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Upper-bound estimates for weighted sums satisfying Cramer's condition

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Abstract. Let $S = w_1S_1 + w_2S_2 + \ldots + w_NS_N$. Here S_j is the sum of identically distributed random variables and $w_j > 0$ denotes weight. We consider the case, when S_j is the sum of independent random variables satisfying Cramer's condition. Upper-bounds for the accuracy of compound Poisson first and second order approximations in uniform metric are established.

Keywords: compound Poisson distribution, signed compound Poisson measure, Kolmogorov distance.

1. Introduction

Let us consider the following complex sampling design: entire population consists of different clusters and probability for each cluster to be selected into the sample is known. The sum of sample elements, then is equal to $S = w_1S_1 + w_2S_2 + \ldots + w_NS_N$. Here S_i are sums of independent identically distributed random variables and w_i denote weights. Weighting can radically change the structural properties of S. For example, even if all S_i are lattice, the sum S is not. In this article, we consider the case of random variables forming a sequence: $X_1, X_2, \ldots$. More formally, the case of sequences will mean that the distribution of X_j in S_n does not depend on n. Sequences of random variables are comparatively well investigated, since then the normal approximation usually is quite sharp, see, for example, the book of Petrov [4]. However, if we have less than one moment, then accompanying distribution might be a better choice for approximation, see [1, 5]. In this article, we extend the research of [1, 5] estimating the effect of smoothing.

2. Notation

Let $\mathcal{F}$ (resp. $\mathcal{M}$) denote the set of probability distributions (resp. finite signed measures) on $\mathbb{R}$. The Dirac measure concentrated at a is denoted by I_a, $I = I_0$. All products and powers of finite signed measures $W \in \mathcal{M}$ are defined in the convolution sense, and $W^0 = I$. The exponential of W is the finite signed measure defined by $\exp\{W\} = \sum_{m=0}^{\infty} W^m/m!$. The Kolmogorov (uniform) norm $|W|$ and the total variation norm $\| W \|$ of $W \in \mathcal{M}$ are defined by $|W| = \sup_{x\in\mathbb{R}} |W((-\infty, x])|$, $\| W \| = W^+(\mathbb{R}) + W^-(\mathbb{R})$, respectively. Here $W = W^+ - W^-$ is the Jordan-Hahn decomposition. Note that $|W| \leqslant \| W \|$. For $F \in \mathcal{F}$, $h \geqslant 0$ Lévy's concentration function is defined by $Q(F, h) = \sup_x F\{[x, x+h]\}$. We denote by $\widehat{W}(t)$ the Fourier–Stieltjes transform of $W \in \mathcal{M}$. Absolute positive constants are denoted by C.

3. Results

We consider random variables $X_1, X_2, \ldots$ having distributions $F_1, F_2, \ldots$ that satisfy the following conditions:

$$\mathbb{E}X_j = 0, \quad \mathbb{E}|X_j|^{1+\delta} < \infty, \quad \lim\sup_{|t|\to\infty} |\widehat{F}_j(t)| < 1 \quad (j = 1, 2, \ldots, N). \tag{1}$$

Note that we used the well-known Cramer's condition, which means that all F_j are not purely discrete distributions. Although we did not formulate our results in terms of $w_j S_j$, it is easy to understand that our case corresponds to the case $w_j X_j \sim F_j$, where $w_j \asymp C$ and X_j satisfies (1).

It is known that then the following estimates hold:

$$\left|F_j^{n_j} - \exp\{n_j(F_j - I)\}\right| \leqslant C(F_j) n_j^{-\delta} \tag{2}$$

and

$$\left|F_j^{n_j} - \exp\{n_j(F_j - I)\}\Bigl(I - \frac{n_j}{2}(F_j - I)^2\Bigr)\right| \leqslant C n_j^{-2\delta}, \tag{3}$$

see [1].

Now we can formulate the main result of this paper.

THEOREM 1. *Let conditions* (1) *be satisfied and let* $n := n_1 + n_2 + \ldots + n_N$. *Then*

$$\left|\prod_{j=1}^{N} F_j^{n_j} - \exp\Bigl\{\sum_{j=1}^{N} n_j(F_j - I)\Bigr\}\right| \leqslant C_1(F, N) n^{-\delta} \tag{4}$$

and

$$\left|\prod_{j=1}^{N} F_j^{n_j} - \exp\Bigl\{\sum_{j=1}^{N} n_j(F_j - I)\Bigr\}\Bigl(I - \frac{1}{2}\sum_{j=1}^{N} n_j(F_j - I)^2\Bigr)\right| \leqslant C_2(F, N) n^{-\delta}. \tag{5}$$

Thus, we see that for the case of sequences the same order of accuracy can be obtained for weighted sums as well as for the sum of identically distributed random variables.

4. Proofs

Everywhere in the proofs, we use the same notation C for all different absolute constants. We will need the following lemmas.

LEMMA 4.1. *Let* $F, G \in \mathcal{F}$, $h > 0$ *and* $a > 0$. *Then*

$$Q(F, h) \leqslant \Bigl(\frac{96}{95}\Bigr)^2 h \int_{|t|\leqslant 1/h} \bigl|\widehat{F}(t)\bigr| \,\mathrm{d}t, \quad Q(FG, h) \leqslant Q(F, h), \tag{6}$$

$$Q(F, h) \leqslant \Bigl(1 + \Bigl(\frac{h}{a}\Bigr)\Bigr) Q(F, a), \quad Q\bigl(\exp\{a(F - I)\}, h\bigr) \leqslant \frac{C}{\sqrt{aF\{|x| > h\}}}. \tag{7}$$

If, in addition, $\widehat{F}(t) \geqslant 0$, *then*

$$h \int_{|t| \leqslant 1/h} \left|\widehat{F}(t)\right| \mathrm{d}t \leqslant C Q(F, h). \tag{8}$$

Lemma 4.1 contains the well-known properties of Levy's concentration function (see, for example, [2]).

We also use the following variant of Esseen's smoothing estimate which is a slight modification of inequality of Le Cam [3], see also [2]. For $h \in (0, \infty)$ and a finite measure G on $\mathbb{R}$, set $|G|_h = \sup_y |G\{[y, y+h]\}|$.

LEMMA 4.2. *Let* $G, M \in \mathcal{F}, W \in \mathcal{M}$ *with* $W(\mathbb{R}) = 0$, *Then, for arbitrary* $h \in (0, \infty)$, *we have*

$$|W| \leqslant C \int_{|t|<1/h} \left|\frac{\widehat{W}(t)}{t}\right| \mathrm{d}t + C \min\left\{|W^+|_h,\ |W^-|_h\right\}$$

$$|F - G| \leqslant C \int_{|t|<1/h} \left|\frac{\widehat{F}(t) - \widehat{G}(t)}{t}\right| \mathrm{d}t + C Q(G, h).$$

Proof of Theorem 1. We will use the following estimates:

$$\left|\widehat{F}_j(t)\right|,\ \left|\exp\{\widehat{F}_j(t) - 1\}\right| \leqslant \mathrm{e}^{-C(F_j)t^2}, \quad j = 1, \ldots, N, \tag{9}$$

where $|t| \leqslant \epsilon$, and

$$\left|\widehat{F}_j(t)\right|,\ \left|\exp\{\widehat{F}_j(t) - 1\}\right| \leqslant \mathrm{e}^{-C(F_j)}, \quad j = 1, \ldots, N \tag{10}$$

where $|t| \geqslant \epsilon$. Here $\epsilon = \epsilon(F_1, F_2, \ldots, F_N)$.

Also, for all $|t|$ the following estimate holds:

$$\left|\widehat{F}_j(t) - 1\right| \leqslant C(F_j)|t|^{1+\delta} \tag{11}$$

and, for $|t| \leqslant \epsilon$:

$$\left|\widehat{F}_j^{n_j} - \exp\{n_j(\widehat{F}_j - 1)\}\right| \leqslant C(F_j)\mathrm{e}^{-C(F_j)n_j t^2} \cdot n_j|t|^{2+2\delta}, \tag{12}$$

see [1], [5].

We then use Lemma 4.2:

$$\left|\prod_{j=1}^{N} F_j^{n_j} - \prod_{j=1}^{N} \exp\{n_j(F_j - I)\}\right|$$
$$\leqslant C \int_0^{\epsilon} \frac{\left|\prod_{j=1}^{N} \widehat{F}_j^{n_j} - \prod_{j=1}^{N} \exp\{n_j(\widehat{F}_j - 1)\}\right|}{|t|} \mathrm{d}t$$
$$+ C \int_{\epsilon}^{T} \frac{\left|\prod_{j=1}^{N} \widehat{F}_j^{n_j} - \prod_{j=1}^{N} \exp\{n_j(\widehat{F}_j - 1)\}\right|}{|t|} \mathrm{d}t$$

$$+ CQ\Big(\prod_{j=1}^{N}\exp\{n_j(\widehat{F}_j-1)\}, \frac{1}{T}\Big)$$
$$= A_1 + A_2 + A_3. \tag{13}$$

Then

$$A_1 \leqslant C(F)\int_0^{\epsilon} \frac{\sum_{j=1}^{N}|\widehat{F}_j^{n_j} - \exp\{n_j(\widehat{F}_j-1)\}|\prod_{l=1}^{j-1}|\widehat{F}_l^{n_l}|\prod_{l=j+1}^{N}|\exp\{n_l(\widehat{F}_l-1)\}|}{|t|}\,\mathrm{d}t$$
$$\leqslant C(F)\int_0^{\epsilon} \frac{\sum_{j=1}^{N}C(F_j)\mathrm{e}^{-C(F_j)n_jt^2}n_j|t|^{2+2\delta}\prod_{l=1}^{j-1}\mathrm{e}^{-C(F_l)n_lt^2}\prod_{l=j+1}^{N}\mathrm{e}^{-C(F_l)n_lt^2}}{|t|}\,\mathrm{d}t$$
$$\leqslant C(F,N)\int_0^{\infty}\mathrm{e}^{-C(F)nt^2}n|t|^{1+2\delta} \leqslant \frac{C(F,N)}{n^{\delta}} = C(F,N)n^{-\delta}. \tag{14}$$

Similarly, we get

$$A_2 \leqslant C(F)\int_{\epsilon}^{T}\frac{\mathrm{e}^{-C(F)n}}{|t|}\,\mathrm{d}t \leqslant T\frac{\mathrm{e}^{-C(F)n}}{\epsilon} \leqslant C(F)n^{-\delta}. \tag{15}$$

Finally, using the properties of the concentration functions, we get the estimate for A_3:

$$A_3 \leqslant \frac{C}{T}\int_{-T}^{T}\Big|\prod_{j=1}^{N}\exp\{n_j(\widehat{F}_j-1)\}\Big|\,\mathrm{d}t$$
$$\leqslant \frac{C}{T}\Big(\int_0^{\epsilon}\Big|\prod_{j=1}^{N}\exp\{n_j(\widehat{F}_j-1)\}\Big|\,\mathrm{d}t + \int_{\epsilon}^{T}\Big|\prod_{j=1}^{N}\exp\{n_j(\widehat{F}_j-1)\}\Big|\,\mathrm{d}t\Big)$$
$$\leqslant \frac{C}{T}\Big(\int_0^{\epsilon}\mathrm{e}^{-Cnt^2}\,\mathrm{d}t + T\mathrm{e}^{-Cn}\Big) \leqslant \frac{C}{T\sqrt{n}} + T\mathrm{e}^{-Cn}. \tag{16}$$

By substituting $T=\sqrt{n}$, we get $A_3 \leqslant C(F)n^{-\delta}$.
From that we easily obtain (4).
For the proof of (5) we use the following estimate:

$$\Big|\widehat{F}_j^{n_j} - \exp\{n_j(\widehat{F}_j-1)\}\Big(1-\frac{n_j(\widehat{F}_j-1)^2}{2}\Big)\Big| \leqslant C\mathrm{e}^{-Cn_jt^2}|t|^{4\delta}, \quad |t|\leqslant\epsilon. \tag{17}$$

Using the the formula of inversion, we have

$$\big|W\big| = \Big|\prod_{j=1}^{N}F_j^{n_j} - \exp\Big\{\sum_{j=1}^{N}n_j\big(F_j-I\big)\Big\}\Big(I-\frac{1}{2}\sum_{j=1}^{N}n_j\big(F_j-I\big)^2\Big)\Big|$$
$$\leqslant C\int_{-T}^{T}\frac{|\widehat{W}(t)|}{|t|}\,\mathrm{d}t + \Big\|I-\frac{1}{2}\sum_{j=1}^{N}n_j(F_j-I)^2\Big\|\,Q\Big(\exp\Big\{\sum_{j=1}^{N}n_j(F_j-I)\Big\}, \frac{1}{T}\Big)$$

$$= B_1 + B_2. \tag{18}$$

As in previous part of the proof, by taking $T = n^{5/2}$, it easy to show that

$$B_2 \leqslant Cn^{-2\delta}. \tag{19}$$

We divide B_1 into two parts:

$$B_1 = \int_0^{\epsilon} \frac{|\widehat{W}(t)|}{|t|}\,\mathrm{d}t + \int_{\epsilon}^{T} \frac{|\widehat{W}(t)|}{|t|}\,\mathrm{d}t. \tag{20}$$

Then we have

$$\int_{\epsilon}^{T} \frac{|\widehat{W}(t)|}{|t|}\,\mathrm{d}t \leqslant \frac{TC}{\epsilon} n\mathrm{e}^{-Cn} \leqslant Cn^{-2\delta}. \tag{21}$$

It only remains to estimate the second integral. For that we define

$$\widehat{A}_j = \exp\{n_j(\widehat{F}_j - 1)\}\Big(1 - \frac{n_j}{2}(\widehat{F}_j - 1)^2\Big). \tag{22}$$

Then

$$\begin{aligned} \Big|\widehat{W}(t)\Big| &\leqslant \Big|\prod_{j=1}^{N} \widehat{F}_j^{n_j} - \prod_{j=1}^{N} \widehat{A}_j\Big| \\ &\quad + \Big|\prod_{j=1}^{N} \widehat{A}_j - \exp\Big\{\sum_{j=1}^{N} n_j(\widehat{F}_j - 1)\Big\}\Big(1 - \frac{1}{2}\sum_{j=1}^{N} n_j\big(\widehat{F}_j - 1\big)^2\Big)\Big|. \end{aligned} \tag{23}$$

From there we have

$$\Big|\prod_{j=1}^{N} \widehat{F}_j^{n_j} - \prod_{j=1}^{N} \widehat{A}_j\Big| \leqslant \sum_{j=1}^{N} \Big|\widehat{F}_j^{n_j} - \widehat{A}_j\Big| \Big|\prod_{l=1}^{j-1} \widehat{F}_l^{n_l} \prod_{l=j+1}^{N} \widehat{A}_l \leqslant C\mathrm{e}^{-Cnt^2}|t|^{4\delta} \tag{24}$$

and

$$\begin{aligned} &\Big|\prod_{j=1}^{N} \widehat{A}_j - \exp\Big\{\sum_{j=1}^{n} n_j(\widehat{F}_j - 1)\Big\}\Big(1 - \frac{1}{2}\sum_{j=1}^{N} n_j\Big(\widehat{F}_j - 1\Big)^2\Big)\Big| \\ &\leqslant \Big|\exp\Big\{\sum_{j=1}^{N} n_j(\widehat{F}_j - 1)\Big\}\Big| \cdot \Big|\prod_{j=1}^{N}\Big(1 - \frac{n_j}{2}\big(\widehat{F}_j - 1\big)^2\Big) - \Big(1 - \frac{1}{2}\sum_{j=1}^{N} n_j\big(\widehat{F}_j - 1\big)^2\Big)\Big| \\ &\leqslant C\mathrm{e}^{-Cnt^2} \sum_{j\neq k} n_j n_k |\widehat{F}_j - 1|^2 \cdot |\widehat{F}_k - 1|^2 \leqslant C\mathrm{e}^{-Cnt^2} \sum_{j\neq k} n_j n_k |t|^{2+2\delta+2+2\delta} \\ &\leqslant C\mathrm{e}^{-Cnt^2}|t|^{4\delta}. \end{aligned} \tag{25}$$

Therefore, by collecting all estimates we obtain (5).

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REZIUMĖ

V. Čekanavičius, A. Elijio. Svertinių sumų, tenkinančių Kramerio sąlygą, įverčiai iš viršaus

Tarkime, kad $S = w_1 S_1 + w_2 S_2 + \ldots + w_N S_N$. Čia S_j – suma nepriklausomų vienodai pasiskirsčiusių atsitiktinių dydžių, tenkinančių Kramerio sąlygą; w_j – svoris. Įverčiai iš viršaus gauti sudėtinėms Puasono aproksimacijoms.