

The existence and uniqueness of the solution of the integral equation driven by a bounded p -variation function

Kęstutis KUBILIUS (MII, VGTU)

e-mail: kubilius@ktl.mii.lt

1. Introduction

In this note we examine the nonlinear integral equation

$$x_t = \alpha + \int_{t_0}^t f(s, x_s) ds + \int_{t_0}^t g(s, x_s) dh_s, \quad t_0 \leq t \leq T, \quad (1)$$

where h is a continuous function of bounded p -variation for some p , $1 < p < 2$, i.e. $h \in CW_p([t_0, T])$. The first integral in (1) is the Riemann integral and the second is the Riemann-Stieltjes (RS) integral which exist for functions f and g considered below.

Lyons [3] considered the integral equation

$$y_t = a + \int_0^t \sum_{i=1}^d f^i(y_s) dh_s^i, \quad (2)$$

where h^i , $i = 1, \dots, d$, are continuous functions which have bounded p -variation for some p , $1 \leq p < 2$. He proved that the equation (2) can be solved by Picard iteration whenever f^i has a derivative $(f^i)'$ satisfying a global Lipschitz condition of order α , where $p < 1 + \alpha \leq 2$. This solution is unique in the space of continuous functions of bounded p -variation. We extend this result to the equation (1) in the case $d = 1$.

Denote by $C^{0,1}([t_0, T] \times \mathbf{R})$ the space of all continuous functions $u(t, x)$ on $[t_0, T] \times \mathbf{R}$ such that the norm

$$\|u\|_T = \sup_{s, x} |u(s, x)| + \sup_{s, x} |\partial_x u(s, x)|.$$

is finite.

Let $H^{\ell/2, \ell}([t_0, T] \times \mathbf{R})$, $\ell = 1 + \alpha$, $0 < \alpha < 1$, be the space of all continuous functions u on $[t_0, T] \times \mathbf{R}$ possessing continuous partial derivative $\partial_x u$ such that the norm

$$\|u\|_T^{(\ell)} = \|u\|_T + \sup_{\substack{(s, x), (s, y) \\ x \neq y}} \frac{|\partial_x u(s, x) - \partial_x u(s, y)|}{|x - y|^\alpha}$$

$$+ \sup_{\substack{(s,x),(t,x) \\ s \neq t}} \frac{|u(s,x) - u(t,x)|}{|s-t|^{(1+\alpha)/2}} + \sup_{\substack{(s,x),(t,x) \\ s \neq t}} \frac{|\partial_x u(s,x) - \partial_x u(t,x)|}{|s-t|^{\alpha/2}}$$

is finite.

Our main result is the following:

Theorem. Let $T > 0$ and let $h \in CW_p([t_0, T])$, $1 < p < 2$, $f \in C^{0,1}([t_0, T] \times \mathbf{R})$, $g \in H^{\ell/2, \ell}([t_0, T] \times \mathbf{R})$, $\ell = 1 + \alpha$, $2(1 - 1/p) < \alpha < 1$. Then the equation (1) has a unique solution in $CW_p([t_0, T])$.

Note. If the functions f and g don't depend on the time variable t then the condition $\alpha > 2(1 - 1/p)$ can be replaced by the condition $\alpha + 1 > p$ which yields Lyons' result in the case $d = 1$.

Further we list few facts concerning the p -variation. The proofs can be found in [1] and [4].

Let f be a real-valued function defined on a closed interval $[a, b]$. Let $Q([a, b])$ be the set of all partitions $\kappa = \{x_i: i = 0, \dots, n\}$ of $[a, b]$ such that $a = x_0 < x_1 < \dots < x_n = b$. The p -variation, $0 < p < \infty$, on $[a, b]$ of f is defined by

$$v_p(f) := v_p(f; [a, b]) := \sup \{s_p(f; \kappa): \kappa \in Q([a, b])\},$$

where $s_p(f; \kappa) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p$ for $\kappa = \{x_i: i = 0, \dots, n\}$. If $v_p(f) < \infty$, f is said to have a bounded p -variation on $[a, b]$. If f is a Hölder function with exponent $0 < \alpha \leq 1$, then it has a bounded $1/\alpha$ -variation.

Denote by $\mathcal{W}_p([a, b])$ the class of all functions defined on $[a, b]$ with a bounded p -variation, that is

$$\mathcal{W}_p([a, b]) := \{f: [a, b] \rightarrow \mathbf{R}: v_p(f; [a, b]) < \infty\}.$$

Define $V_p(f) := V_p(f; [a, b]) = v_p^{1/p}(f)$ which is 0 if and only if f is a constant. If $p \geq 1$ and $f, g \in \mathcal{W}_p$ then

$$V_p(f + g) \leq V_p(f) + V_p(g). \quad (3)$$

For each f , $V_p(f)$ is a non-increasing function of p , i.e., if $q < p$ then $V_p(f) \leq V_q(f)$. This follows from the inequality

$$\left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^q \right)^{1/q} \quad (4)$$

valid for $0 < q < p < \infty$ and any $\{a_k\}$.

Let $p \geq 1$ and

$$V_{p,\infty}(f) = V_{p,\infty}(f; [a, b]) = V_p(f; [a, b]) + |f|_{\infty, [a, b]},$$

where $|f|_{\infty, [a, b]} = \sup_{a \leq x \leq b} |f(x)|$. Then $V_{p, \infty}(f)$ is the norm, and $\mathcal{W}_p([a, b])$ equipped with the p -variation norm is a Banach space.

Let $f \in CW_p([a, b])$, $1 < p < \infty$, and $\varepsilon > 0$. Then there exists $\{x_i: i = 0, \dots, n\} \in Q([a, b])$ such that $\max_{1 \leq i \leq n} v_p(f; [x_{i-1}, x_i]) < \varepsilon$ ([1], Lemma 2.20, p.94).

The existence of the RS integral in (1) follows by the Love–Young inequality. Now we formulate it. Let $f \in \mathcal{W}_q([a, b])$ and $h \in \mathcal{W}_p([a, b])$ with $p \geq 1$, $q \geq 1$, $1/p + 1/q > 1$. If f and h have no common discontinuities then the Riemann–Stieltjes integral $\int_a^b f dh$ exists and the inequality

$$\left| \int_a^b f dh - f(\xi)[h(b) - h(a)] \right| \leq C_{p,q} V_q(f; [a, b]) V_p(h; [a, b]), \quad (5)$$

holds for any $\xi \in [a, b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n \geq 1} n^{-s}$.

Let $f \in \mathcal{W}_q([a, b])$ and $h \in CW_p([a, b])$. From (5) it follows that

$$V_p\left(\int_a^\cdot f dh; [a, b]\right) \leq [C_{p,q} V_q(f; [a, b]) + |f|_{\infty, [a, b]}] V_p(h; [a, b]). \quad (6)$$

If the function $h \in CW_p([a, b])$ then the indefinite integral $\int_a^y f dh$, $y \in [a, b]$, is a continuous function ([2], Lemma 3.23, p. 124).

2. Proof

Lemma 1 (see [2]). Let $g \in H^{\ell/2, \ell}([a, b] \times \mathbf{R})$. Then for any $a < b$ and $q = 2/\alpha$, $0 < \alpha < 1$, $p \geq 1$ and $x, y \in \mathcal{W}_q$

$$\begin{aligned} V_q(g(\cdot, x) - g(\cdot, y); [a, b]) &\leq \|g\|_T^{(\ell)} V_q(x - y; [a, b]) \\ &\quad + \|g\|_T^{(\ell)} |x - y|_{\infty, [a, b]} \left((b - a)^{\alpha/2} + V_2^\alpha(y; [a, b]) \right). \end{aligned}$$

Proof. By the mean value theorem, we have

$$\begin{aligned} &[g(t, x_t) - g(t, y_t)] - [g(s, x_s) - g(s, y_s)] \\ &= [g(t, y_t + (x_t - y_t)) - g(t, y_t + (x_s - y_s))] \\ &\quad + \left\{ [g(t, y_t + (x_s - y_s)) - g(t, y_t)] - [g(s, y_t + (x_s - y_s)) - g(s, y_t)] \right\} \\ &\quad + \left\{ [g(s, y_t + (x_s - y_s)) - g(s, y_t)] - [g(s, y_s + (x_s - y_s)) - g(s, y_s)] \right\} \\ &= [g(t, y_t + (x_t - y_t)) - g(t, y_t + (x_s - y_s))] \\ &\quad + \int_0^{x_s - y_s} [\partial_x g(t, y_t + u) - \partial_x g(s, y_t + u)] du \end{aligned}$$

$$+ \int_0^{x_s - y_s} [\partial_x g(s, y_t + v) - \partial_x g(s, y_s + v)] dv$$

for any $a \leq s < t \leq b$. Therefore

$$\begin{aligned} & \left| [g(t, x_t) - g(t, y_t)] - [g(s, x_s) - g(s, y_s)] \right| \\ & \leq |\partial_x g|_\infty |(x_t - y_t) - (x_s - y_s)| + \|g\|_T^{(\ell)} |x_s - y_s| [(t - s)^{\alpha/2} + |y_t - y_s|^\alpha]. \end{aligned}$$

Now it is easy to finish the proof of the lemma.

Next we construct the Picard iteration

$$y_m(t) = \alpha + \int_{t_0}^t f(s, y_{m-1}(s)) ds + \int_{t_0}^t g(s, y_{m-1}(s)) dh_s, \quad m \geq 1$$

and $y_0(t) = \alpha$, for the equation (1).

Lemma 2. Let $2(1 - 1/p) < \alpha < 1$ and $q = 2/\alpha$. If for some $t > t_0$

$$V_p(h; [t_0, t]) < (2 \cdot C_{p,q} |\partial_x g|_{\infty, [t_0, T]})^{-1} \quad (7)$$

then for each $m \geq 0$,

$$\begin{aligned} V_p(y_{m+1}; [t_0, t]) & < 2 \left\{ |f|_{\infty, [t_0, t]} (t - t_0) + [C_{p,q} \|g\|_T^{(\ell)} (t - t_0)^{(1+\alpha)/2} \right. \\ & \quad \left. + |g|_{\infty, [t_0, t]} V_p(h; [t_0, t]) \right\}. \end{aligned}$$

Proof. For any $t_0 \leq a < b \leq T$, by properties of the seminorm V_p and by inequality (6), we get

$$\begin{aligned} & V_p(y_{m+1}; [a, b]) \\ & \leq V_1 \left(\int_{t_0}^{\cdot} f(s, y_m(s)) ds; [a, b] \right) + V_p \left(\int_{t_0}^{\cdot} g(s, y_m(s)) dh_s; [a, b] \right) \\ & \leq |f|_{\infty, [a, b]} (b - a) + [C_{p,q} V_q(g(\cdot, y_m); [a, b]) + |g|_{\infty, [a, b]}] V_p(h; [a, b]) \\ & \leq |f|_{\infty, [a, b]} (b - a) + \left\{ C_{p,q} \left[\|g\|_T^{(\ell)} (b - a)^{(1+\alpha)/2} \right. \right. \\ & \quad \left. \left. + |\partial_x g|_{\infty, [a, b]} V_p(y_m; [a, b]) \right] + |g|_{\infty} \right\} V_p(h; [a, b]). \end{aligned} \quad (8)$$

Denote

$$A = |f|_{\infty, [t_0, t]} (t - t_0) + [C_{p,q} \|g\|_T^{(\ell)} (t - t_0)^{(1+\alpha)/2} + |g|_{\infty, [t_0, t]}] V_p(h; [t_0, t]).$$

By inequalities (7) and (8) we get

$$\begin{aligned} V_p(y_{m+1}; [t_0, t]) &< \frac{1}{2} V_p(y_m; [t_0, t]) + A \\ &< \frac{1}{2^{m+1}} V_p(y_0; [t_0, t]) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^m}\right) A < 2A \end{aligned}$$

since $V_p(y_0; [t_0, t]) = 0$. The proof of the lemma is complete.

As a consequence of Lemma 2 we get that $y_m \in \mathcal{W}_p([t_0, T])$ for each $m \geq 1$. Moreover $y_m \in C\mathcal{W}_p([t_0, T])$ since $h \in C\mathcal{W}_p([t_0, T])$.

Let $z_m(t) := y_m(t) - y_{m-1}(t)$, $m \geq 1$. Now we estimate $V_p(z_m; [t_0, t])$. Put $q = 2/\alpha$. Similarly as in Lemma 2 we get

$$\begin{aligned} V_p(z_m; [t_0, t]) &\leq V_1 \left(\int_{t_0}^{\cdot} [f(s, y_{m-1}(s)) - f(s, y_{m-2}(s))] ds; [t_0, t] \right) \\ &\quad + V_p \left(\int_{t_0}^{\cdot} [g(s, y_{m-1}(s)) - g(s, y_{m-2}(s))] dh_s; [t_0, t] \right) \\ &\leq |\partial_x f|_{\infty, [t_0, t]} (t - t_0) |y_{m-1}(s) - y_{m-2}(s)|_{\infty, [t_0, t]} \\ &\quad + \left\{ C_{p,q} V_q(g(\cdot, y_{m-1}) - g(\cdot, y_{m-2}); [t_0, t]) \right. \\ &\quad \left. + |g(\cdot, y_{m-1}) - g(\cdot, y_{m-2})|_{\infty, [t_0, t]} \right\} V_p(h; [t_0, t]). \end{aligned}$$

By Lemma 1 and inequality

$$\sup_{t_0 \leq s \leq t} |y_{m-1}(s) - y_{m-2}(s)| \leq V_p(z_{m-1}; [t_0, t])$$

we get

$$\begin{aligned} V_p(z_m; [t_0, t]) &\leq \left[|\partial_x f|_{\infty, [t_0, t]} + (C_{p,q} + 1) \|g\|_T^{(\ell)} \right. \\ &\quad \left. + C_{p,q} \|g\|_T^{(\ell)} ((t - t_0)^{\alpha/2} + V_p^\alpha(y_{m-2}; [t_0, t])) \right] \\ &\quad \times V_p(z_{m-1}; [t_0, t]) \cdot \max \left\{ (t - t_0), V_p(h; [t_0, t]) \right\}. \end{aligned}$$

Denote

$$\begin{aligned} K_t &= |\partial_x f|_{\infty, [t_0, t]} + (C_{p,q} + 1) \|g\|_T^{(\ell)} \\ &\quad + C_{p,q} \|g\|_T^{(\ell)} \left\{ (t - t_0)^{\alpha/2} + 2^\alpha [|f|_{\infty, [t_0, t]} (t - t_0) \right. \\ &\quad \left. + (C_{p,q} \|g\|_T^{(\ell)} (t - t_0)^{(1+\alpha)/2} + |g|_{\infty, [t_0, t]}) V_p(h; [t_0, t])]^\alpha \right\}. \end{aligned}$$

Let $\tau > t_0$ be such that (7) holds. Then for $t \in [t_0, \tau]$ by Lemma 2 we get

$$V_p(z_m; [t_0, t]) < K_t V_p(z_{m-1}; [t_0, t]) \cdot \max \left\{ (t - t_0), V_p(h; [t_0, t]) \right\}.$$

Now we take $t_1 \leq \tau$ such that

$$\max \left\{ (t_1 - t_0), V_p(h; [t_0, t_1]) \right\} < \frac{1}{2} K_\tau^{-1}.$$

Then for $t \in [t_0, t_1]$

$$\begin{aligned} V_p(z_m; [t_0, t]) &< \frac{1}{2} V_p(z_{m-1}; [t_0, t]) < \frac{1}{2^{m-1}} V_p(z_1; [t_0, t]) \\ &= \frac{1}{2^{m-1}} V_p(y_1; [t_0, t]). \end{aligned}$$

Thus the series $\sum_{k=1}^{\infty} V_p(z_k; [t_0, t_1])$ converges. Now by the routine arguments one can prove that there is an element $\hat{y} \in C\mathcal{W}_p([t_0, t_1])$ such that $\hat{y}$ is the solution of the equation (1) in the interval $[t_0, t_1]$.

Further we can repeat the whole construction for the equation

$$\tilde{y}(t) = \hat{y}(t_1) + \int_{t_1}^t f(s, \tilde{y}(s)) ds + \int_{t_1}^t g(s, \tilde{y}(s)) dh_s, \quad t \in [t_1, T],$$

and prove the existence of the solution in some interval $[t_1, t_2]$. Then the function

$$\bar{y}(t) = \begin{cases} \hat{y}(t), & \text{for } t \in [t_0, t_1], \\ \tilde{y}(t), & \text{for } t \in [t_1, t_2] \end{cases}$$

is the solution of the equation (1) in $[t_0, t_2]$. After finitely number of steps we get the solution y on the whole interval $[t_0, T]$ and $y \in C\mathcal{W}_p([t_0, T])$.

Now we prove the uniqueness of the solution of equation (1). Let z be another solution of (1). Similarly to the estimate of $V_p(z_m; [t_0, t])$, we get

$$\begin{aligned} V_p(y - z; [t_0, t]) &\leq \left[|\partial_x f|_{\infty, [t_0, t]} + (C_{p,p} + 1) \|g\|_T^{(\ell)} \right. \\ &\quad \left. + C_{p,q} \|g\|_T^{(\ell)} ((t - t_0)^{\alpha/2} + V_p^\alpha(y; [t_0, t])) \right] \\ &\quad \times V_p(y - z; [t_0, t]) \cdot \max \left\{ (t - t_0), V_{p,\infty}(h; [t_0, t]) \right\}. \end{aligned}$$

Similarly to above we get

$$\begin{aligned} |y - z|_{\infty, [t_0, t]} &\leq \left[|\partial_x f|_{\infty, [t_0, t]} + (C_{p,q} + 1) \|g\|_T^{(\ell)} + C_{p,q} \|g\|_T^{(\ell)} ((t - t_0)^{\alpha/2} \right. \\ &\quad \left. + V_p^\alpha(y; [t_0, t])) \right] V_p(y - z; [t_0, t]) \cdot \max \left\{ (t - t_0), V_{p,\infty}(h; [t_0, t]) \right\}. \end{aligned}$$

Therefore one can find a $\delta > 0$ such that

$$V_{p,\infty}(y - z; [t_0, t]) < \frac{1}{2} V_{p,\infty}(y - z; [t_0, t])$$

for $t \in [t_0, t_0 + \delta]$. From the last inequality we get $V_{p,\infty}(y - z; [t_0, t_0 + \delta]) = 0$. Since $V_{p,\infty}$ is the norm then $y(t) - z(t) = 0$ on the interval $[t_0, t_0 + \delta]$. After finitely many steps we get $y(t) = z(t)$ on $[t_0, T]$. The proof of the Theorem is complete.

References

- [1] R.M. Dudley, R. Norvaiša, Differentiability of six operators on nonsmooth functions and p -variation, *Lecture Notes Math.*, Springer, vol. 1703 (1999).
- [2] R.M. Dudley, Picard iteration and p -variation: the work of Lyons (1994), *Mini-proceedings: Workshop on Product Integrals and Pathwise Integration*, MaPhySto (1999).
- [3] T. Lyons, Differential equations driven by rough signals (I): An extension of an inequality of L.C. Young, *Mathematical Research Letters*, **1**, 451–464 (1994).
- [4] L.C. Young, An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.*, **67**, 251–282 (1936).

Integralinės lygties generuotos funkcijos, turinčios baigtinę p -variaciją, sprendinio egzistavimas ir vienatis

K. Kubilius

Tarkime, kad funkcija $f(t, x)$ apibrėžta aibėje $[t_0, T] \times \mathbf{R}$ yra aprėžta ir turi aprėžtą dalinę išvestinę x atžvilgiu, o $g \in H^{\ell/2, \ell}([t_0, T] \times \mathbf{R})$, $p < \alpha/2 + 1 < 2$. Įrodyta, kad lygtis

$$x_t = \alpha + \int_{t_0}^t f(s, x_s) ds + \int_{t_0}^t g(s, x_s) dh_s, \quad t_0 \leq t \leq T,$$

čia h yra tolydi funkcija turinti baigtinę p -variaciją tam tikram p , $1 < p < 2$, turi vienintelį sprendinį tolydžių baigtinės p -variacijos funkcijų klasėje.