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On recursive stopping of decimation of discrete-time bandlimited signals

Rimantas Pupeikis[1,2]

[1] *Institute of Mathematics and Informatics, Vilnius University*
Akademijos 4, LT-08663 Vilnius
[2] *Faculty of Electronics, Vilnius Gediminas Technical University*
Naugarduko 41, LT-03227 Vilnius

E-mail: rimantas.pupeikis@mii.vu.lt

Abstract. For each non-decimated as well as decimated realization discrete-time Fourier series coefficient values, located at Nyquist frequency are calculated, using original speedy recursive expressions based on reverse order processing of the given realizations. The criterion for stopping of multifold decimation of discrete-time bandlimited signals has been developed. The simulation results for the bandlimited signal with a triangularshaped spectrum are presented.

Keywords: Digital signal processing (DSP), Fourier coefficients, Nyquist frequency, decimation, discrete-time signals, stopping.

1 Introduction

The sampling operation of an analogous signal $X_a(t)$ with a sampling frequency F_s significantly higher than twice the highest signal's frequency B is frequently used while processing $X_a(t)$ digitally [2]. There are some reasons for performing such an oversampling. One of the main ones of them is a less complex and inexpensive anti-aliasing filter: a signal can be filtered digitally, and afterwards, downsampled to the desired sampling frequency by reducing a large digital data set considerably. A time-scaling operation is used here that is equivalent to changing the sampling rate of an analogous signal $U(t)$ from $1/T_s$ to $1/PT_s$, where T_s is a sampling period and its reciprocal $1/T_s = F_s$, i.e., decreasing the sampling rate by factor P. On the other hand, the number of samples N to be processed decreases P times, as well. In general, the basic sampling frequency F_s could be decreased by varying the integer number of times if not the fact that the data decimation process ought to be finished before the frequency content of the downsampled signal is above the new Nyquist frequency [2]. In the opposite case, the spectrum $X(F/F_s)$ of the discrete-time signal $X_a(kT_s)\ \forall k \in \overline{0, \mathrm{N}-1}$ contains aliased frequency components of the spectrum $X_a(F)$ of an analogous signal $X_a(t)$, because the downsampling in the frequency domain leads to the spread of signal's spectrum by the same factor P [2]. That is why there arises a question: how much can we downsample the discrete-time bandlimited signals in order to reduce the amount of samples to be processed, without loss of information.

2 Statement of the problem

We consider a discrete-time bandlimited signal $X_a(kT_s)$ $\forall k \in \overline{0, \mathrm{N}-1}$ that is obtained by uniform sampling with the sampling frequency F_s its continuous-time counterpart $X_a(t)$ having a bandwidth $[-B, B]$ [3, 4]. Here N, divisible n times by 2, i.e., $\mathrm{N} = 2^n$, is the general number of samples of the basic signal $X_a(kT_s)$ $\forall k \in \overline{0, \mathrm{N}-1}$ under consideration. The maximal frequencies of some realizations, obtained after repeated decimating, could be higher than the varying new Nyquist frequencies leading to overlapping of respective signal frequencies when the spectrum replicates. Thus, the set $\mathbf{\Omega}$ of awailable realizations could be subdivided in turn, into two subsets: a subset of non-aliased realizations $\mathbf{\Omega}_1$, and a subset of aliased ones $\mathbf{\Omega}_2$. The last realization of subset $\mathbf{\Omega}_1$ is the very last non-aliased realization, after which there follows the first aliased one from the subset $\mathbf{\Omega}_2$.

The aim of the given paper is to choose some criterion that could be used to recognize the very last downsampled realization in the given set $\mathbf{\Omega}$ with the maximal frequency that is still below the new Nyquist frequency.

3 Recursive expressions

For a discrete-time bandlimited signal, Fourier series coefficients are nonzero inside the band $[-B, B]$, and zero outside the same band [2]. Therefore, the values of coefficients appearing far from the zero value for frequencies outside the bandwidth of some decimated realization could show us that it is time to finish the decimation process. Calculations could be significantly reduced if for each realization from the set $\mathbf{\Omega}$, only the values of the Fourier coefficient $A(2^{(n-1)})$, located at Nyquist frequency $F_s/2$, which corresponds to the normalized frequency π, were calculated. Let us formulate now the corollary on calculation of $A(2^{(n-1)})$, assuming, for simplicity, that the basic T_s for each decimated realization is increased by 2 times.

Corollary 1. *The value of the coefficient $A(2^{(n-i)})$ is calculated in reverse order, using the recursive expression of the form*

$$A\big(2^{(n-i)}\big) = \frac{1}{2^{(n-i+1)}}\Bigg\{2^{(n-i)}A\big(2^{(n-i-1)}\big) + 2\sum_{k=1}^{2^{(n-i-1)}} x\big(2^{(i)}(2k-1)\big) - \sum_{k=1}^{2^{(n-i)}} x\big(2^{(i-1)}(2k-1)\big)\Bigg\}. \tag{1}$$

Here $A(2^{(n-i)})$, $A(2^{(n-i-1)})$ $\forall i \in \overline{1, n-2}$ are current and previous values of the Fourier coefficient.

Proof of Corollary 1. For the initial non-decimated realization $x(k) \equiv x(kT_s)$ $\forall k \in \overline{0, 2^n-1}$, the Fourier coefficient could be calculated by

$$A(2^{(n-1)}) = \frac{1}{2^n}\sum_{k=0}^{2^{\mathrm{n}}-1}(-1)^k x(k) = \frac{1}{2^n}\Bigg\{\sum_{k=0}^{2^{(n-1)}-1} x(2k) - \sum_{k=1}^{2^{(n-1)}} x(2k-1)\Bigg\}. \tag{2}$$

Then, for the first and for the second downsampled realization the coefficients $A(2^{(n-2)})$, $A(2^{(n-3)})$ are of the form

$$A\big(2^{(n-2)}\big) = \frac{1}{2^{(n-1)}} \sum_{k=0}^{2^{(n-1)}-1} (-1)^k x(2k), \qquad A\big(2^{(n-3)}\big) = \frac{1}{2^{(n-2)}} \sum_{k=0}^{2^{(n-2)}-1} (-1)^k x(4k), \tag{3}$$

respectively. Continuing the procedure, one could obtain the formulas:

$$\ldots, A\big(2^2\big) = \frac{1}{2^3} \sum_{k=0}^{2^3-1} (-1)^k x\big(2^{(n-3)}k\big), \qquad A(2) = \frac{1}{2^2} \sum_{k=0}^{2^2-1} (-1)^k x\big(2^{(n-2)}k\big). \tag{4}$$

Expressions can be rewritten in reverse order as follows:

$$A\big(2^2\big) = \frac{1}{2^3}\left\{2^2 A(2) + 2\sum_{k=1}^{2} x\big(2^{(n-2)}(2k-1)\big) - \sum_{k=1}^{2^2} x\big(2^{(n-3)}(2k-1)\big)\right\}, \tag{5}$$

$$A\big(2^3\big) = \frac{1}{2^4}\left\{2^3 A\big(2^2\big) + 2\sum_{k=1}^{2^2} x\big(2^{(n-3)}(2k-1)\big) - \sum_{k=1}^{2^3} x\big(2^{(n-4)}(2k-1)\big)\right\}, \ldots,$$
$$A\big(2^{(n-2)}\big) = \frac{1}{2^{(n-1)}}\left\{2^{(n-2)} A\big(2^{(n-3)}\big) + 2\sum_{k=1}^{2^{(n-3)}} x\big(2^2(2k-1)\big) - \sum_{k=1}^{2^{(n-2)}} x\big(2(2k-1)\big)\right\}. \tag{6}$$

Finally, equation (2) can also be rewritten in a recursive form:

$$A\big(2^{(n-1)}\big) = \frac{1}{2^n}\left\{2^{(n-1)} A\big(2^{(n-2)}\big) + 2\sum_{k=1}^{2^{(n-2)}} x\big(2(2k-1)\big) - \sum_{k=1}^{2^{(n-1)}} x(2k-1)\right\}. \tag{7}$$

Thus, the general expression for calculating $A(2^{(n-i)})\ \forall i \in \overline{1, n-2}$ is of the form (1). □

Now let us choose the value of the form

$$c_i = \frac{\|A(2^{(n-i)}) - A(2^{(n-i-1)})\|_E^2}{\|A(2^{(n-i-1)})\|_E^2} 100\% \quad \forall i \in \overline{1, n-2} \tag{8}$$

as the criterion of recognition of the very last non-aliased realization assuming that the values of $A(2^{(n-i)})$, $A(2^{(n-i-1)})\ \forall i \in \overline{1, n-2}$ will be calculated recursively in reverse order, i.e., first, we calculate $A(2^{(n-i-1)})$, afterwards, $A(2^{(n-i)})$. We finish the decimation when the criterion achieves approximately 100%.

In a theoretical discussion of sampling theory, it is usual to represent the signal of interest with a triangularshaped Fourier spectrum [1]. Using the sinc() function in MATLAB, the signal $U(kT_s)\ \forall k \in \overline{0, \mathrm{N}-1}$, that for $T_s = 1$ is of the form

$$U(k) = \frac{1}{4}\,\mathrm{sinc}\left(\frac{1}{4}(k-512)\right)^2 \quad \forall k \in \overline{0, 2^{10}-1}, \tag{9}$$

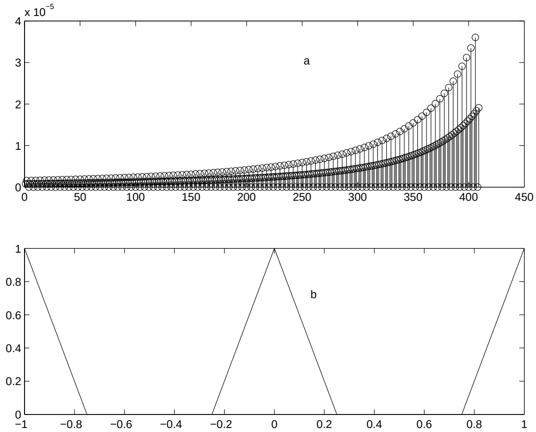

Fig. 1. A short segment of the simulated initial realization (9) to be processed (a) depending on the number of observations, and its unit height spectrum (b) depending on the normalized frequency $\omega/2\pi$.

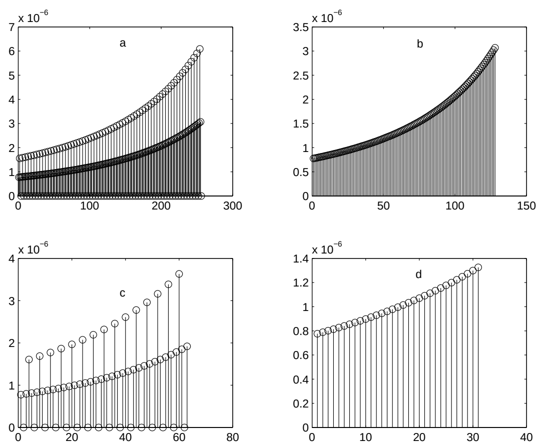

Fig. 2. A set of realizations depending on the number of observations: initial (9), (a) initial realization downsampled by 2 (b), by 3 (c), and by 4 samples (d).

has been generated (Fig. 1(a), (b)). The initial realization of signal (9) was decimated by 2, 4, 8, 16 samples. Decimated realizations are shown in Fig. 2 while their spectras are given in Fig. 3. Note that Fig. 2(a), (b), (c), (d) corresponds to Fig. 3(a), (b), (c), (d), respectively. The values of the recognition criterion (8) have been calculated recursively in reverse order, as follows: $c_4 = 0.01\%$, $c_3 = 14.92\%$, $c_2 = 99.99\%$, and $c_1 = 2.70\%$. Thus, with a decrease in the sampling rate F_s by $P > 2$ there appears aliasing of frequencies (see Fig. 3(c), (d)) because values of corresponding Fourier coefficients already are not equal zero. Therefore, the decimation of the signal (9) ought to be finished before the F_s is decreased more than twice.

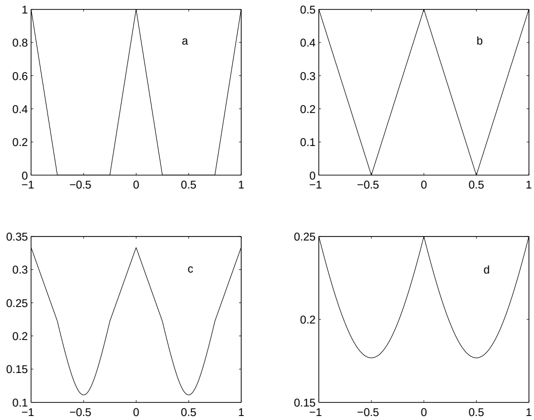

Fig. 3. Spectras of the initial realization (a), and decimated versions (b, c, d) depending on the frequency $\omega/2\pi$. In such a case Nyquist frequency is equal 0.5.

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REZIUMĖ

Apie rekurentinį baigtinės dažnių juostos diskrečiojo laiko signalų decimacijos stabdymą

R. Pupeikis

Straipsnyje nagrinėjamas daugkartinės baigtinės dažnių juostos diskrečiojo laiko signalo decimacijos rekurentinio stabdymo uždavinys. Pateikti signalo su trikampiu spektru modeliavimo bei jo decimuotos realizacijos, nepersiklojusios dažnių srityje, rekurentinio išrinkimo rezultatai.

Raktiniai žodžiai: Skaitmeninis signalų apdorojimas, Furje koeficientai, Naikvisto dažnis, decimacija, diskrečiojo laiko signalai.