

An approximation of the solution of Stratanovich integral equation driven by a continuous p -semimartingale

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Introduction

Consider the Stratonovich integral equation

$$X_t = \eta + (S) \int_0^t f(X_s) dZ_s, \quad t \in [0, T], \quad (1)$$

or equivalent equation

$$X_t = \eta + \int_0^t f(X_s) dZ_s + \frac{1}{2} \int_0^t f f'(X_s) ds, \quad t \in [0, T],$$

where $Z = W + B^H$, W is a standard Brownian motion, B^H is a fractional Brownian motion (fBm) with Hurst index $1/2 < H < 1$. For short, we shall write $f f'(X_s)$ instead of $f(X_s) f'(X_s)$.

In compute simulation of the solution of the equation (1) it is useful to find a good approximation. The main problem is to approximate fBm B^H . Several schemes of an approximations of the fBm are considered in [1,3–5].

Assume that the self similarity index H satisfies $H > 1/2$. In this case we have the following kernel representation of B^H with respect to the standard Brownian motion

$$B_t^H = \int_0^t K_H(t, s) dW_s$$

with a deterministic kernel

$$K_H(t, s) = c_H \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du,$$

where c_H is the normalizing constant.

Let $\varkappa^n = \{t_k^n: 0 \leq k \leq n\}$, $n \geq 1$, be a sequence of partitions of the interval $[0, T]$, i.e., $0 = t_0^n < t_1^n < \dots < t_n^n = T$, $t_k^n = Tk/n$.

For partition \mathcal{r}^n define $\rho^n(t) = \max\{t_k^n: t_k^n \leq t\}$ and $r^n(t) = \max\{k: t_k^n \leq t\}$, $t \in [0, T]$. For every $x \in D([0, T])$ the sequence $(x^{\mathcal{r}^n})$ denotes the following discretizations of x :

$$x_t^{\mathcal{r}^n} = x(t_k^n) \text{ for } t \in [t_k^n, t_{k+1}^n), \quad 0 \leq k \leq n, \quad n \in \mathbb{N}.$$

Define

$$A_t^n = \sum_{k=1}^{r^n(t)} K_H(\rho^n(t), t_k^n) (W(t_k^n) - W(t_{k-1}^n)), \quad M_t^n = W_t^{\mathcal{r}^n}.$$

Let \widehat{Z}^n and \widetilde{Z}^n be linear approximations of the processes $Z^n = M^n + A^n$ and $Z = W + B^H$ correspondingly, i.e.,

$$\widehat{Z}_t^n = Z^n(t_{k-1}^n) + \frac{t - t_{k-1}^n}{t_k^n - t_{k-1}^n} (Z^n(t_k^n) - Z^n(t_{k-1}^n)) \text{ for } t \in [t_{k-1}^n, t_k^n),$$

$$\widetilde{Z}_t^n = Z(t_{k-1}^n) + \frac{t - t_{k-1}^n}{t_k^n - t_{k-1}^n} (Z(t_k^n) - Z(t_{k-1}^n)) \text{ for } t \in [t_{k-1}^n, t_k^n),$$

where $n \in \mathbb{N}$, $1 \leq k \leq n$. Define the approximations

$$Y_t^n = \eta + \int_0^t f(Y_s^n) d\widehat{Z}_s^n, \quad \widetilde{Y}_t^n = \eta + \int_0^t f(\widetilde{Y}_s^n) d\widetilde{Z}_s^n, \quad t \in [0, T].$$

Now we formulate our results.

THEOREM 1. *Let f' be a continuous function and $f(x) > 0$ for all x . Then $\sup_{t \leq T} |Y_t^n - X_t| \xrightarrow{\mathbf{P}} 0$ as $n \rightarrow \infty$.*

THEOREM 2. *Let $f \in C_b^3(\mathbb{R})$. Then*

$$n^{1/q} (1 + \ln n)^{-1/2} \sup_{t \leq T} |\widetilde{Y}_t^n - X_t| \xrightarrow{\mathbf{P}} 0 \text{ as } n \rightarrow \infty.$$

1. Proofs

LEMMA 1. *We have $\sup_{t \leq T} |A_t^n - B_t^H| \xrightarrow{\mathbf{P}} 0$ as $n \rightarrow \infty$.*

Proof. For every fix $t > 0$ and $\varepsilon > 0$

$$\mathbf{E} |A_t^n - B_t^H|^2$$

$$\leq 4\mathbf{E} \left| \sum_{k=1}^{r^n(\varepsilon)} K_H(\rho^n(t), t_k^n) (W(t_k^n) - W(t_{k-1}^n)) \right|^2 + 4\mathbf{E} \left| \int_0^{\rho^n(\varepsilon)} K_H(t, s) dW_s \right|^2$$

$$+4\mathbf{E}\left|\sum_{k=r^n(\varepsilon)+1}^{r^n(t)}\left(K_H(\rho^n(t), t_k^n)(W(t_k^n) - W(t_{k-1}^n)) - \int_{t_{k-1}^n}^{t_k^n} K_H(t, s) dW_s\right)\right|^2$$

$$+4\mathbf{E}\left|\int_{\rho^n(t)}^t K_H(t, s) dW_s\right|^2.$$

By martingale property and inequality

$$|K_H(t, s)| \leq c_H(H - 1/2)^{-1} s^{1/2-H} := \widehat{c}_H s^{1/2-H}$$

we get

$$\mathbf{E}|A^n - B_t^H|^2 \leq 4n^{-1} \sum_{k=1}^{r^n(\varepsilon)} K_H^2(\rho^n(t), t_k^n) + 4 \int_0^{\rho^n(\varepsilon)} K_H^2(t, s) ds$$

$$+ 4 \sum_{k=r^n(\varepsilon)+1}^{r^n(t)} \int_{t_{k-1}^n}^{t_k^n} (K_H(\rho^n(t), t_k^n) - K_H(t, s))^2 ds + 4 \int_{\rho^n(t)}^t K_H^2(t, s) ds$$

$$\leq 8\widehat{c}_H^2 \frac{\varepsilon^{2-2H}}{2-2H} + \frac{4c_H^2}{(H-1/2)^2 n^{2H-1}} \left\{ 3 \left(\frac{1}{\rho^n(\varepsilon)} \right)^{H-1/2} + 1 \right\}^2 + \frac{4\widehat{c}_H^2}{2-2H} \frac{1}{n^{2-2H}}.$$

Thus $\mathbf{E}|A_t^n - B_t^H|^2 \rightarrow 0$ as $n \rightarrow \infty$ for every $t > 0$.

By simple calculations we get

$$\mathbf{E}|A^n(t) - A^n(s)|^2 \leq \frac{2c_H}{(H-1/2)^2} \frac{1}{2-2H} (\rho^n(t) - \rho^n(s))^{2H-1}.$$

By tightness criterium formulated in Theorem 6.4.1 [2] we get that the sequence (A^n) is tight. The sequence $A^n - B^H$ is tight as a difference of two tight sequences (see Corollary 3.3.3 in section 6 [2]). Thus we obtain the statement of the lemma.

Proof of Theorem 1. From [6] one can get that

$$|Y_t^n - X_t| \leq \left(1 + \sup_{t \leq T} |X_t|\right) \left(\exp\{C_1 |\widehat{Z}_t^n - Z_t|\} - 1\right),$$

where C_1 is a constant. Since

$$\sup_{t \leq T} |\widehat{Z}_t^n - \widehat{Z}_t^{n, \mathcal{X}^n}| \leq \max_{1 \leq k \leq n} |Z^n(t_k^n) - Z^n(t_{k-1}^n)|$$

$$\leq \max_{1 \leq k \leq n} |W(t_k^n) - W(t_{k-1}^n)| + \max_{1 \leq k \leq n} |A^n(t_k^n) - A^n(t_{k-1}^n)|$$

and

$$\mathbf{E} \max_{1 \leq k \leq n} |A^n(t_k^n) - A^n(t_{k-1}^n)| \leq C_2 n^{1/2-H} (1 + \ln n)^{1/2},$$

then $\sup_{t \leq T} |\widehat{Z}_t^n - Z_t| \xrightarrow{\mathbf{P}} 0$ as $n \rightarrow \infty$. Thus the proof is completed.

Proof of Theorem 2. Let

$$X_t^n = \eta + \int_0^t f(X_s^{n, \mathcal{X}^n}) dZ_s + \frac{1}{2} \int_0^t ff'(X_s^{n, \mathcal{X}^n}) ds \\ + \int_0^t ff'(X_s^{n, \mathcal{X}^n}) \left(\int_{\rho_s^n}^s dW_u + \int_{\rho_s^n}^s dB_u^H \right) dB_s^H.$$

Similarly as in [3] one can show that

$$n^{1/q} (1 + \ln n)^{-1/2} V_q(X - X^n; [0, T]) \xrightarrow{\mathbf{P}} 0 \quad n \rightarrow \infty, \quad \text{if } q > 2.$$

It is easy to show that

$$\mathbf{E} \sup_{t \leq T} |X_t^n - X_t^{n, \mathcal{X}^n}| \leq C_3 n^{-1/2} (1 + \ln n)^{1/2},$$

$$\mathbf{E} \sup_{t \leq T} |Y_t^n - Y_t^{n, \mathcal{X}^n}| \leq C_4 \{n^{-H} + n^{-1/2}\} (1 + \ln n)^{1/2}.$$

It still remains to prove that

$$n^{1/2} (1 + \ln n)^{1/2} \sup_{t \leq T} |X_t^{n, \mathcal{X}^n} - Y_t^{n, \mathcal{X}^n}| \xrightarrow{\mathbf{P}} 0 \quad \text{as } n \rightarrow \infty.$$

The proof of this fact is too long for this note.

References

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REZIUMĖ

K. Kubilius. *Stratanovičiaus integralinės lygties sprendinio, valdomo tolydaus p -semimartingalo, aproksimacija*

Konstruojamos dvi Vong-Zakai tipo aproksimacijos. Gautos sąlygos, kada pirmoji aproksimacija konverguoja pagal tikimybę į Stratanovičiaus integralinės lygties sprendinį. Rastas antrosios aproksimacijos konvergavimo greitis.