

On the uniform distribution of endomorphisms of s -dimensional torus, II

Birutė KRYŽIENĖ (VGTU), Gintautas MISEVIČIUS (VU)

e-mail: gintautas.misevicius@maf.vu.lt

This article is a continuation of the article [3]. The object of our investigation is a uniform distribution of endomorphisms of s -dimensional torus Ω_s . Here we prove that the uniform distribution is attained under different conditions than in [3].

Let $\Omega = \Omega_s$ ($s \geq 2$) be a s -dimensional torus, and its endomorphisms $T: \Omega_s \rightarrow \Omega_s$ are defined by

$$T\vec{x} = \vec{x}W \pmod{1}$$

where W is a non-singular matrix with integer elements, and \vec{x} is a point of a parametric curve

$$\vec{\xi} = \vec{\xi}(\varphi_1(t), \dots, \varphi_s(t)), \quad a \leq t \leq b. \quad (1)$$

Let the functions $\varphi_i(t)$ satisfy the condition

$$|\varphi_i'''(t)| > \gamma_0 > 0, \quad i = 1, \dots, s. \quad (2)$$

In what follows c_1, c_2, \dots are positive constants, $\|a\|$ denotes the distance of a to the closest integer.

Theorem. *Let the characteristic polynomial of the matrix W have distinct integer roots $\theta_1, \dots, \theta_s$. Let there exist sufficiently small $c > 0$ and sufficiently large $\omega > 0$ for which the inequality*

$$\|m_1\varphi_1(t) + \dots + m_s\varphi_s(t)\| \geq \frac{c}{(\bar{m}_1 \dots \bar{m}_s)^\omega}, \quad a \leq t \leq b, \quad (3)$$

holds. Here the non-zero vector $\vec{m} = (\bar{m}_1, \dots, \bar{m}_s)$ has integer coordinates,

$$\bar{m}_k = \begin{cases} 1, & \text{if } m_k = 0, \\ |m_k|, & \text{if } m_k \neq 0. \end{cases}$$

Then the sequence $\{\vec{\xi}W^k\}$, $k = 1, 2, \dots$, is uniformly distributed in the torus Ω_s for almost all $t \in [a, b]$.

Proof. We make use of the expansion

$$\vec{\xi} W^m = \sum_{k=1}^s L_k(\varphi_1(t), \dots, \varphi_s(t)) \theta_k^m \vec{w}_k, \quad (4)$$

where \vec{w}_k is the eigenvector corresponding to the eigenvalue θ_k and L_k is the linear form of the variables z_1, \dots, z_s :

$$L_k(z_1, \dots, z_s) = \sum_{l=1}^s v_{kl} z_l, \quad k = 1, \dots, s,$$

where v_{kj} ($k, l = 1, \dots, s$) are real numbers defined by the matrix W and independent of m . We get from (4)

$$\vec{m} \cdot \vec{\xi} W^m = \sum_{k=1}^s L_k(\varphi_1(t), \dots, \varphi_s(t)) (\vec{m} \cdot \vec{w}_k) \theta_k^m. \quad (5)$$

Since at least one product $\vec{m} \cdot \vec{w}_k$ is not zero, suppose that $\vec{m} \cdot \vec{w}_1 \neq 0$. Assume that $|\theta_1| > |\theta_2|$. In this case the numbers $\vec{m} \cdot \vec{w}_1$ and v_{kl} are rational, and $v_{1k} \neq 0$ for some k . Let R be the least common denominator of fractions v_{1k} and $\vec{m} \cdot \vec{w}_1$. Denote

$$\varphi(t) = \sum_{k=1}^s R v_{1k} (\vec{m} \cdot \vec{w}_1) \cdot \varphi_k(t) = \sum_{k=1}^s n_k \cdot \varphi_k(t),$$

where $n_k = R v_{1k} \cdot (\vec{m} \cdot \vec{w}_1)$ are integers. Then we get from (3) that

$$\|\varphi(t)\| \geq \frac{c}{(\bar{n}_1 \cdots \bar{n}_s)^\omega} \geq \frac{c_1}{(\bar{m}_1 \cdots \bar{m}_s)^{s\omega}}, \quad (6)$$

where c_1 does not depend on \vec{m} . The equality (5) and inequality (6) imply now that

$$\vec{m} \cdot \vec{\xi} W^m = \theta_1^m \frac{\varphi(t)}{R} \left(1 + O\left(\left(\frac{\theta_2}{\theta_1} \right)^m \cdot (\bar{m}_1 \cdots \bar{m}_s)^{s\omega+1} \right) \right). \quad (7)$$

The vector \vec{m} can be chosen so that

$$(\bar{m}_1 \cdots \bar{m}_s)^{s\omega+1} \leq \left| \frac{\theta_1}{\theta_2} \right|^{m/2} = e^{c_2 m}, \quad c_2 > 0, \quad (8)$$

hold. Then we obtain from (7)

$$I_m = \int_a^b \exp(2\pi i \vec{m} \cdot \vec{\xi} W^m) dt = \int_a^b \exp\left(\frac{2\pi i}{R} \theta_1^m \varphi(t) (1 + O(e^{-c_2 m})) \right) dt. \quad (9)$$

Assume that $|\varphi'(t)| \geq c_3 > 0$, otherwise the calculation is analogous to [1]. After the change of variables in (9) and some calculation we obtain the inequality

$$|I_m| \leq c_4 e^{-c_3 m}. \quad (10)$$

Consider a function $g(\cdot)$ belonging to the Korobov class of functions $E_s^\alpha(c)$, $\alpha > 2$ (see [1]). Then this function can be expressed in the following way:

$$g(\vec{x}) = \int \cdots \int_{\Omega_s} g(\vec{x}) d\vec{x} + \sum_{m_1, \dots, m_s = -\infty}^{\infty *} c(m_1, \dots, m_s) \exp(2\pi i \vec{m} \cdot \vec{x})$$

where (*) denotes the summation over s -tuples (m_1, \dots, m_s) for which at least one of $m_1, \dots, m_s \neq 0$. This implies that

$$\begin{aligned} & \left| \int_a^b g(\vec{\xi} W^m) dt - (b-a) \int \cdots \int_{\Omega_s} g(\vec{x}) d\vec{x} \right| \\ & \leq c_5 \sum_{\bar{m}_1, \dots, \bar{m}_s > A} (\bar{m}_1 \cdots \bar{m}_s)^{-\alpha} \\ & \quad + c_6 \sum_{\bar{m}_1, \dots, \bar{m}_s \leq A} (\bar{m}_1 \cdots \bar{m}_s)^{-\alpha} \left| \int_a^b \exp(2\pi i \vec{m} \cdot \vec{\xi} W^m) dt \right| \end{aligned}$$

where $A = |\theta_1/\theta_2|^{c_3 m}$.

The first sum on the right-hand side is evaluated using the corresponding lemma in [1]. We obtain

$$\sum_{\bar{m}_1, \dots, \bar{m}_s > A} (\bar{m}_1 \cdots \bar{m}_s)^{-\alpha} < c_7 \ln^2 m e^{-c_3 m}.$$

From (10) we get that the second sum in the same expression does not exceed

$$c_4 e^{-c_3 m} \sum_{\bar{m}_1, \dots, \bar{m}_s \leq A} (\bar{m}_1 \cdots \bar{m}_s)^{-\alpha} \leq c_8 e^{-c_3 m}$$

These estimates imply

$$\int_a^b g(\vec{\xi} W^m) dt = (b-a) \int \cdots \int_{\Omega_s} g(\vec{x}) d\vec{x} + c_9 \ln^2 m e^{-c_3 m}.$$

The rest of the proof is analogous to the proof the theorem in [3].

References

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Apie s -mačio toro endomorfizmų tolygų pasiskirstymą, II

B. Kryžienė, G. Misevičius

Straipsnis yra [3] darbo tęsinys. Įrodoma, kad tolygus pasiskirstymas galioja esant skirtingoms, negu minėtame straipsnyje sąlygoms. Parametrinei kreivei, nusakytai (1) formule ir funkcijomis $\varphi_i(t)$, $i = 1, \dots, s$, tenkinančiomis (2) sąlygą, įrodoma

Teorema. Tegų matricos W charakteristinis polinomas turi sveikąsias šaknis $\theta_1, \dots, \theta_s$. Jei egzistuoja toks pakankamai mažas skaičius $c > 0$ ir pakankamai didelis skaičius $\omega > 0$, kuriems galioja (3) nelygė, tai seka $\{\xi W^m\}$ yra tolygiai pasiskirsčiusi tore Ω_s beveik visiems $t \in [a, b]$.