $c_{13} = 3$, $c_{21} = 4$, $c_{31} = 5$, $c_{32} = 6$, $c_{33} = 7$, $c_{41} = 8, \dots$. So we have obtained two sets, upper C^* and lower C_*, where zeta function's strings' lengths are equal to $1/n$ or zero,

$$C^* = \{c_{kl} \colon k \text{ is odd}, l \in \mathbb{N}\},$$

$$C_* = \{c_{kl} \colon k \text{ is even}, l \in \mathbb{N}\}.$$

Dirac's delta function is a linear functional from a space (commonly taken as a Schwartz space S or the space of all smooth functions of compact support D) of test functions f

$$\delta(x-a)=0, \quad \text{as } x \neq a, \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a).$$

The Heaviside step function is defined by

$$H(x)=\begin{cases} 1, & \text{if } x \geqslant 0; \\ 0, & \text{if } x < 0. \end{cases}$$

The Dirac delta function can be viewed as the derivative of the Heaviside step function [3]

$$\frac{\mathrm{d}}{\mathrm{d}x}H(x)=\delta(x).$$

Now we can construct the numbers

$$\eta_k = c_{k1} - \omega,$$

where ω is infinitesimal number, bigger than zero, but less than any positive real number [2].

THEOREM 1. *For $\sigma > 1$ and $M = \{r\colon\ n \in c_{rp}\}$ we have*

$$\zeta_{\mathrm{BM}}(s)=\sum_{n=1}^{\infty}\sum_{k=1}^{M}\frac{(-1)^{k-1}H(n-\eta_k)}{n^s}.$$

Proof. Let's investigate two cases: $n \in C_*$ and $n \in C^*$. Let n be fixed. In the first case when $n \in C_*$ we have $H(n-\eta_k)=1$ for $n > \eta_k$, and the number of such terms will be even. From this it follows

$$\sum_{k=1}^{M}(-1)^{k-1}H(n-\eta_k)=0,$$

and we get $a_n = 0$ if $n \in C_*$.

In the second case when $n \in C^*$ we have $H(n-\eta_k)=1$ for $n > \eta_k$, and the number of such terms will be odd. Because of the first term is positive, the last term of the sequence $(-1)^{k-1}H(n-\eta_k)$ is positive. From this we have that

$$\sum_{k=1}^{M}(-1)^{k-1}H(n-\eta_k)=1,$$

and we get $a_n = 1$ if $n \in C^*$.

If we will sum more terms, than M, then we get $n < \eta_k$, $H(n-\eta_k)=0$ and these terms will not contribute the sum. This completes the proof.

For our purposes will be useful the following statement.

LEMMA 1. *Let*

$$\sum_{n \leqslant x} b_n = Kx + R(x),$$

where $R(x) = \mathrm{O}(x^\alpha)$ *with* $0 \leqslant \alpha < 1$. *Then we have*

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \frac{Ks}{s-1} + \int_1^\infty \frac{R(u)\,\mathrm{d}u}{u^{s+1}}$$

for $\sigma > \alpha$.

Proof can be found in [5].

Let a_n satisfy the hypothesis of Lemma 1

$$\sum_{n \leqslant x} a_n = Kx + \mathrm{O}(x^\alpha),$$

with $0 \leqslant \alpha < 1$. For example, we can take $a_{2^n-1} = 1$ for all $n \in \mathbb{N}$. Then the equality $\sum_{n \leqslant x} a_n = [\log_2(x+1)]$ holds.

For such a_n we have the following statement.

THEOREM 2. *The function* $\zeta_{\mathrm{BM}}(s)$ *is analytically continuable to the region* $\sigma > \alpha$, *except, maybe, for a simple pole at* $s = 1$ *with residue* K.

Proof. Summing by parts we find that

$$\sum_{n \leqslant x} \frac{a_n}{n^s} = K\Big(\frac{x^{1-s}}{1-s} - \frac{s}{1-s}\Big) + s\int_1^x \frac{R(u)\,\mathrm{d}u}{u^{s+1}} + \mathrm{O}(x^{\delta-\sigma}).$$

Taking $\sigma > 1$ and letting x to infinity, whence we obtain

$$\zeta_{\mathrm{BM}}(s) = \frac{Ks}{s-1} + s\int_1^\infty \frac{R(u)\,\mathrm{d}u}{u^{s+1}}.$$

The integral here converges uniformly in $\sigma \geqslant \alpha + \varepsilon$ for each $\varepsilon > 0$. Therefore the last equality gives the analytic continuation of the function $\zeta_{\mathrm{BM}}(s)$ to the half-plane $\sigma > \alpha$. In this half-plane $\zeta_{\mathrm{BM}}(s)$ is regular if $K = 0$. In case $K \neq 0$ the point $s = 1$ is its simple pole with residue K.

Now we can evaluate the case given by the Lemma 1, and we have

$$\zeta_{\mathrm{BM}}(s) = \frac{Ks}{s-1} + \mathrm{O}\Big(\frac{|s|}{\alpha - \sigma}\Big).$$

References

1. P. Clote, E. Kranakis, *Boolean Functions and Computation Models*, Springer, Berlin (2002).
2. H. Gonshor, *An Introduction to Surreal Numbers* (London Mathematical Society Lecture Note Series), Cambridge University Press, London (1986).
3. A.N. Kolmogorov, S.V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover Publications, New York (1999).
4. M.L. Lapidus, M. van Frankenhuysen, *Fractal Geometry and Number Theory*, Birkhauser, New York (2000).
5. A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht (1996).

REZIUMĖ

A. Javtokas. Dvejetainė dzeta funkcija

Straipsnyje apibrėžiama dvejetainė dzeta funkcija. Suformuluojamos dvi teoremos, kuriose dvejetainė dzeta funkcija išreiškiama Heavisaido funkcija ir pratęsiama į sritį $\sigma > \alpha$, kai $0 \leqslant \alpha < 1$.