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A discrete limit theorem on the complex plane for one class of general Dirichlet series*

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Abstract. In the paper a discrete limit theorem in the sense of weak convergence on the complex plane for general Dirichlet series with improved condition is presented.

Keywords: general Dirichlet series, probability measure, random variable, weak convergence.

1. Introduction

We denote by $\mathbb{N}, \mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, real numbers, and complex numbers, respectively. Let $\{a_m\colon\ m \in \mathbb{N}\}$ be a sequence of complex numbers, and let $\{\lambda_m\colon\ m \in \mathbb{N}\}$ be an increasing sequence of positive numbers such that $\lim_{m\to\infty} \lambda_m = +\infty$. We denote by $s = \sigma + it$ a complex variable. The series

$$\sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s} \tag{1}$$

is called a general Dirichlet series. Suppose that series (1) absolutely converges for $\sigma > \sigma_a$ to the sum $f(s)$. Then the function $f(s)$ is regular in the half-plane $\sigma > \sigma_a$. In [2], we investigated the discrete value-distribution of series (1) by probabilistic methods and we proved limit theorems in the sense of weak convergence of probability measures on the complex plane.

Let $N \in \mathbb{N}$, and let

$$\mu_N(\ldots) = \frac{1}{N+1}\#\{0 \leqslant m \leqslant N\colon\ \ldots\},$$

where in place of dots a condition satisfied by m is to be written. Let $\mathcal{B}(S)$ denote the class of Borel sets of the space S. We suppose that the function $f(s)$ is meromorphically continuable to the region $\sigma > \sigma_1$, $\sigma_1 < \sigma_a$, and that all poles in this region are in a compact set. Moreover, we suppose that, for $\sigma > \sigma_1$, the estimates

$$f(s) = B|t|^{\alpha}, \quad |t| \geqslant t_0, \quad \alpha > 0, \tag{2}$$

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and

$$\int_{-T}^{T} \left| f(\sigma + it) \right|^2 \mathrm{d}t = BT, \quad T \to \infty, \tag{3}$$

are satisfied. Here and in the sequel, B denotes a quantity (not always the same) bounded by some constant.

Let $h > 0$ be fixed. Define, for $\sigma > \sigma_1$, a probability measure

$$P_N(A) = \mu_N\big(f(\sigma + imh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

In [2], it was noted that, in the case of limit theorems on the complex plane for the function $f(s)$, we can suppose without loss of generality that $f(s)$ is regular for $\sigma > \sigma_1$. Then, in [2], two limit theorems on the complex plane for the function $f(s)$ were obtained.

THEOREM 1. *Suppose that conditions* (2) *and* (3) *for the function* $f(s)$ *are satisfied. Then there exists a probability measure* P *on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *such that the measure* P_N *weakly converges to* P *as* $N \to \infty$.

Proof of this theorem can be found in [2].

Now define the infinite-dimensional torus

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma = \{s \in \mathbb{C}\colon\ |s| = 1\}$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication, Ω is a compact topological Abelian group. Therefore, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ exists, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_m. Define, for $\sigma > \sigma_1$,

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m \sigma}, \tag{4}$$

and additionally assume that the exponents λ_m satisfy

$$\lambda_m \geqslant c(\log m)^{\delta} \tag{5}$$

with some $c > 0$ and $\delta > 0$. Then in [1] it was proved that $f(\sigma, \omega)$ is a complex-valued random variable defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by P_f its distribution, i.e.,

$$P_f(A) = m_H\big(\omega \in \Omega\colon\ f(\sigma, \omega) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Let $h > 0$ be fixed and such that $\exp\{\frac{2\pi}{h}\}$ is a rational number.

THEOREM 2. *Suppose that the function $f(s)$ satisfies conditions* (2) *and* (3), $\{\lambda_m\}$ *is a sequence of algebraic numbers linearly independent over the field of rational numbers and satisfies condition* (5). *Then the measure P_N weakly converges to P_f as $N \to \infty$.*

The aim of this paper is to extend the choice of the sequence $\{\lambda_m\}$ in Theorem 2, i.e., to replace condition (5) by the convergence of the series

$$\sum_{m=1}^{\infty} |a_m|^2 \mathrm{e}^{-2\lambda_m\sigma} \log^2 m. \tag{6}$$

Proof of Theorem 2 with replaced condition (6) we obtain in the same way as with the condition (5) (see, [2]). As it was noted above, condition (5) is necessary only for the proof on the existence of the complex-valued random variable.

2. The random variable $f(\sigma, \omega)$

We will prove that if series (6) converges, then for $\sigma > \sigma_1$, $f(\sigma, \omega)$ is a complex-valued random element, i.e., the series (4) converges for almost all $\omega \in \Omega$ with respect to the Haar measure m_H.

LEMMA 1. $f(\sigma, \omega)$ *is a complex-valued random variable defined on the probability space* $(\Omega, \mathcal{B}(\Omega), m_H)$.

For the proof of this lemma we need the following results. Let, as usual, $\mathrm{E}\varphi$ stand for the mean of the random element φ.

LEMMA 2. *Let the random variables $X_1, X_2, \ldots$ be orthogonal, and assume that*

$$\sum_{m=1}^{\infty} \mathrm{E}|X_m|^2 (\log m)^2 < \infty.$$

Then the series

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely.

The lemma is Rademacher's theorem, its proof can be found, for example, in [3].

Proof of Lemma 1. Let, for $\sigma > \sigma_1$,

$$\varphi_m(\omega) = a_m \omega(m) \mathrm{e}^{-\lambda_m \sigma}, \quad m \in \mathbb{N}.$$

Then $\{\varphi_m\}$ is a sequence of complex-valued random variables on $(\Omega, \mathcal{B}(\Omega), m_H)$. It is not difficult to see that

$$\mathrm{E}|\varphi_m|^2 = |a_m|^2 \mathrm{e}^{-2\lambda_m\sigma}, \tag{7}$$

and

$$\mathrm{E}(\varphi_m\overline{\varphi}_k) = \int_\Omega \varphi_m(\omega)\overline{\varphi}_k(\omega)\,\mathrm{d}m_H = a_m\overline{a}_k\mathrm{e}^{-(\lambda_m+\lambda_k)\sigma}\int_\Omega \omega(m)\overline{\omega(k)}\,\mathrm{d}m_H$$
$$= \begin{cases} 0, & \text{if } m \neq k, \\ |a_m|^2\mathrm{e}^{-2\lambda_m\sigma}, & \text{if } m = k. \end{cases}$$

Consequently, $\{\varphi_m\}$ is a sequence of pairwise orthogonal random elements. In view of the convergence of series (6) and equality (7), we have that

$$\sum_{m=1}^{\infty} \mathrm{E}|\varphi_m|^2\log^2 m < \infty.$$

Therefore, by Lemma 2, the series $\sum_{m=1}^{\infty}\varphi_m$ converges almost surely, i.e., the series

$$\sum_{m=1}^{\infty} a_m\omega(m)\mathrm{e}^{-\lambda_m\sigma}$$

converges for almost all $\omega \in \Omega$ with respect to the Haar measure. This shows that $f(\sigma,\omega)$ is a random variable on $(\Omega, \mathcal{B}(\Omega), m_H)$.

References

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REZIUMĖ

R. Macaitienė. Diskreti ribinė teorema bendrosioms Dirichlet eilutėms kompleksinėje plokštumoje su pagerinta sąlyga

Pateikta diskreti ribinė teorema su pagerinta sąlyga bendrosioms Dirichlet eilutėms kompleksinėje plokštumoje tikimybinių matų silpnojo konvergavimo prasme.