

A two-dimensional limit discrete theorem for Mellin transforms of the Riemann zeta-function

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Abstract. In the paper two-dimensional limit theorem for the modified Mellin transform of the Riemann zeta-function is obtained.

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Let $\zeta(s)$, $s = \sigma + it$, as usual, denote the Riemann zeta-function. The modified Mellin transforms $\mathcal{Z}_k(s)$ of powers $|\zeta(\frac{1}{2} + it)|^{2k}$, $k \geq 0$, are defined, for $\sigma \geq \sigma_0(k) > 1$, by

$$\mathcal{Z}_k(s) = \int_1^\infty \left| \zeta\left(\frac{1}{2} + ix\right) \right|^{2k} x^{-s} dx.$$

In [2] and [3], discrete limit theorems on the complex plane for $\mathcal{Z}_1(s)$ and $\mathcal{Z}_2(s)$, respectively, were proved. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S , and, for $N \in \mathbb{N} \cup \{0\}$, put

$$\mu_N(\dots) = \frac{1}{N+1} \sum_{\substack{0 \leq m \leq N \\ \dots}} 1,$$

where in place of dots a condition satisfied by m is to be written. Let $h > 0$ be a fixed number.

THEOREM 1 [2]. *Let $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\mu_N(\mathcal{Z}_1(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $N \rightarrow \infty$.

THEOREM 2 [3]. *Let $\frac{7}{8} < \sigma < 1$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\mu_N(\mathcal{Z}_2(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $N \rightarrow \infty$.

The aim of this note is a two-dimensional limit discrete theorem for the functions $\mathcal{Z}_1(s)$ and $\mathcal{Z}_2(s)$. Define

$$P_{N,\sigma_1,\sigma_2} = \mu_N((\mathcal{Z}_1(\sigma_1 + imh), \mathcal{Z}_2(\sigma_2 + imh)) \in A), \quad A \in \mathcal{B}(\mathbb{C}^2).$$

THEOREM 3. *Suppose that $\sigma_1 > \frac{1}{2}$ and $\frac{7}{8} < \sigma_2 < 1$. Then on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ there exists a probability measure P_{σ_1,σ_2} such that the measure P_{N,σ_1,σ_2} converges weakly to P_{σ_1,σ_2} as $N \rightarrow \infty$.*

Let $a > 1$ be a fixed number, for $y \geq 1$, $\sigma_0 > \frac{1}{2}$, $v(x, y) = \exp\{-\left(\frac{x}{y}\right)^{\sigma_0}\}$, and

$$\mathcal{Z}_{k,a,y}(s) = \int_1^a \left| \zeta\left(\frac{1}{2} + ix\right) \right|^{2k} x^{-s} v(x, y) dx, \quad k = 1, 2.$$

We begin the proof of Theorem 3 with a limit theorem for the vector

$$\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, t) = (\mathcal{Z}_{1,a,y}(\sigma_1 + it), \mathcal{Z}_{2,a,y}(\sigma_2 + it)).$$

For this aim, we apply a limit theorem on the torus

$$\Omega_a = \prod_{u \in [1, a]} \gamma_u,$$

where $\gamma_u = \{s \in \mathbb{C} : |s| = 1\} \stackrel{\text{def}}{=} \gamma$ for all $u \in [1, a]$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω_a is a compact topological Abelian group. On $(\Omega_a, \mathcal{B}(\Omega_a))$, define the probability measure

$$Q_{N,a}(A) = \mu_N((u^{imh} : u \in [1, a]) \in A).$$

LEMMA 4. *On $(\Omega_a, \mathcal{B}(\Omega_a))$, there exists a probability measure Q_a such that the probability measure $Q_{N,a}$ converges weakly to Q_a as $N \rightarrow \infty$.*

Proof of the lemma is given in [3], Theorem 5.

Now let

$$P_{N,a,y,\sigma_1,\sigma_2}(A) = \mu_N(\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh) \in A), \quad A \in \mathcal{B}(\mathbb{C}^2).$$

THEOREM 5. *On $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$, there exists a probability measure $P_{a,y,\sigma_1,\sigma_2}$ such that the probability measure $P_{N,a,y,\sigma_1,\sigma_2}$ converges weakly to $P_{a,y,\sigma_1,\sigma_2}$ as $N \rightarrow \infty$.*

Proof. Define a function $h_{a,y} : \Omega_a \rightarrow \mathbb{C}^2$ by the formula

$$h_{a,y}(\{y_x : x \in [1, a]\}) = \left(\int_1^a \left| \zeta\left(\frac{1}{2} + ix\right) \right|^2 x^{-\sigma_1} v(x, y) \widehat{y}_x^{-1} dx, \right. \\ \left. \int_1^a \left| \zeta\left(\frac{1}{2} + ix\right) \right|^4 x^{-\sigma_2} v(x, y) \widehat{y}_x^{-1} dx \right),$$

where

$$\widehat{y}_x = \begin{cases} y_x & \text{if } y_x \text{ is integrable over } [1, a], \\ \text{an arbitrary integrable over } [1, a] \text{ circle function} & \text{otherwise.} \end{cases}$$

Then the function $h_{a,y}$ is continuous, and

$$\begin{aligned} & h_{a,y}(\{x^{imh}: x \in [1, a]\}) \\ &= \left(\int_1^a \left| \zeta \left(\frac{1}{2} + ix \right) \right|^2 x^{-(\sigma_1 + imh)} v(x, y) dx, \int_1^a \left| \zeta \left(\frac{1}{2} + ix \right) \right|^4 x^{-(\sigma_2 + imh)} v(x, y) dx \right) \\ &= \underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh). \end{aligned}$$

Therefore, the theorem is a consequence of Lemma 4 and Theorem 5.1 of [1].

By [2], [3], the integrals

$$\mathcal{Z}_{k,y}(s) = \int_1^\infty \left| \zeta \left(\frac{1}{2} + ix \right) \right|^{2k} x^{-s} v(x, y) dx, \quad k = 1, 2,$$

converges absolutely for $\sigma > \frac{1}{2}$ and $\sigma > \frac{7}{8}$, respectively.

Let

$$\underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, t) = (\mathcal{Z}_{1,y}(\sigma_1 + it), \mathcal{Z}_{2,y}(\sigma_2 + it)),$$

and

$$P_{N,y,\sigma_1,\sigma_2}(A) = \mu_N(\underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, mh) \in A), \quad A \in \mathcal{B}(\mathbb{C}^2).$$

THEOREM 6. *Let $\sigma_1 > \frac{1}{2}$ and $\frac{7}{8} < \sigma_2 < 1$. Then on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ there exists a probability measure P_{y,σ_1,σ_2} such that the probability measure $P_{N,y,\sigma_1,\sigma_2}$ converges weakly to P_{y,σ_1,σ_2} as $N \rightarrow \infty$.*

Proof. Let $M > 0$ be arbitrary number. Then we have that

$$\begin{aligned} & \limsup_{N \rightarrow \infty} P_{N,a,y,\sigma_1,\sigma_2}(\{|\underline{z}| \in \mathbb{C}^2: |\underline{z}| > M\}) \\ &= \limsup_{N \rightarrow \infty} \mu_N(|\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh)| > M) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{M(N+1)} \sum_{m=0}^N |\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh)| \\ &\leq \frac{1}{M} \sup_{a \geq 1} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \left(\sum_{k=1}^2 |\mathcal{Z}_{k,a,y}(\sigma_k + imh)|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{M} \sup_{a \geq 1} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \left(\sum_{m=0}^N \sum_{k=1}^2 |\mathcal{Z}_{k,a,y}(\sigma_k + imh)|^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \frac{1}{M} \sum_{k=1}^2 |\mathcal{Z}_{k,y}(\sigma_k)|^2 \leq \frac{R}{M}.$$

Hence, taking $M = R\varepsilon^{-1}$, we find that

$$\limsup_{N \rightarrow \infty} P_{N,a,y,\sigma_1,\sigma_2}(\{\underline{z} \in \mathbb{C}^2: |\underline{z}| > M\}) \leq \varepsilon.$$

Therefore, we obtain that the family of probability measures $\{P_{a,y,\sigma_1,\sigma_2}: a \geq 1\}$ is tight, and relatively compact. Thus, there exists a subsequence $\{P_{a_1,y,\sigma_1,\sigma_2}\} \subset \{P_{a,y,\sigma_1,\sigma_2}\}$ such that $P_{a_1,y,\sigma_1,\sigma_2}$ converges weakly to some measure P_{y,σ_1,σ_2} as $a_1 \rightarrow \infty$.

On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, define a random variable θ_N by

$$\mathbb{P}(\theta_N = hm) = \frac{1}{N+1}, \quad m = 0, 1, \dots, N,$$

and put

$$\underline{X}_{N,a,y}(\sigma_1, \sigma_2) = \underline{Z}_{a,y}(\sigma_1, \sigma_2, \theta_N).$$

Then by Theorem 5,

$$\underline{X}_{N,a,y}(\sigma_1, \sigma_2) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \underline{X}_{a,y}(\sigma_1, \sigma_2), \quad (1)$$

where $\underline{X}_{a,y}(\sigma_1, \sigma_2)$ is a \mathbb{C}^2 -valued random element with the distribution $P_{a,y,\sigma_1,\sigma_2}$. Moreover, from above we have that

$$\underline{X}_{a_1,y}(\sigma_1, \sigma_2) \xrightarrow[a_1 \rightarrow \infty]{\mathcal{D}} P_{y,\sigma_1,\sigma_2}. \quad (2)$$

Denoting by ρ the metric on \mathbb{C}^2 , we obtain that

$$\begin{aligned} & \lim_{a \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \rho(\underline{Z}_{a,y}(\sigma_1, \sigma_2, mh), \underline{Z}_y(\sigma_1, \sigma_2, mh)) \\ & \leq \lim_{a \rightarrow \infty} \limsup_{N \rightarrow \infty} \sum_{k=1}^2 \frac{1}{N+1} \sum_{m=0}^N |\mathcal{Z}_{k,a,y}(\sigma_k + imh) - \mathcal{Z}_{k,y}(\sigma_k + imh)| = 0. \end{aligned} \quad (3)$$

Now let $\underline{X}_{N,y}(\sigma_1, \sigma_2) = \underline{Z}_y(\sigma_1, \sigma_2, \theta_N)$. Then, in view of (3), for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{a \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}(\rho(\underline{X}_{N,a,y}(\sigma_1, \sigma_2), \underline{X}_{N,y}(\sigma_1, \sigma_2)) \geq \varepsilon) \\ & = \lim_{a \rightarrow \infty} \limsup_{N \rightarrow \infty} \mu_N(\rho(\underline{Z}_{a,y}(\sigma_1, \sigma_2, mh), \underline{Z}_y(\sigma_1, \sigma_2, mh)) \geq \varepsilon) = 0. \end{aligned}$$

This, (1), (2) and Theorem 4.2 of [1] prove the theorem.

Proof of Theorem 3. In view of Theorem 6, it remains to pass from the vector $\underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, mh)$ to

$$\underline{\mathcal{Z}}(\sigma_1, \sigma_2, mh) = (\mathcal{Z}_1(\sigma_1 + imh), \mathcal{Z}_2(\sigma_2 + imh)).$$

In [2] it was proved that, for $\sigma > \frac{1}{2}$,

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N |\mathcal{Z}_1(\sigma + imh) - \mathcal{Z}_{1,y}(\sigma + imh)| = 0,$$

and in [3] it was obtained that, for $\sigma > \frac{7}{8}$,

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N |\mathcal{Z}_2(\sigma + imh) - \mathcal{Z}_{2,y}(\sigma + imh)| = 0.$$

Hence, it follows that

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \rho(\underline{\mathcal{Z}}(\sigma_1, \sigma_2, mh), \underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, mh)) = 0.$$

Therefore, putting

$$\underline{X}_N(\sigma_1, \sigma_2) = \underline{\mathcal{Z}}(\sigma_1, \sigma_2, \theta_N),$$

we derive that, for every $\varepsilon > 0$,

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}(\rho(\underline{X}_{N,y}(\sigma_1, \sigma_2), \underline{X}_N(\sigma_1, \sigma_2)) \geq \varepsilon) = 0. \quad (4)$$

By Theorem 6, we have that

$$\underline{X}_{N,y}(\sigma_1, \sigma_2) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \underline{X}_y(\sigma_1, \sigma_2), \quad (5)$$

where $\underline{X}_y(\sigma_1, \sigma_2)$ is a \mathbb{C}^2 -valued random element with the distribution P_{y,σ_1,σ_2} . Similarly, as in the proof of Theorem 6, we find that the family of probability measures $\{P_{y,\sigma_1,\sigma_2} : y \geq 1\}$ is tight. Hence, it is relatively compact. Therefore, there exists a subsequence $\{P_{y_1,\sigma_1,\sigma_2}\} \subset \{P_{y,\sigma_1,\sigma_2}\}$ such that $P_{y_1,\sigma_1,\sigma_2}$ converges weakly to some probability measure P_{σ_1,σ_2} as $y_1 \rightarrow \infty$. Hence

$$\underline{X}_{y_1}(\sigma_1, \sigma_2) \xrightarrow[y_1 \rightarrow \infty]{\mathcal{D}} P_{\sigma_1,\sigma_2}.$$

This, (4), (5) and Theorem 4.2 of [1] again show that

$$\underline{X}_N(\sigma_1, \sigma_2) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} P_{\sigma_1,\sigma_2}$$

and the theorem is proved.

References

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REZIUOMĖ

V. Balinskaitė. Dvimatė diskreti ribinė teorema Rymano dzeta funkcijos Melino transformacijoms

Įrodyta dvimatė diskreti ribinė teorema Rymano dzeta funkcijos antrojo ir ketvirtojo laipsnio Melino transformacijoms.