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On the Mellin transforms of the fourth power of the Riemann zeta-function

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Let $\zeta(s), s=\sigma+it$, denote the Riemann zeta-function. In the theory of $\zeta(s)$, the modified Mellin transforms

$$\mathcal{Z}_k(s)=\int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s}\,\mathrm{d}x,$$

$\sigma \geqslant \sigma_0(k)>1,\ k\geqslant 0$, play an important role. They were introduced and studied by Y. Motohashi [8], and A. Ivič, M. Jutila and Y. Motohashi [3]. The functions $\mathcal{Z}_k(s)$ are applied in the investigations of the moments

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \mathrm{d}t.$$

In [4], [5] and [6] the first limit theorems in the sense of weak convergence of probability measures for the functions $\mathcal{Z}_1(s)$ and $\mathcal{Z}_2(s)$ were proved. The mentioned theorems are continuous because they deal with probability measures defined by continuous shifts of $\mathcal{Z}_k(s), k=1,2$. The aim of this note are discrete limit theorems for $\mathcal{Z}_2(s)$. In these theorems, the probability measures are defined by discrete shifts $\mathcal{Z}_2(s+imh)$.

Let, for $N\in\mathbb{N}\cup\{0\}$,

$$\mu_N(\ldots)=\frac{1}{N+1}\sum_{\substack{0\leqslant m\leqslant N\\ \ldots}} 1,$$

where in place of dots a condition satisfied by m is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Let, as usual, $\mathbb{C}$ denote the complex plane, and let h be an arbitrary fixed positive number. Then we have the following statements.

THEOREM 1. *Suppose that* $\frac{7}{8}<\sigma<1$. *Then on* $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ *there exists a probability measure* P_σ *such that the probability measure*

$$\mu_N\big(\mathcal{Z}_2(\sigma+imh)\in A\big),\quad A\in\mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ *as* $N\to\infty$.

Now let $D=\{s\in\mathbb{C}\colon \frac{7}{8}<\sigma<1\}$, and $H(D)$ be the space of analytic on D functions equipped with the topology of uniform convergence on compacta.

THEOREM 2. *On* $(H(D), \mathcal{B}(H(D)))$ *there exists a probability measure* P *such that the probability measure*

$$\mu_N\big(\mathcal{Z}_2(s+imh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P *as* $N \to \infty$.

In this short paper we give only a sketch of the proof of Theorems 1 and 2. Their full proof will be given elsewhere.

First we consider an integral over a finite interval. Let $\sigma_1 > \frac{1}{2}$ be fixed, and, for $y \geqslant 1$,

$$v(x, y) = \exp\Big\{-\Big(\frac{x}{y}\Big)\Big\}^{\sigma_1}.$$

For $\infty > a > 1$, consider the function

$$\mathcal{Z}_{2,a,y}(s) = \int_1^a \Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^4 v(x,y)\, x^{-s}\,\mathrm{d}x.$$

LEMMA 3. *On* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *there exists a probability measure* $P_{\sigma,a,y}$ *such that the probability measure*

$$P_{N,\sigma,a,y}(A) \stackrel{\text{def}}{=} \mu_N\big(\mathcal{Z}_{2,a,y}(\sigma+imh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{\sigma,a,y}$ *as* $N \to \infty$.

LEMMA 4. *On* $(H(D), \mathcal{B}(H(D)))$ *there exists a probability measure* $P_{a,y}$ *such that the probability measure*

$$P_{N,a,y}(A) \stackrel{\text{def}}{=} \mu_N\big(\mathcal{Z}_{2,a,y}(s+imh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{a,y}$ *as* $N \to \infty$.

Lemmas 3 and 4 follow easily from Theorem 5.1 of [1] and the following assertion. Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ and

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \gamma$ for all $u \in [1, a]$. By the Tikhonov theorem the torus Ω_a is a compact topological Abelian group. On $(\Omega_a, \mathcal{B}(\Omega_a))$ define the probability measure $Q_{N,a}$ by

$$Q_{N,a}(A) = \mu_N\Big(\big(u^{imh}\colon u \in [1,a]\big) \in A\Big).$$

PROPOSITION 5. *On* $(\Omega_a, \mathcal{B}(\Omega_a))$ *there exists a probability measure* Q_a *such that the probability measure* $Q_{N,a}$ *converges weakly to* Q_a *as* $N \to \infty$.

Proof. The dual group of Ω_a is $\oplus_{u\in[1,a]}\mathbb{Z}_u$, where $\mathbb{Z}_u=\mathbb{Z}$ for all $u\in[1,a]$. Hence it follows that the Fourier transform $g_N(\underline{k})$ of the probability measure $Q_{N,a}$ is

$$g_N(\underline{k}) = \int_{\Omega_a}\prod_{u\in[1,a]} x_u^{k_u}\,\mathrm{d}Q_{N,a}$$

$$= \frac{1}{N+1}\sum_{m=0}^{N}\exp\left\{imh\sum_{u\in[1,a]}k_u\log u\right\},$$

where only a finite number of integers k_u are non-zero. Therefore, we find without difficult that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \sum_{u\in[1,a]}k_u\log u = \frac{2\pi r}{h} \text{ for some } r\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

This and Theorem 1.4.2 of [2] prove the proposition.

Now define the function

$$\mathcal{Z}_{2,y}(s) = \int_1^{\infty}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^4 v(x,y)x^{-s}\,\mathrm{d}x.$$

A simple application of the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\Gamma(s)a^{-s}\,\mathrm{d}s = \mathrm{e}^{-a}, \quad a,b>0,$$

shows that the integral for $\mathcal{Z}_{2,y}(s)$ converges absolutely for $\sigma>\frac{1}{2}$.

LEMMA 6. *Suppose that $\sigma>\frac{1}{2}$. Then on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_{\sigma,y}$ such that the probability measure*

$$\mu_N\big(Z_{2,y}(\sigma+it)\in A\big), \quad A\in\mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{\sigma,y}$ as $N\to\infty$.

Proof. First we observe that the family of probability measures $\{P_{\sigma,a,y}\}$ is tight for fixed σ and y. Therefore, it is relative compact, and there exists a subsequence $\{P_{\sigma,a_1,y}\}\subset\{P_{\sigma,a,y}\}$ such that $P_{\sigma,a_1,y}$ converges weakly to some measure $P_{\sigma,y}$ on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ as $a_1\to\infty$. Hence,

$$X_{a_1,y}(\sigma) \underset{a_1\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} P_{\sigma,y}, \tag{1}$$

where $X_{a,y}(\sigma)$ is a complex-valued random variable having the distribution $P_{\sigma,a,y}$.

Let θ_N be a random variable defined on a certain probability space $(\Omega,\mathcal{B}(\Omega),\mathbb{P})$ and having the distribution

$$\mathbb{P}(\theta_N=hm) = \frac{1}{N+1}, \quad m=0,1,\ldots,N.$$

We set $X_{N,a,y}(\sigma) = \mathcal{Z}_{2,a,y}(\sigma + i\theta_N)$. Then in view of Lemma 3

$$X_{N,a,y}(\sigma) \xrightarrow[N\to\infty]{\mathcal{D}} X_{a,y}(\sigma). \tag{2}$$

Since

$$\lim_{a\to\infty} \mathcal{Z}_{2,a,y}(s) = \mathcal{Z}_{2,y}(s)$$

uniformly in t, we find that, for $\sigma > \frac{1}{2}$ and every $\varepsilon > 0$,

$$\lim_{a\to\infty} \limsup_{N\to\infty} \mu_N\Big(\big|\mathcal{Z}_{2,y}(\sigma + imh) - \mathcal{Z}_{2,a,y}(\sigma + imh)\big| \geqslant \varepsilon\Big)$$

$$\leqslant \lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{m=0}^{\infty} \big|\mathcal{Z}_{2,y}(\sigma + imh) - \mathcal{Z}_{2,a,y}(\sigma + imh)\big| = 0.$$

Thus, denoting $X_{N,y}(\sigma) = \mathcal{Z}_{2,y}(\sigma + i\theta_N)$, we have the relation

$$\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big(\big|X_{N,y}(\sigma) - X_{N,a,y}(\sigma)\big|\Big) \geqslant \varepsilon) = 0.$$

This, (1), (2) and Theorem 4.2 of [1] prove the lemma.

LEMMA 7. *On $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure P_y such that the probability measure*

$$\mu_N\big(\mathcal{Z}_{2,y}(s + imh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_y as $N \to \infty$.

Proof. We apply the same way as in the proof Lemma 6, only in place of the Euclidean metric we use the metric which induces the topology of uniform convergence on compacta.

To prove Theorems 1 and 2 it remains to pass from function $\mathcal{Z}_{2,y}(s)$ to $\mathcal{Z}_2(s)$ and apply Lemmas 6 and 7. For this, we need the approximation in the mean of the function $\mathcal{Z}_2(s)$ by $\mathcal{Z}_{2,y}(s)$.

LEMMA 8. *Let K be a compact subset of the strip D. Then*

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s\in K} \big|\mathcal{Z}_2(s + imh) - \mathcal{Z}_{2,y}(s + imh)\big| = 0.$$

The proof of Lemma 8 is based on the mean square estimate

$$\int_0^T \big|\mathcal{Z}_2(\sigma + it)\big|^2 \,\mathrm{d}t = \mathrm{O}(T), \quad T \to \infty,$$

which is valid for $\frac{7}{8} < \sigma < 1$ and follows from [8]. This and the Gallagher lemma, see, for example, [7], for the same σ, imply the estimate

$$\sum_{m=0}^{N} \left|\mathcal{Z}_2(\sigma + imh)\right|^2 = \mathrm{O}(N), \quad N \to \infty.$$

Now the later bound and application of the contour integration complete the proof of the lemma.

Proof of Theorem 1. We argue similarly to the proof of Lemma 6. We start with the proof that the family of probability measures $\{P_{\sigma,y}\}$ is tight for a fixed σ. Hence it is relatively compact, and we obtain that on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ and $y_1 \to \infty$ such that

$$X_{y_1}(\sigma) \underset{y_1 \to \infty}{\overset{\mathcal{D}}{\longrightarrow}} P_\sigma. \tag{3}$$

Here $X_y(\sigma)$ is a complex-valued random element with the distribution $P_{\sigma,y}$ defined in Lemma 6. Moreover, in view of Lemma 6

$$X_N(\sigma) \underset{N \to \infty}{\overset{\mathcal{D}}{\longrightarrow}} P_\sigma, \tag{4}$$

where $X_{N,y}(\sigma)$ is defined in the proof of Lemma 6. Let $X_N(\sigma) = \mathcal{Z}_2(\sigma + i\theta_N)$. Then from Lemma 8 we deduce that, for $\sigma > \frac{1}{2}$ and every $\varepsilon > 0$,

$$\lim_{y \to \infty} \limsup_{N \to \infty} \mathbb{P}\Big(\left|X_N(\sigma) - X_{N,y}(\sigma)\right| \geqslant \varepsilon\Big) = 0.$$

This, (3), (4) and Theorem 4.2 of [1] prove the theorem.

Proof of Theorem 2 is similar to that of Theorem 1.

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REZIUMĖ

V. Balinskaitė. Diskrečios ribinės teoremos Rymano dzeta funkcijos ketvirtojo laipsnio Melino transformacijai

Gautos diskrečios ribinės teoremos silpnojo tikimybinių matų konvergavimo prasme Rymano dzeta funkcijos ketvirtojo laipsnio Melino transformacijai.