

On the uniform distribution of endomorphisms of s -dimensional torus, II

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Let $\Omega = \Omega_s$ ($s \geq 2$) be s -dimensional torus, i.e., the set of points

$$\mathbf{x} = (x_1, \dots, x_s), \quad 0 \leq x_i < 1, \quad i = 1, \dots, s.$$

An endomorphism of torus $T: \Omega \rightarrow \Omega$ is defined by

$$T\mathbf{x} = \mathbf{x}W \pmod{1}, \quad \mathbf{x} \in \Omega,$$

where W is a nonsingular matrix with integer elements.

We continue our work [1] on the investigation of the conditions on the initial point \mathbf{x} and the matrix W for the sequence $\mathbf{x}W, \mathbf{x}W^2, \dots, \mathbf{x}W^k, \dots$ be uniformly distributed on torus Ω . Therefore all the notations are the same as in [1]. Here we remind some of them:

$$\xi = \xi(t) = (\varphi_1(t), \dots, \varphi_s(t)), \quad a \leq t \leq b, \quad (1)$$

is a parametric curve on Ω_s , functions $\varphi_i(t)$ have bounded derivatives of order $s - 1$, $W(t)$ is the Wronskian of these functions, $W(t) \neq 0$ for $t \in [a, b]$, the characteristic polynomial of the matrix W satisfies certain conditions on its roots.

D. Moskvina [2] proved that the sequence

$$\xi W, \xi W^2, \dots, \xi W^k, \dots \quad (2)$$

is uniformly distributed on torus Ω_s for almost all $t \in [a, b]$ in the sense of the Lebesgue measure μ .

In [1] the condition

$$|\varphi_i''(t)| \geq \kappa > 0, \quad i = 1, \dots, s \quad (3)$$

was used instead of $|W(t)| \neq 0$, and a restricted condition was imposed on the roots $\theta_1, \dots, \theta_s$ of the characteristic polynomial of the matrix W .

In this paper the condition (3) is replaced by another one.

The following theorem is proved.

THEOREM. *Let W be nonsingular matrix with integer elements, $\theta_1, \theta_2, \dots, \theta_s$ be its eigenvalues, $|\theta_1| > |\theta_2| > \dots > |\theta_s|$, $\mathbf{w}_i = (w_{i1}, \dots, w_{is})$ be the corresponding eigen-*

vectors, and $g(t) = \mathbf{w}_1 \boldsymbol{\xi}(t)$,

$$\kappa^2 = (\varphi_1''(t))^2 + \cdots + (\varphi_s''(t))^2 > 0, \quad t \in [a, b].$$

If for each $t_0 \in [a, b]$ such that $g(t_0) = 0$, there exists k , $1 < k \leq s$, $g^{(k)}(t_0) \neq 0$, then the sequence (2) is uniformly distributed on Ω_s for almost all $t \in [a, b]$ in the sense of the Lebesgue measure.

Consider the linear combination

$$g(t) = w_{11}\varphi_1(t) + w_{12}\varphi_2(t) + \cdots + w_{1s}\varphi_s(t).$$

The following auxiliary result for this function is true.

LEMMA. If $g(t_0) = 0$, $t_0 \in [a, b]$, and $g^{(k)}(t_0) \neq 0$ for some k , $1 < k \leq s$, then there exist two sufficiently small constants λ , λ^* such that

$$|g(t)| \geq \lambda |t - t_0|^{s-1} \quad \text{for } |t - t_0| \leq \lambda^*.$$

Proof of Lemma. According to the Taylor formula in the neighbourhood of $t = t_0$,

$$g(t) = \sum_{k=1}^{s-1} \frac{g^{(k)}(t_0)}{k!} (t - t_0)^k + \frac{g^{(s)}(t^*)}{s!} (t - t_0)^s, \quad t^* \in [a, b]. \quad (4)$$

Let $g'(t_0) \neq 0$. Then

$$g(t) = (t - t_0)(g'(t_0) + (t - t_0)B_1(t))$$

with a function $B_1(t)$ bounded for $t \in [a, b]$, $\max_t |B_1(t)| = B_1$. Therefore the inequality

$$|g(t)| \geq \frac{|g'(t_0)|}{2B_1} |t - t_0|$$

is true in the interval $|t - t_0| \leq \frac{|g'(t_0)|}{2B_1}$, and the statement of Lemma is true.

Now let $g'(t_0) = 0$, but $g''(t_0) \neq 0$. In the same manner we obtain from (4) that

$$g(t) = (t - t_0)^2 (g''(t_0) + (t - t_0)B_2(t))$$

with a function $B_2(t)$ bounded for $t \in [a, b]$, $\max_t |B_2(t)| = B_2$, and the inequality

$$|g(t)| \geq \frac{|g''(t_0)|}{2B_2} |t - t_0|^2$$

is true in the interval $|t - t_0| \leq \frac{|g''(t_0)|}{2B_2}$.

After a final number of similar steps, since $g^{(k)}(t_0) \neq 0$ for some k , we get the proof of Lemma.

So, every zero of the function $g(t)$ is isolated and the number of zeroes is finite.

Proof of Theorem. The eigenvectors \mathbf{w}_i , $i = 1, \dots, s$, corresponding to different eigenvalues form the complete basis in the space \mathbf{R}_s . So, any vector can be written as follows:

$$\xi W^m = \sum_{i=1}^s \left(\sum_{j=1}^s w_{ij} \varphi_j(t) \right) \theta_i^m \mathbf{w}_i = \sum_{i=1}^s L_i(t) \theta_i^m \mathbf{w}_i. \quad (5)$$

The function $L_1(t)$ as well as $L_1'(t)$ have finite number of isolated zeroes. Let these zeroes be t_i , $i = 1, 2, \dots, r$. According to Lemma

$$|L_1(t)| \geq \lambda_i |t - t_i|^{s-1}, \quad i = 1, \dots, r.$$

For a sufficiently small δ , $\delta > 0$, we take nonintersecting intervals of length 2δ :

$$\Delta_i = (t_i - \delta, t_i + \delta), \quad i = 1, \dots, r.$$

For every t , $t \notin \bigcup \Delta_i$, i.e., $t \in [a, b] \setminus (\bigcup \Delta_i) = A$, analogously to [2] the inequality

$$|L_1(t)| \geq d \delta^{s-1}, \quad d = \min_i \lambda_i \quad (6)$$

can be proved.

The set A is a union of intervals, and let $[a_1, b_1]$ be one of them. Consider the integral

$$I_m = \int_{a_1}^{b_1} g(\xi(t) W^m) dt, \quad g(\mathbf{x}) \in E_s^\alpha(c),$$

where $E_s^\alpha(c)$ is the Korobov class of functions [3]. We obtain from (5) and (6) that

$$\begin{aligned} \xi W^m &= L_1(t) \theta_1^m \mathbf{w}_1 + L_2(t) \theta_2^m \mathbf{w}_2 + \dots \\ &= (L_1(t) \theta_1^m w_{11} + L_2(t) \theta_2^m w_{21} + \dots, \\ &\quad L_1(t) \theta_1^m w_{12} + L_2(t) \theta_2^m w_{22} + \dots, \dots) \\ &= (L_1(t) w_{11} (\theta_1^m + O(\delta^{-s} \theta_2^m)), L_1(t) w_{12} (\theta_1^m + O(\delta^{-s} \theta_2^m)), \dots) \\ &= (w_{11} L_1(t), w_{12} L_2(t), \dots) (\theta_1^m + O(\delta^{-s} \theta_2^m)) \end{aligned} \quad (7)$$

Let $t = t_0$ be such that $L_1'(t_0) = 0$. Then the integral I_m can be divided into three parts and each of them is evaluated separately:

$$I_m = \left(\int_{a_1}^{t_0-\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^{b_1} \right) g(\xi(t) W^m) dt = I_{m1} + I_{m2} + I_{m3}.$$

It is evident that the middle term $I_{m2} = O(\delta)$. Both I_{m1} and I_{m3} have the same estimation. Let us take I_{m1} :

$$I_{m1} = \int_{a_2}^{b_2} g(\xi(t) W^m) dt, \quad \text{where } a_2 = a_1, \quad b_2 = t_0 - \delta.$$

According to (7) we can write:

$$I_{m1} = \int_{a_2}^{b_2} g(w_{11}\theta_1^m L_1(t)(1 + O(\delta^{-s}\varrho)), \dots) dt, \quad \text{where } \varrho = \left| \frac{\theta_2}{\theta_1} \right|^m.$$

After the change of variables $u = \theta_1^m L_1(t)$ we obtain

$$I_{m1} = \frac{1}{\theta_1^m} \int_{D_m}^{D'_m} g(w_{11}u(1 + O(\varrho\delta^{-s})), \dots) \Phi(u) du,$$

with $D_m = \theta_1^m L_1(a_2)$, $D'_m = \theta_1^m L_1(b_2)$, $\Phi(u) = \frac{d}{du} L_1^{-1}\left(\frac{u}{\theta_1^m}\right)$.

Suppose D_m and D'_m are integers. Otherwise the estimation of I_{m1} differs only in $O(\theta_1^m \delta^{s-1})$. Thus by the Abel transformation and analogously to [4] we get

$$\begin{aligned} I_{m1} &= \frac{1}{\theta_1^m} \sum_{k=D_m}^{D'_m-1} \int_k^{k+1} g(\dots) \Phi(u) du = \frac{1}{\theta_1^m} \sum_{k=D_m}^{D'_m-1} \int_0^1 g(\dots) \Phi(u+k) du \\ &= \frac{1}{\theta_1^m} \int_0^1 \sum_{k=D_m}^{D'_m-1} (\Phi(u+k) - \Phi(u+k+1)) \sum_{l=D_m}^k g(\dots) du \\ &= \frac{1}{\theta_1^m} \int_0^1 \sum_{k=D_m}^{D'_m-1} (\Phi(u+k) - \Phi(u+k+1))(k - D_m + 1) \\ &\quad \times \left\{ \int_{\Omega_s} g(\mathbf{x}) d\mathbf{x} + O\left(\frac{1}{k - D_m + 1} + (k - D_m + 1)(\varrho\delta^{-s})^{\frac{\alpha-1}{1+\varepsilon}} \right) \right\} \end{aligned}$$

with

$$\alpha = 1 + \frac{2(1 + \varepsilon) \ln |\theta_1|}{\ln |\theta_1| - \ln |\theta_2|}, \quad \varepsilon > 0.$$

From (6) we get the estimate

$$|\Phi(u+k) - \Phi(u+k+1)| = O(\theta_1^{-m} \delta^{-3s})$$

and then we have the equality

$$I_{m1} = (b_2 - a_2) \int_{\Omega_s} g(\mathbf{x}) d\mathbf{x} + O\left(\delta + \frac{1}{\theta_1^m \delta^{3s}} + \frac{\theta_1^m}{\delta^{3s}} \left(\left| \frac{\theta_2}{\theta_1} \right| \frac{1}{\delta} \right)^{\frac{\alpha-1}{1+\varepsilon}} \right).$$

The remaining part of the proof of Theorem is the same as in [4] (see also [1]).

References

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REZIUMĖ

B. Kryžienė, G. Misevičius. s -mačio toro endomorfizmų tolygus pasiskirstymas, II

Darbe apibendrinama D. Moskvino teorema apie s -mačio toro Ω_s endomorfizmų (mod 1) tolygų pasiskirstymą. Vietoje apribojimo – funkcijų $\varphi_1(t), \dots, \varphi_s(t)$ vronskijanas $W(t) \neq 0$, $t \in [a, b]$, naudojama kita sąlyga $(\varphi_1''(t))^2 + \dots + (\varphi_s''(t))^2 > 0$, $t \in [a, b]$.