

p -variation of Ornstein–Uhlenbeck type processes*

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Abstract. The p -variation of sample paths of Ornstein–Uhlenbeck type processes is investigated. It is shown that the p -variation index of such a process is the same as the p -variation index of the driving Lévy process, provided this process is of unbounded total variation.

Keywords: Ornstein–Uhlenbeck type process, Lévy process, p -variation.

1. Introduction

Assume that $X = \{X_t, t \geq 0\}$ is a real valued Lévy process, i.e., a process with stationary and independent increments, which starts at the origin at $t = 0$ and has almost all càdlàg trajectories. Fix a constant $\lambda > 0$ and consider the following stochastic differential equation

$$dY_u = -\lambda Y_u du + dX_u, \quad u \geq 0, \quad (1)$$

which can be understood in the sense of semimartingales (see [6]), since such are Lévy processes. Multiplying both sides of (1) by $e^{\lambda u}$ and integrating from 0 to t , we easily obtain the solution as

$$Y_t = e^{-\lambda t} Y_0 + e^{-\lambda t} \int_0^t e^{\lambda u} dX_u, \quad (2)$$

which is called an Ornstein–Uhlenbeck type (OU-type) process, generated by X (see [7, §17], [2, §15.3]). The stochastic integral in (2) can also be understood *pathwise* (see [1, Thm. 7.14 and Prop. 3.9.1]) in the refinement–Riemann–Stieltjes sense (see [4, p. 2]) and thus allows applications of several results related to the path properties of Y_t . In fact, we will show that the p -variation properties of Y_t are the same as for X_t , provided X_t is of unbounded total variation.

2. Preliminaries and results

Throughout we will fix a $T > 0$ and will omit it in the notations, since the value of T will be unimportant. We will say that a process $X_s, 0 \leq s \leq T$ (or, in particular, a function $f: [0, T] \rightarrow \mathbb{R}$) has a finite p -variation index $v(X)$ almost surely if

$$v(X) = \inf \{p > 0: v_p(X) < \infty \text{ almost surely}\} \quad (3)$$

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is finite, where

$$v_p(X) = \sup \left\{ \sum_{k=0}^{m-1} |X_{t_{k+1}} - X_{t_k}|^p : 0 = t_0 < t_1 < \dots < t_m = T, m = 1, 2, \dots \right\}$$

is the p -variation of X . Whenever $p = 1$, we have the usual total variation.

Here is the main result of this paper.

THEOREM 1. *Let $X_t, t \geq 0$ be a real-valued Lévy process of almost surely unbounded total variation. Then the p -variation index $v(Y)$ of the corresponding OU-type process $Y_t, t \geq 0$ is equal to $v(X)$.*

3. Proofs

We begin by proving two auxiliary lemmas for real valued functions. Later we will apply them to the sample paths of an OU-type process.

LEMMA 2. *Let $f: [a, b] \rightarrow \mathbb{R}$ ($a < b$) be a function of bounded total variation. Then for any function $g: [a, b] \rightarrow \mathbb{R}$ with a finite p -variation index $v(g)$ we have $v(f + g) \vee 1 = v(g) \vee 1$, where $x \vee y = \max\{x, y\}$.*

Proof. First consider the case $v(g) \leq 1$. Clearly, $v(g) \vee 1 = 1 \leq v(f + g) \vee 1$. Moreover, for any $p > 1$ we have $v_p(g) < +\infty$ and

$$v_p^{1/p}(f + g) \leq v_p^{1/p}(f) + v_p^{1/p}(g) < +\infty, \quad (4)$$

since $v_p^{1/p}(f) \leq v_1(f) < +\infty$ for any $p \geq 1$. And so $v(f + g) \leq p$. Letting $p \downarrow 1$ we obtain $v(f + g) \leq 1$ and $v(f + g) \vee 1 = 1$.

In the case $v(g) > 1$ we have $v(g) \vee 1 = v(g)$ and for any $p > v(g)$ by (4) we obtain $v(f + g) \leq p$. Now taking $p = p_n = v(g) + 1/n$ and letting $n \rightarrow \infty$ we get $v(f + g) \leq v(g)$. If $v(f + g) = v(g)$ we are done; otherwise, there exists a $p \in [1 \vee v(f + g), v(g))$ such that $v_p(f + g) < +\infty$. Then $g = (f + g) - f$ and

$$v_p^{1/p}(g) \leq v_p^{1/p}(f + g) + v_p^{1/p}(f) < +\infty,$$

a contradiction, since $v_p(g) = +\infty$. Therefore, $1 \vee v(f + g) \geq v(g)$, and the proof is completed.

Remark 3. It is obvious that the maximum with 1 cannot be omitted. As an example, take $f(x) = x$, $g(x) = 1 - x, x \in [0, 1]$. Then $v(f) = v(g) = 1$ but $v(f + g) = 0$, since $f + g \equiv 1$.

Let $\|f\|_p := v_p^{1/p}(f) + \|f\|_\infty, p \in [1, +\infty)$.

LEMMA 4. *Let $f: [a, b] \rightarrow \mathbb{R}$ ($a < b$) be a function of bounded total variation. Then for any function $g: [a, b] \rightarrow \mathbb{R}$ with the p -variation index $v(g) \geq 1$ we have*

$v(fg) \leq v(g)$. If, in addition, g is of unbounded total variation, i.e., $v_1(g) = +\infty$, and the function $1/f$ is well-defined and of bounded total variation, then $v_1(fg) = +\infty$ and $v(fg) = v(g)$.

Proof. Pick any $\varepsilon > 0$ and consider any $p \in (v(g), v(g) + \varepsilon)$. Then, by definition of $v(g)$, we get $v_p(g) < +\infty$. Since $p \geq 1$, applying a result of Krabbe [5] we obtain

$$\|fg\|_{[p]} \leq \|f\|_{[p]} \|g\|_{[p]} < +\infty,$$

and so $v_p(fg) < +\infty$. Moreover, $v(fg) \leq p < v(g) + \varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain $v(fg) \leq v(g)$.

Now suppose that $v_1(g) = +\infty$ and $v_1(1/f) < +\infty$. If the function fg were of bounded total variation or of bounded p -variation for some $p < v(g)$ then, applying the above mentioned result of Krabbe, we would get

$$\|g\|_{[q]} \leq \|1/f\|_{[q]} \|fg\|_{[q]} < +\infty,$$

where we take $q = 1$ in the former case and $q = p$ in the latter case. But this is a contradiction in either case, since $v_1(g) = +\infty$ (applied in case $v(g) = 1$), and $v_q(g) = +\infty$ for any $q < v(g)$ (applied in case $1 < v(g)$). Therefore, $v(fg) \geq 1 = v(g)$ in case $v(g) = 1$, and $v(fg) \geq p$ for any $p < v(g)$ in case $v(g) > 1$. So either way we get $v(fg) \geq v(g)$, which completes the proof.

Remark 5. Without the assumption $v_1(g) = +\infty$ the equality $v(fg) = v(g)$ would no longer hold, in general, as one can take, for example, $f(x) = e^{-x}$, $g(x) = e^x$ for any $x \in [1, 2]$, so that $fg \equiv 1$ and $v(fg) = 0 < v(f) = v(g) = 1$.

We now proceed to the proof of the main theorem.

Proof of Theorem 1. Consider a Lévy process X_t of almost surely unbounded total variation and a corresponding OU-type process Y_t which we can write as

$$Y_t = e^{-\lambda t} X_0 + e^{-\lambda t} \int_0^t e^{\lambda u} dX_u = Z_t^{(1)} + e^{-\lambda t} Z_t^{(2)}.$$

Clearly, the function $e^{-\lambda t}$ and the process $Z_t^{(1)}$ are of bounded total variation on any finite interval $[a, b] \subset [0, \infty)$. So in order to show that $v(Y) = v(X)$ almost surely, by Lemmas 2 and 4, it suffices to show that $1 \leq v(X) = v(Z^{(2)})$ almost surely.

Since the function $h(u) = e^{\lambda u}$, $u \in [0, T]$, is a continuous function of bounded total variation and the process X_u , $u \in [0, T]$, being a Lévy process is a semimartingale of bounded p -variation for any $p > 2$, by [4, Thm. 4.26] or [3, Thm. II.3.27] we get that the integral defining $Z_t^{(2)}$ exists in the Riemann–Stieltjes sense and by [4, Cor. 4.28] or [3, Prop. II.3.32] defines a function of bounded p -variation for any $p > v(X) \geq 1$. Hence, the inequality $v(Z^{(2)}) \leq v(X)$ holds almost surely.

To show that $v(Z^{(2)}) \geq v(X)$ consider any $p \geq 1$ such that $v_p(X) = +\infty$ almost surely. At least one such p exists, since X is assumed to be of unbounded total variation. For any $\omega \in \Omega$ such that $v_p(X(\omega)) = +\infty$ consider a sequence of

partitions of $[0, t]$, say $\{\pi_n\}_{n=1}^\infty$, such that $v_p(X(\omega), \pi_n) \uparrow \infty$, as $n \rightarrow \infty$. Denote $\pi_n = \{t_i^n: 0 = t_0^n < t_1^n < \dots < t_{m_n}^n = t\}$ and $\Delta_{i,n}f = f(t_i^n) - f(t_{i-1}^n)$ so that

$$v_p(X(\omega), \pi_n) = \sum_{i=1}^{m_n} |\Delta_{i,n}X(\omega)|^p$$

and

$$v_p(Z^{(2)}(\omega), \pi_n) = \sum_{i=1}^{m_n} |\Delta_{i,n}Z^{(2)}(\omega)|^p = \sum_{i=1}^{m_n} \left| \int_{t_{i-1}^n}^{t_i^n} e^{\lambda u} dX_u(\omega) \right|^p.$$

Integration by parts (see [3, Thm. I.4.8]) yields:

$$\begin{aligned} \Delta_{i,n}Z^{(2)}(\omega) &= e^{\lambda u} X_u(\omega) \Big|_{u=t_{i-1}^n}^{u=t_i^n} - \int_{t_{i-1}^n}^{t_i^n} X_u(\omega) \lambda e^{\lambda u} du \\ &= e^{\lambda t_i^n} \Delta_{i,n}X(\omega) + X_{t_{i-1}^n}(\omega) \Delta_{i,n}h - \int_{t_{i-1}^n}^{t_i^n} X_u(\omega) \lambda e^{\lambda u} du, \end{aligned}$$

where $h(x) = e^{\lambda x}$. By the triangle inequality,

$$\begin{aligned} |\Delta_{i,n}Z^{(2)}(\omega)| &\geq e^{\lambda t_i^n} |\Delta_{i,n}X(\omega)| - |\Delta_{i,n}h| |X_{t_{i-1}^n}(\omega)| \\ &\quad - \lambda(t_i^n - t_{i-1}^n) \sup_{t_{i-1}^n \leq u \leq t_i^n} e^{\lambda u} |X_u(\omega)| \\ &\geq |\Delta_{i,n}X(\omega)| - 2\lambda e^{\lambda t_i^n} (t_i^n - t_{i-1}^n) \sup_{t_{i-1}^n \leq u \leq t_i^n} |X_u(\omega)|, \end{aligned}$$

since $|\Delta_{i,n}h| \leq \lambda(t_i^n - t_{i-1}^n) \sup_{t_{i-1}^n \leq u \leq t_i^n} e^{\lambda u}$. Letting $M(\omega) = \sup_{0 \leq u \leq t} |X_u(\omega)|$, which is finite almost surely, since $X_u, u \in [0, t]$ is regulated, we get, for $p \geq 1$ as above,

$$\begin{aligned} \sum_{i=1}^{m_n} |\Delta_{i,n}X(\omega)|^p &\leq \sum_{i=1}^{m_n} \left(|\Delta_{i,n}Z^{(2)}(\omega)| + 2\lambda e^{\lambda t} (t_i^n - t_{i-1}^n) M(\omega) \right)^p \\ &\leq 2^{p-1} \sum_{i=1}^{m_n} \left(|\Delta_{i,n}Z^{(2)}(\omega)|^p + (2\lambda e^{\lambda t} (t_i^n - t_{i-1}^n) M(\omega))^p \right). \end{aligned}$$

Since $\sum_{i=1}^{m_n} (t_i^n - t_{i-1}^n)^p \leq \max_i (t_i^n - t_{i-1}^n)^{p-1} \sum_{i=1}^{m_n} (t_i^n - t_{i-1}^n) \leq t^p$, and $v_p(X(\omega), \pi_n) \uparrow +\infty$, we obtain $v_p(Z^{(2)}(\omega), \pi_n) \uparrow +\infty$, as $n \rightarrow \infty$. Thus, $v_p(Z^{(2)}) = +\infty$, so long as $v_p(X) = +\infty$. This implies $v(Z^{(2)}) \geq v(X)$ almost surely.

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REZIUMĖ

M. Manstavičius. Ornšteino–Ulenbeko tipo procesų *p*-variacija

Straipsnyje nagrinėjama Ornšteino–Ulenbeko tipo procesų trajektorijų *p*-variacija. Įrodyta, kad bet kurio tokio proceso *Y* *p*-variacijos indeksas $v(Y)$ sutampa su ši procesą generuojančio Levy proceso *X* *p*-variacijos indeksu $v(X)$, jei *X* trajektorijos yra beveik tikrai neapbrėžtos pilnosios variacijos.