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Note on arithmetical functions and multiples

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Abstract. The existence of the logarithmic and number-theoretic densities of some sets related to arithmetical functions is investigated. The Dirichlet convolution is used for the representation of these functions.

Keywords: arithmetical functions, Dirichlet convolution, multiples.

The set of arithmetical functions $\mathcal{A} = \{f : f : \mathbb{N} \to \mathbb{R}\}$ with the Dirichlet convolution

$$(f * g)(n) = \sum_{d|n} f(n) g\Big(\frac{n}{d}\Big)$$

is a ring of functions with the unity element $e(n)$, where $e(1) = 1$ and $e(n) = 0$ if $n > 1$. We denote as usual by $\mu(n)$ the Möbius function, and by $\omega(n), \Omega(n)$ the numbers of primes dividing n counted without and with multiplicity. We use the concepts of additive and multiplicative functions in the usual number-theoretic sense.

An arbitrary arithmetical function f can be viewed as a result of convolution of some arithmetical function w and the constant function $I(n) = 1$:

$$f(n) = (w * I)(n) = \sum_{d|n} w(d), \quad w(d) = (f * \mu)(d).$$

We use this representation as generic and write $f(n) = f(n|w)$. It is our aim to investigate some relations between the conditions set on w and properties of $f(n|w)$.

It is easy to find out which functions $w(n)$ generate additive or multiplicative functions $f(n|w)$.

THEOREM 1. *The function $f(n|w)$ is multiplicative if and only if $w(n)$ is multiplicative.*

The function $f(n|w)$ is additive if and only if $w(n) = 0$ for all n with the condition $\omega(n) \neq 1$.

Proof. The first statement can be found in most textbooks of number theory.

Let us prove the second statement. It is obvious, that the conditions on w imply additivity of $f(n|w)$. We prove that these conditions are necessary. It can be done easily by induction over the values of $\Omega(n)$. Obviously, $f(1) = w(1) = 0$. If $\Omega(n) = \omega(n) = 2$, then $n = pq$, where p, q are both primes. If $f(n|w)$ is additive, then

$$f(n|w) = f(p|w) + f(q|w) = w(p) + w(q) = w(p) + w(q) + w(pq),$$

and $w(n) = 0$ follows. Let the statement be true for all n with the condition $2 \leqslant \omega(n) \leqslant \Omega(n) \leqslant m$. Let for some n, $\omega(n) \geqslant 2$, $\Omega(n) = m+1$. Then $n = n'p^a$, where p is prime and $(n', p) = 1$. We have

$$f(n|w) = f(n') + f(p^a) = \sum_{d'|n'} w(d') + \sum_{b \leqslant a} w(p^b) + \sum_{\substack{\delta|n', \delta>1 \\ 1 \leqslant b \leqslant a}} w(\delta p^b).$$

The last sum is zero and for each δp^b, except for the largest $\delta p^b = n'p^a$, the condition $2 \leqslant \omega(\delta p^b) \leqslant \Omega(\delta p^b) \leqslant m$ is satisfied. Hence $w(\delta p^b) = 0$, and $w(n) = w(n'p^a) = 0$. The theorem is proved.

For an arbitrary subset of natural numbers $A \subset \mathbb{N}$ we denote the set of multiples

$$\mathcal{M}(A) = \bigcup_{a \in A} \{n \in \mathbb{N}: n \equiv 0 \pmod a\}.$$

If $w(d) \in \{0, 1\}$ and $A_w = \{d: w(d) = 1\}$, then $f(n|w) > 0$ holds only if $n \in \mathcal{M}(A_w)$. The value of $f(n|w)$, if $f(n|w) > 0$, can be interpreted as the „weight" of the multiple n in the obvious sense.

We introduce two systems of densities. If $A \subset \mathbb{N}$ and $x > 1$, let us denote

$$\nu_x\{A\} = \frac{\#(A \cap (0, x])}{\lfloor x \rfloor}, \quad \lambda_x\{A\} = L^{-1} \sum_{\substack{n \in A \\ n \leqslant x}} \frac{1}{n}, \quad L = \sum_{n \leqslant x} \frac{1}{n}.$$

We denote the lower and the upper limits of $\nu_x\{A\}, \lambda_x\{A\}$, as $x \to \infty$, by $\underline{\nu}\{A\}, \overline{\nu}\{A\}$, $\underline{\lambda}\{A\}, \overline{\lambda}\{A\}$, respectively. It is well known that for all subsets $A \subset \mathbb{N}$

$$\underline{\nu}\{A\} \leqslant \underline{\lambda}\{A\} \leqslant \overline{\lambda}\{A\} \leqslant \overline{\nu}\{A\}.$$

If $\underline{\nu}\{A\} = \overline{\nu}\{A\}$, we denote this value by $\nu\{A\}$ and say that A possess the number-theoretic density. If $\underline{\lambda}\{A\} = \overline{\lambda}\{A\} = \lambda\{A\}$, we say that A has the logaritmic density.

We are going to prove some facts about the existence of densities for the sets $\{n: f(n|w) \geqslant z\}$.

THEOREM 2. *If the function $w(n)$ satisfies*

$$\sum_{w(d) \neq 0} \frac{1}{d} < \infty,$$

then for any z the density $\nu\{n: f(n|w) \geqslant z\}$ exists.

Proof. Let $d_1 < d_2 < \ldots$ be the sequence of all numbers with the property $w(d) \neq 0$. Let $\epsilon > 0$ and N be some number such that

$$\sum_{j > N} \frac{1}{d_j} \leqslant \epsilon.$$

Define a function w^* taking $w^*(d) = w(d)$, if $d = d_j$ with $j \leqslant N$, and $w^*(d) = 0$ otherwise. Then

$$\overline{\nu}\{n\colon f(n|w) \neq f(n|w^*)\} \leqslant \sum_{j>N} \frac{1}{d_j} \leqslant \epsilon,$$

and

$$\underline{\nu}\{f(n|w^*) \geqslant z\} - \epsilon \leqslant \underline{\nu}\{f(n|w) \geqslant z\} \leqslant \overline{\nu}\{f(n|w) \geqslant z\} \leqslant \overline{\nu}\{f(n|w^*) \geqslant z\} + \epsilon.$$

Hence, it suffices to show the existence of $\nu\{f(n|w^*) \geqslant z\}$.

Let $\mathbb{D} = \{d_1, d_2, \ldots, d_N\}$. For each non-empty subset $D \subset \mathbb{D}$ we denote by $m(D)$ the least common multiple of numbers in D. The numbers $m(D_1), m(D_2)$ indexed by different subsets D_1, D_2 are not necessarily different. We avoid repetitions in the following way: if a is some number in the sequence, find all numbers $m(D_j) = a$ and remove them, except the number indexed by $\cup D_j$, i. e. leave the number $a = m(\cup D_j)$. Let $\mathbb{M}$ be the set of all remaining (different) numbers with $M = m(\mathbb{D})$ the largest of them.

If $n \equiv m \pmod{M}$ with some $m \in \mathbb{M}$, then $f(n|w^*) = f(m|w^*)$; if $m \notin \mathbb{M}$, then $f(n|w^*) = 0$. Hence for all values a the densities $\nu\{f(n|w^*) = a\}$ exist, and this suffices for the proof.

We turn now to the question of existence $\lambda\{f(n|w) \geqslant z\}$. We use in our reasoning the fact established by Erdös and Davenport: for any subset $A \subset \mathbb{N}$ the set of multiples $\mathcal{M}(A)$ has the logarithmic density (see [3]; [4] Th. 12, p.258; [5] Th. 02, p.5).

THEOREM 3. *If the function $w(n)$ satisfies*

$$\sum_{w(d)<0} \frac{1}{d} < \infty,$$

then for any z the density $\lambda\{n\colon f(n|w) \geqslant z\}$ exists.

Proof. As in the proof of the previous theorem we reduce the proof to the case of function w with the finite number of d satisfying $w(d) < 0$. Let $\mathbb{D}$ be the set of all d such that $w(d) < 0$. We repeat all the arguments of the proof of the previous theorem leading from the set $\mathbb{D}$ to the set of different multiples $\mathbb{M}$ and $M = m(\mathbb{D})$.

Let $w_+(d) = \max\{w(d), 0\}$, $w_-(d) = \min\{w(d), 0\}$. Then

$$f(n|w) = f(n|w_-) + f(n|w_+).$$

The function $f(n|w_+)$ has a nice property: for every u

$$\mathcal{M}(\{n\colon f(n|w_+) \geqslant u\}) = \{n\colon f(n|w_+) \geqslant u\}.$$

Hence we get from the Erdös-Davenport result that $\lambda\{n\colon f(n|w_+) \geqslant u\}$ exists.

Note, that if $m \in \mathbb{M}$ and $n \equiv m \pmod{M}$, then

$$f(n|w) = f(n|w_-) + f(n|w_+) = f(m|w_-) + f(m|w_+), \quad f(m|w_-) < 0. \tag{1}$$

If $m \notin \mathbb{M}$, then (1) holds with $f(m|w_-) = 0$, too. This gives a chance to split the set $\{n\colon f(n|w) \geqslant z\}$ into disjunctive parts:

$$\{n\colon f(n|w) \geqslant z\} = \bigcup_{m=0}^{M-1} \{n\colon n \equiv m \pmod M),\ f(n|w_+) \geqslant z - f(m|w_-)\}.$$

We conclude the proof using the following helpful fact: if $A \subset \mathbb{N}$ and q, Q are some natural numbers, then the logarithmic density

$$\lambda\{n\colon n \equiv q \pmod Q),\ n \in \mathcal{M}(A)\}$$

exists. In the case $(q, Q) = 1$ it is proved in [5] (Lemma 1.17, p.61). To show, that it holds as $(q, Q) > 1$, is easy. Observe now that with $u = z - f(m|w_-)$ in the definition of the set

$$\{n\colon n \equiv m \pmod M),\ f(n|w_+) \geqslant u\},$$

the condition $f(n|w_+) \geqslant u$ can be replaced by $n \in \mathcal{M}(\{n\colon f(n|w_+) \geqslant u\})$; hence this set has the logarithmic density. The proof is complete.

Now we look for an example of function such that for $A_z = \{n\colon f(n|w) \geqslant z\}$ the density $\lambda\{A_z\}$ exists, but $\overline{\nu}\{A_z\} - \underline{\nu}\{A_z\} > 0$ for each z. In the construction of such function we use the following result of Erdös ([2]):

$$\nu\{\mathcal{M}([T; 2T))\} \to 0, \quad T \to \infty, \tag{2}$$

here $[T; 2T)$ means the set of natural numbers in this interval. The existence of densities in (2) can be proved using the combinatorial including-excluding principle, which works because of finitness of $[T; 2T)$.

Let $k \geqslant 1$ be some natural number. We have, obviously, that

$$\mathcal{M}([T; 2^k T)) = \bigcup_{j=0}^{k-1} \mathcal{M}([2^j T; 2^{j+1} T)).$$

It follows then from (2) that

$$\nu\{\mathcal{M}([T; 2^k T))\} \to 0, \quad T \to \infty. \tag{3}$$

THEOREM 4. *Let $c > 0$ and $0 < \delta < 1$ be some real numbers. There exists some function $f(n|w)$ such that for all $z \geqslant c$ the densities $\lambda\{n\colon f(n|w) \geqslant z\}$ exist, and $\overline{\nu}\{n\colon f(n|w) \geqslant z\} - \underline{\nu}\{n\colon f(n|w) \geqslant z\} \geqslant \delta$.*

Proof. Let k be some natural number such that $1 - 2^{-k} \geqslant (1+\delta)/2$ and $\epsilon = (1-\delta)/2$. According to (3) we can choose the sequence of natural numbers $T_m, T_{m+1} > 2^k T_m$ with the conditions

$$\sum_m \nu\{\mathcal{M}([T_m; 2^k T_m))\} < \epsilon/2,$$

$$\nu_x\{\mathcal{M}([T_m; 2^k T_m)\} < 2 \cdot \nu\{\mathcal{M}([T_m; 2^k T_m)\} \text{ as } x \geqslant T_{m+1}.$$

Let $z_1 = c, z_1 < z_2 < \ldots$ be an arbitrary unbounded sequence. We define a function $w(d)$ taking $w(d) = z_m$, if $d \in [T_m; 2^k T_m)$, and $w(d) = 0$, if $d \notin \cup_m [T_m; 2^k T_m)$. The existence of $\lambda\{f(n|w) \geqslant z\}$ follows from the previous theorem.

For fixed $z \geqslant c$ find some z_m such that $z \leqslant z_m$. Obviously,

$$\nu_x\{f(n|w) \geqslant z_m\} \leqslant \nu_x\{f(n|w) \geqslant z\} \leqslant \nu_x\{f(n|w) \geqslant c\}.$$

We show that $\overline{\nu}\{n\colon f(n|w) \geqslant z_m\} \geqslant 1 - 2^{-k}$ and $\underline{\nu}\{n\colon f(n|w) \geqslant c\} \leqslant \epsilon$. The second inequality follows from

$$\nu_{T_m}\{n\colon f(n|w) \geqslant c\} = \nu_{T_m}\{\cup_{j=1}^{m-1} \mathcal{M}([T_j, 2^k T_j))\} \leqslant 2 \sum_{j<m} \nu\{\mathcal{M}([T_j; 2^k T_j))\} < \epsilon.$$

We obtain the first one using the bound

$$\begin{aligned}\nu_{2^k T_{m+j}}\{n\colon f(n|w) \geqslant z_m\} &\geqslant \nu_{2^k T_{m+j}}\{\mathcal{M}([T_{m+j}; 2^k T_{m+j}))\} \\ &= \frac{2^k T_{m+j} - T_{m+j}}{2^k T_{m+j}} = 1 - 2^{-k}.\end{aligned}$$

This suffices for the proof.

We are now going to interpret the Behrend inequality for the set of multiples in the context of arithmetical functions. Let A, B be arbitrary subsets of natural numbers. The Behrend inequality is

$$1 - \lambda\{\mathcal{M}(A \cup B)\} \geqslant (1 - \lambda\{\mathcal{M}(A)\}) \cdot (1 - \lambda\{\mathcal{M}(B)\}), \tag{4}$$

see [1]; [5] Th. 012, p.15.

Let now $f(n|w_1), f(n|w_2)$ be two functions and $w_i(d) \geqslant 0$ for all d. With some fixed z_1, z_2 denote

$$A = \{n\colon f(n|w_1) \geqslant z_1\}, \quad B = \{n\colon f(n|w_2) \geqslant z_2\}.$$

Because of $\mathcal{M}(A) = A, \mathcal{M}(B) = B, \mathcal{M}(A \cup B) = A \cup B$, the sets possess the logaritmic densities. We have

$$\begin{aligned}1 - \lambda\{A\} &= \lambda\{n\colon f(n|w_1) < z_1\}, \\ 1 - \lambda\{B\} &= \lambda\{n\colon f(n|w_2) < z_2\}, \\ 1 - \lambda\{A \cup B\} &= \lambda\{n\colon f(n|w_1) < z_1, f(n|w_2) < z_2\}.\end{aligned}$$

Now from (4) we obtain

THEOREM 5. *If $w_1(d) \geqslant 0, w_2(d) \geqslant 0$, then for all z_1, z_2*

$$\lambda\{f(n|w_1) < z_1\} \cdot \lambda\{f(n|w_2) < z_2\} \leqslant \lambda\{f(n|w_1) < z_1, f(n|w_2) < z_2\}. \tag{5}$$

Evidently, (5) can be rewritten for more than two functions.

For which additive or multiplicative functions $f_1(n) = f(n|w_1)$, $f_2(n|w_2)$ inequality (5) holds? Having in mind Theorem 1, we derive quickly the sufficient condition: it suffices that for any prime p

$$0 \leqslant f_i(p) \leqslant f_i(p^2) \leqslant \dots, \quad i = 1, 2, \dots,$$

holds. If the sets $\{n: f_i(n) < z_i\}, \{n: f_1(n) < z_1,\ f_2(n) < z_2\}$ possess the number-theoretic densities they can be used in (5) instead of logaritmic ones. For example, let P_1, P_2 be some arbitrary subsets of prime numbers; define the additive functions

$$f_i(n) = \sum_{p \in P_i,\, p^\alpha || n} (\alpha - 1), \quad i = 1, 2.$$

Then for all z_1, z_2

$$\nu\{n: f_1(n) < z_1\} \cdot \nu\{n: f_2(n) < z_1\} \leqslant \nu\{n: f_1(n) < z_1,\ f_2(n) < z_2\}.$$

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REZIUMĖ

V. Stakėnas. Pastaba apie aritmetines funkcijas ir kartotinius

Straipsnyje nagrinėjamas kartotinių aibių ir aritmetinių funkcijų ryšys. Įrodomi teiginiai apie aritmetinių funkcijų reikšmių asimptotinius dažnius.

Raktiniai žodžiai: aritmetinės funkcijos, kartotiniai.