

On limit theorems of Fortet–Kac type

Birutė KRYŽIENĖ (VGTU), Gintautas MISEVIČIUS (VU)

e-mail: gintas.misevicius@mif.vu.lt

Abstract. We get the theorem of large deviations for sums of type $\sum f(T^j t)$ satisfying the conditions weaker than in [5] (see [5, pp. 221–227]).

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Let T be a mapping of $[0, 1]$ into itself defined by $T(t) = \{2t\}$ where the braces mean the fractional part of a number. Let the point $t \in [0, 1]$ be expressed by

$$t = \frac{\varepsilon_1(t)}{2} + \frac{\varepsilon_2(t)}{2^2} + \dots + \frac{\varepsilon_k(t)}{2^k} + \dots.$$

Then the coefficients $\varepsilon_i = \varepsilon_i(t)$, $i = 1, \dots, k$, are the so-called Rademacher functions, and are independent identically distributed random variables taking values 0 and 1 with equal probabilities.

This expression for irrational numbers is infinite, numbers $t \in (0, 1)$ are in a one-to-one correspondence with the sequences $\varepsilon_1(t), \varepsilon_2(t), \dots$, and every measurable function $f(t)$ can be represented in the form

$$f(t) = f(\varepsilon_1, \varepsilon_2, \dots).$$

So this function is a random variable, measurable with respect of the σ -algebra, generated by ε_j , $j = 1, 2, \dots$. The variables $f_k(T^k t) = f_k = f(2^k t) = f(\varepsilon_k, \varepsilon_{k+1}, \dots)$ form a stationary sequence. For details see [2].

Let $S_n = S_n(t) = \sum_{j=1}^n f(2^j t)$, $f(t)$ being a measurable periodic function with period 1, $\Phi(x)$ denote the normal distribution function.

We suppose that

$$\begin{aligned} \mathbf{E}f(t) &= \int_0^1 f(t) dt = 0, \quad \mathbf{E}|f(t)| \geq \delta > 0, \\ \mathbf{E}|f(t)|^k &= \int_0^1 |f(t)|^k dt \leq H_1 H_2^{k-2} (k-2)! \end{aligned} \tag{1}$$

and

$$\int_0^1 |f(t+h) - f(t)|^2 dt \leq H_3 h^\gamma, \tag{2}$$

$k = 3, 4, \dots, s + 2$, $\sqrt{s} \leq \frac{H_0 \sqrt{n}}{\rho(n) \ln n} \stackrel{\text{def}}{=} \Delta_n$, H_0, H_1, H_2, \dots are positive constants, c_1, c_2, \dots are constants, $\rho(n)$ is a slowly growing function,

$$B_n^2 = \mathbf{D}S_n = \int_0^1 S_n^2(t) dt = \sigma^2 n + c_1. \quad (3)$$

This paper deals with probabilities of large deviations similar to classical results (see [3,4]) for the above defined sums S_n with the condition (2) instead of the following condition in [5]:

$$\int_0^1 |f(t+h) - f(t)|^p dt \leq H_3^p h^\delta p!, \quad p = 2, 3, \dots$$

This leads to a narrower zone of validity of large deviations.

THEOREM. *Let $f(t)$ be the above defined Lebesgue measurable function satisfying conditions (1), (2) and $\sigma \neq 0$. Then for $1 \leq x \leq \sqrt{s}$ the following relations of large deviations hold*

$$\frac{\text{meas}_{t \in [0,1]} \{S_n > x B_n\}}{1 - \Phi(x)} = \exp\{L(x)\} \left(1 + \theta_1 f_1(x) \frac{x+1}{\sqrt{s}}\right),$$

$$\frac{\text{meas}_{t \in [0,1]} \{S_n < -x B_n\}}{\Phi(-x)} = \exp\{L(-x)\} \left(1 + \theta_2 f_2(x) \frac{x+1}{\sqrt{s}}\right),$$

where $f_1(x), f_2(x)$ are bounded functions, $|\theta_i| \leq 1$, $i = 1, 2$, the power series $L(x) = \sum l_k x^{k+3}$ converges for $|x| < \Delta_n$, the coefficients l_k , $k \geq 0$, can be expressed by cumulants, and for $k \leq s - 3$ these coefficients coincide with coefficients of the classical Cramér–Petrov series.

Some auxiliary statements. Following [1] let us introduce the conditional mean values, i.e., new random variables

$$\zeta_j^{(m)} = [f]_j^{(m)} = [f]_j^{(m)}(t) = \mathbf{E}(f(T^j t) | \varepsilon_{j+1}, \dots, \varepsilon_{j+m}) \quad (4)$$

and

$$\eta_j^{(m)} = \frac{\varepsilon_{j+1}}{2} + \dots + \frac{\varepsilon_{j+m}}{2^m}. \quad (5)$$

The function $[f]_j^{(k)}$ is measurable by the σ -algebra generated by $\varepsilon_1, \dots, \varepsilon_k$. It is evident that $\eta_j^{(m)}$ defines a unique set $\varepsilon_{j+1}, \dots, \varepsilon_{j+m}$, therefore the σ -algebra generated by random variables $\eta_{j_1}^{(m)}, \dots, \eta_{j_k}^{(m)}$ coincides with the σ -algebra generated by random variables $\varepsilon_{j_1}, \dots, \varepsilon_{j_k+m}$.

LEMMA 1 (see [5]). *The random variables $\eta_j^{(m)}$, $j = 1, 2, \dots$, make the Markov chain. Therefore the random variables $[f]_j^{(m)}$, $j = 1, 2, \dots$, are connected into the Markov chain.*

Let \mathcal{F}_a^b be the σ -algebra generated by $\{\eta_k^{(m)}, k = a, a + 1, \dots, b\}$ and Ω_k be the space of $\eta_k^{(m)}$ values. The coefficients of ergodicity in our case are

$$\alpha_{l,k}^{(m)} = \begin{cases} 0 & \text{for } k - l \leq m, \\ 1 & \text{for } k - l \geq m + 1, \end{cases}$$

since $\varepsilon_1, \varepsilon_2, \dots$ are independent identically distributed random variables.

Let us consider the “modified” sum

$$S'_n = \sum_{j=1}^n [f]_j^{(m)}.$$

LEMMA 2. *Under the conditions (1) and (2), the following relations are valid:*

$$B_n^2 = \mathbf{D}S_n = \sigma^2 n + \theta_n, \quad |\theta_n| < c_1, \quad (6)$$

$$B_n'^2 = \mathbf{D}S'_n = \sigma_m^2 n + \theta_m, \quad |\theta_m| < c_2, \quad (7)$$

and

$$|\sigma - \sigma_m| \leq \frac{c_3}{(\sqrt{2})^{m\gamma}}, \quad (8)$$

where

$$\sigma^2 = \int_0^1 f^2(t) dt + 2 \sum_{k=1}^{\infty} f(t) f(2^k t) dt \neq 0. \quad (9)$$

Proof. Following the reasoning of I.A. Ibragimov [1] we get the estimation

$$\mathbf{E}^{1/2} |\zeta_j - \zeta^{(m)}|^2 \leq c_4 2^{-m\gamma}, \quad (10)$$

and using the reasoning in §2 and §3 of Chapter XVIII in [2], we find

$$\mathbf{D}S_n = n \mathbf{E}\zeta_1^2 + \sum_{j=1}^n (n - j) \mathbf{E}\zeta_1 \zeta_j$$

with the estimation

$$|\mathbf{E}\zeta_1 \zeta_{k+1}| \leq 4 \cdot 2^{-\frac{\gamma}{2}k} c_\gamma \mathbf{E}^{1/2} \zeta_1^2. \quad (11)$$

An analogous estimation takes place for $\mathbf{D}S'_n$ and $|\mathbf{E}\zeta_1^{(m)} \zeta_j^{(m)}|$.

Thus we get

$$\mathbf{D}S_n = n \left(\mathbf{E}\zeta_1^2 + \sum_{j=1}^{\infty} \mathbf{E}\zeta_1 \zeta_j \right) + n \sum_{j=n+1}^{\infty} \mathbf{E}\zeta_1 \zeta_j - \sum_{j=1}^n \mathbf{E}\zeta_1 \zeta_j$$

and the second and third members do not exceed

$$\left| \frac{4 \cdot 2^{\frac{\gamma}{2}} c_{\gamma} \mathbf{E}^{1/2} \zeta_1^2}{2^{\frac{\gamma}{2} n}} \right| \leq c_5 \quad \text{and} \quad c_6 = \sum_{k=1}^{\infty} \frac{4 \cdot 2^{\frac{\gamma}{2}} c_{\gamma} \mathbf{E} \zeta_1^2}{2^{k \frac{\gamma}{2} - 1}} < \infty.$$

Thus we obtain (6), and in a similar manner one can get (7).

Further we get by the Minkovsky inequality

$$|\mathbf{E} \zeta_1^2 - \mathbf{E}(\zeta_1^{(m)})^2| \leq 8c_{\gamma} \mathbf{E} \zeta_1^2 2^{-\frac{\gamma}{2} m},$$

and

$$|\mathbf{E} \zeta_1 \zeta_{k+1} - \mathbf{E} \zeta_1^{(m)} \zeta_{k+1}^{(m)}| \leq \min \left(\frac{8 \cdot 2^{\frac{\gamma}{2}} c_{\gamma} \mathbf{E}^{1/2} \zeta_1^2}{k^{\frac{\gamma}{2}}}, \frac{8 \mathbf{E} \zeta_1^2 c_{\gamma}}{2^{\frac{\gamma}{2} m}} \right).$$

So we obtain

$$\begin{aligned} |\sigma - \sigma_m| &\leq |\mathbf{E} \zeta_1^2 - \mathbf{E} \zeta_1^{(m)2}| + \sum_{j=1}^m |\mathbf{E} \zeta_1 \zeta_j - \mathbf{E} \zeta_1^{(m)} \zeta_j^{(m)}| \\ &\quad + \sum_{j=m+1}^{\infty} |\mathbf{E} \zeta_1 \zeta_j - \mathbf{E} \zeta_1^{(m)} \zeta_j^{(m)}| \leq \frac{c_3}{2^{m \frac{\gamma}{2} - 1}}, \end{aligned}$$

where $c_3 = 9(2^{\frac{\gamma}{2}} c_{\gamma} \mathbf{E}^{1/2} \zeta_1^2 + c_{\gamma} \mathbf{E} \zeta_1^2)$.

Lemma is proved.

LEMMA 3 (see [3]). *Let a random variable ζ with $\mathbf{E} \zeta = 0$ and $\mathbf{E} \zeta^2 = 1$ satisfy the inequality*

$$\Gamma_k(\zeta) = (k-2)!/\Delta^{k-2}, \quad k = 3, 4, \dots, s+2,$$

where s is even and $s \leq 2\Delta^2$.

Then for x , $0 \leq x \leq \sqrt{s}/3\sqrt{e}$, the following relation of large deviations is valid:

$$\frac{1 - F_{\zeta}(x)}{1 - \Phi(x)} = \exp \{ \tilde{L}(x) \} \left(1 + \theta_s f_1(x) \frac{x+1}{\sqrt{s}} \right), \quad x > 0,$$

and a similar one for negative x .

The power series

$$\tilde{L}(x) = \sum_{k=0}^{\infty} \tilde{l}_k x^{k+3}$$

is convergent for $|x| < \sqrt{2}\Delta/3\sqrt{e}$ and $f_1(x)$ is a bounded function.

Proof of Theorem. Taking in (8) $m = \frac{\rho(n) \ln n}{\gamma}$ we get

$$|\sigma - \sigma_m| \leq \frac{c_7}{n^{\rho(n)}}.$$

By the conditions (1) and (2), using the Schwartz inequality we get

$$\int_0^1 |f(t+h) - f(t)| dt \leq \left(\int_0^1 |f(t+h) - f(t)|^2 dt \right)^{1/2} \\ \times \left(\int_0^1 f(t+h) - f(t) dt \right)^{1/2} \leq h^\alpha 2^k H_5^k k!,$$

and we deduce, using the Hölder and Jensen inequalities, that

$$|\mathbf{E}|f(T^j t)|^k - \mathbf{E}|\zeta_j^{(m)}|^k| \leq \frac{c_8}{n^{\rho(n)}}, \\ \mathbf{E}^{1/k}|[f_j]^{(m)}|^k \leq \mathbf{E}^{1/k}|f(t)|^k + \frac{H_1}{n^{\rho(n)}}, \\ \mathbf{E}|[f]_j^{(m)}| \leq \tilde{H}_{12} H_{13}^{k-2} k!, \quad j = 1, 2, \dots \quad (12)$$

Using (12) it is enough to get the estimate of cumulants of S'_n

$$\left| \Gamma_k \left(\frac{S'_n}{\sigma_m \sqrt{n}} \right) \right| \leq k! H_{14} \left(\frac{H_{15} \rho(n) \ln n}{\sigma_m \sqrt{n}} \right)^{k-2}.$$

Further conclusions coincide with those of [4], and the difference

$$\frac{S_n}{\sigma_m \sqrt{n}} - \frac{S'_n}{\sigma_m \sqrt{n}}$$

can be evaluated following [5] word by word.

Theorem is proved.

References

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REZIUMĖ

B. Kryžienė, G. Misevičius. Ribinės teoremos Forte–Kaco tipo sumoms

Darbe įrodyta didžiųjų nuokrypių ribinė teorema sumoms $\sum f(T^j t)$ su silpnesniais apribojimais periodinei funkcijai $f(t)$ negu ankstesniuose autorių darbuose (žr. [5]).

Raktiniai žodžiai: dvimatis toras, didieji nuokrypiai, semiinvariantai.