Numerical Analysis of the Eigenvalue Problem for One-Dimensional Differential Operator with Nonlocal Integral Conditions

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Abstract. In this paper the eigenvalue problem for one-dimensional differential operator with nonlocal integral conditions is investigated numerically. The special cases of general problem are analyzed and hypothesis about the dependence of the spectral structure of that problem on the coefficient of differential operator and the parameters of nonlocal conditions are formulated.

Keywords: numerical analysis, eigenvalue problem, one-dimensional differential operator, nonlocal integral conditions.

1 Introduction. Statement of the problem

We consider the eigenvalue problem for one-dimensional differential operator with given nonlocal integral boundary conditions,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \lambda u, \quad 0 < x < 1, \tag{1}$$

$$u(0) = \gamma_0 \int_0^1 u(x)\,\mathrm{d}x, \tag{2}$$

$$u(1) = \gamma_1 \int_0^1 u(x)\,\mathrm{d}x, \tag{3}$$

where $p(x) > p_0 \geq 0$, $\gamma_0, \gamma_1 \in \mathbb{R}$, and the corresponding eigenvalue problem for difference operator. Such values of λ that problem (1)–(3) has the non-trivial solution

$u(x) \not\equiv 0$ are called *eigenvalues* and set of all eigenvalues is called *spectrum of the problem*.

The analysis of the spectrum of a difference operator with a nonlocal conditions permits us to analyze the stability of difference schemes [1, 2], justify the convergence of iterative methods for finite-difference equations [3, 4] and is also of interest in itself.

When $p(x) \equiv 1$, $\gamma_0 = \gamma_1 = 0$, we get the well-known problem with classical boundary conditions (see, e.g., [5]). The eigenvalues of such problem are real, positive and simple:

$$\lambda_k = (k\pi)^2, \quad k = 1, 2, \ldots .$$

When, for example, $\gamma_1 \neq 0$ and instead of classical boundary condition at the point $x = 1$ we have nonlocal Bitsadze-Samarskii type boundary condition

$$u(1) = \gamma_1 u(\xi), \quad 0 < \xi < 1,$$

the spectral structure of the problem is more complicated: subject to parameters γ_1 and ξ, the eigenvalues can be both real (positive or non-positive) and complex numbers (see [6]).

In papers [7, 8] the eigenvalue problems for one-dimensional differential operator with $p(x) \equiv 1$ and various nonlocal integral conditions are investigated analytically. However, such problems with a non-constant coefficient $p(x)$ are met in the literature quite rarely and are considerably less investigated.

The main aim of the present paper is to investigate special cases of problem (1)–(3) numerically and to formulate the hypothesis about the dependence of qualitative structure of the spectrum of the problem on coefficient $p(x)$ and parameters γ_0, γ_1.

The paper is organized as follows. In Section 2 we describe the finite-difference scheme and formulate equivalent eigenvalue problem for the transition matrix. Technical details and results of numerical analysis of the special cases of the problem are presented and discussed in Section 3. Some remarks in Section 4 conclude the paper.

2 Problem discretization

To solve the differential problem (1)–(3), we can apply the finite-difference method [5].

Let us define the uniform discrete grid on the interval $(0, 1)$:

$$\omega^h = \{x_i\colon\ x_i = ih,\ i = 1, 2, \ldots, N-1,\ h = 1/N\}.$$

We approximate the differential problem (1)–(3) by the following finite-difference scheme:

$$-\frac{p_{i-1/2}U_{i-1} - (p_{i-1/2} + p_{i+1/2})U_i + p_{i+1/2}U_{i+1}}{h^2} = \lambda U_i, \quad x_i \in \omega^h, \tag{4}$$

$$U_0 = \gamma_0 h \left(\frac{U_0 + U_N}{2} + \sum_{i=1}^{N-1} U_i \right), \tag{5}$$

$$U_N = \gamma_1 h \left(\frac{U_0 + U_N}{2} + \sum_{i=1}^{N-1} U_i \right), \tag{6}$$

where $p_{i\pm 1/2} = p(\frac{x_i + x_{i\pm 1}}{2})$.

We rewrite integral conditions (5) and (6) in the form of a system of two linear algebraic equations with unknowns U_0 and U_N:

$$\begin{cases} \left(1 - \dfrac{\gamma_0 h}{2}\right) U_0 - \dfrac{\gamma_0 h}{2} U_N = \gamma_0 h \displaystyle\sum_{i=1}^{N-1} U_i, \\ -\dfrac{\gamma_1 h}{2} U_0 + \left(1 - \dfrac{\gamma_1 h}{2}\right) U_N = \gamma_1 h \displaystyle\sum_{i=1}^{N-1} U_i. \end{cases} \tag{7}$$

This system has a unique solution if its determinant is not equal to zero, i.e.,

$$D = \begin{vmatrix} 1 - \dfrac{\gamma_0 h}{2} & -\dfrac{\gamma_0 h}{2} \\ -\dfrac{\gamma_1 h}{2} & 1 - \dfrac{\gamma_1 h}{2} \end{vmatrix} \neq 0.$$

If $M_1 = \max\{|\gamma_0|, |\gamma_1|\} < \infty$ and the grid step

$$h < \frac{1}{M_1}, \tag{8}$$

then

$$D = 1 - \frac{\gamma_0 h}{2} - \frac{\gamma_1 h}{2} \geq 1 - \frac{M_1 h}{2} - \frac{M_1 h}{2} = 1 - M_1 h > 0.$$

We can write the solution of system (7) in the form

$$U_0 = \frac{\gamma_0 h}{D} \sum_{i=1}^{N-1} U_i, \quad U_N = \frac{\gamma_1 h}{D} \sum_{i=1}^{N-1} U_i.$$

Now we define the square matrix of order $(N-1)$

$$A = h^{-2} \begin{pmatrix} b_1 + \delta_0 & a_1 + \delta_0 & \delta_0 & \dots & \delta_0 & \delta_0 & \delta_0 \\ a_1 & b_2 & a_2 & \dots & 0 & 0 & 0 \\ 0 & a_2 & b_3 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \dots & a_{N-3} & b_{N-2} & a_{N-2} \\ \delta_1 & \delta_1 & \delta_1 & \dots & \delta_1 & a_{N-2} + \delta_1 & b_{N-1} + \delta_1 \end{pmatrix},$$

where

$$a_i = -p_{i+1/2}, \quad b_i = -(a_{i-1} + a_i), \quad \delta_0 = a_0 \frac{\gamma_0 h}{D}, \quad \delta_1 = a_{N-1} \frac{\gamma_1 h}{D}.$$

One can see, that the difference eigenvalue problem (4)–(6) is equivalent to the eigenvalue problem

$$AU = \lambda U \tag{9}$$

for the matrix A.

3 Numerical results and discussion

3.1 Technical details of numerical analysis

In order to analyze the qualitative structure of the spectrum of the differential eigenvalue problem (1)–(3) numerically, the eigenvalue problem (9) for the matrix A was solved. For numerical analysis MATLAB (The MathWorks, Inc.) software package [9] was used.

Numerical experiments were executed with different types of coefficient $p(x)$ and various values of parameters γ_0, γ_1. We executed series of numerical solution procedures for problem (9) with concrete coefficients $p(x)$ and points (γ_0, γ_1) from discrete set selected in the rectangle with restrictions

$$\gamma_0^* < \gamma_0 < \gamma_0^{**}, \quad \gamma_1^* < \gamma_1 < \gamma_1^{**}.$$

In order to ensure the validity of inequality (8), in all numerical experiments we used

$$N = 500, \quad \gamma_0^* = \gamma_1^* = -500, \quad \gamma_0^{**} = \gamma_1^{**} = 500.$$

Note that opportunities of symbolic calculations give us quite efficient way to avoid some undesirable effects of numerical calculations when observing the behaviour of the function

$$P(\lambda; \gamma_0, \gamma_1) = \det(A - \lambda E),$$

where E is the identity matrix.

Now let us analyze a couple of cases of the coefficient $p(x)$.

3.2 Case 1: $p(x) = \alpha(x-1)x + \left(1 + \frac{\alpha}{4}\right)$, $\alpha > -4$

In this case $p(x)$ are concave ($-4 < \alpha < 0$) or convex ($\alpha > 0$) and symmetrical in respect of the line $x = 1/2$ quadratic functions. When $\alpha = 0$, we have the well-known case $p(x) \equiv 1$ (see [7]).

The results of numerical analysis of the problem in Case 1 with $\alpha = -7/2, -3, -2, -1, -1/2, 1/2, 1, 2, 3, 5, 10, 50, 100, 500, 1000$ allow us to formulate the following generalization:

All the eigenvalues in Case 1 of the considered problem are real numbers. If $\gamma_0 + \gamma_1 < 2$, then all eigenvalues are positive. When $\gamma_0 + \gamma_1 = 2$, the number $\lambda = 0$ is an eigenvalue of the problem. Lastly, there exists only one negative eigenvalue, when $\gamma_0 + \gamma_1 > 2$.

These results coincide with analytical results in case $p(x) \equiv 1$ (see [7]).

Note that when $\lambda = 0$, the general solution of equation (1) in Case 1 is

$$u(x) = c_1 + c_2 \arctan\left((x - 1/2)\sqrt{\alpha}\right) \tag{10}$$

and the condition $\gamma_0 + \gamma_1 = 2$ for $\lambda = 0$ being an eigenvalue of the differential problem can be easily derived analytically. Indeed, by substituting solution (10) into boundary conditions (2) and (3), we get the system of two linear algebraic equations with unknowns c_1 and c_2:

$$\begin{cases} (1-\gamma_1)c_1 + \arctan\left(-\sqrt{\alpha}/2\right)c_2 = 0, \\ (1-\gamma_2)c_1 + \arctan\left(\sqrt{\alpha}/2\right)c_2 = 0. \end{cases}$$

Thus, this system has a unique solution if its determinant is equal to zero, i.e. $\gamma_1 + \gamma_2 = 2$.

3.3 Case 2: $p(x) = \alpha x + \left(1 - \frac{\alpha}{2}\right), \ -2 < \alpha < 2, \ \alpha \neq 0$

We have monotonically decreasing ($-2 < \alpha < 0$) or increasing ($0 < \alpha < 2$) linear functions $p(x)$ in this case.

We numerically analyzed problem with $\alpha = -3/2, -1, -1/2, 1/2, 1, 3/2$. The results of analysis are presented in Fig. 1.

When $\lambda = 0$, the general solution of equation (1) in Case 2 is

$$u(x) = c_1 + c_2 \ln\left(x + \frac{2-\alpha}{2\alpha}\right).$$

As marked in all graphs of Fig. 1, in this case the problem has eigenvalue $\lambda = 0$ when parameters γ_0 and γ_1 belongs to the straight line, which is denoted as dashed line. Using the same technique as in Case 1, it is easy to make sure, that for every α, $-2 < \alpha < 2$, $\alpha \neq 0$, the equation of the certain straight line is

$$\left((\alpha-2)\ln\frac{2-\alpha}{2+\alpha} - 2\alpha\right)\gamma_0 + \left((\alpha+2)\ln\frac{2-\alpha}{2+\alpha} + 2\alpha\right)\gamma_1 = 2\alpha\ln\frac{2-\alpha}{2+\alpha}. \quad (11)$$

When γ_0 and γ_1 are located somewhere above the straight line in the northeast direction from the origin of coordinates, the considered problem has exactly one negative eigenvalue.

The parameters γ_0 and γ_1 when exists non-negative multiple eigenvalue are located on the hyperbola-like curves. As we can see from Fig. 1, the directions of branches of hyperbolas are different in cases of decreasing and increasing functions $p(x)$. Each hyperbola divide the coordinate plane (γ_0, γ_1) into three unbounded regions. All the eigenvalues are real, when point (γ_0, γ_1) belongs to the region between two branches of the hyperbola. However, when (γ_0, γ_1) belongs to one of two other regions, the considered problem has complex eigenvalue.

The structure of the spectrum is rather complicated when the parameters γ_0 and γ_1 are located near one of the half-branches of the hyperbola (see quadrants in the southeast and the northwest directions from the origin of coordinates in graphs Fig. 1(a), 1(b), 1(c) and Fig. 1(d), 1(e), 1(f), respectively). When (γ_0, γ_1) belongs to the region between the dashed straight line and the ray which start point lies in that straight line and which is denoted as dash-dot line with circles in the graphs of Fig. 1, there exist two negative eigenvalues. Moreover, when γ_0 and γ_1 are located somewhere between two rays which are denoted as dotted line with crosses and dash-dot line with circles, the considered problem has conjugate complex eigenvalues with negative real parts.

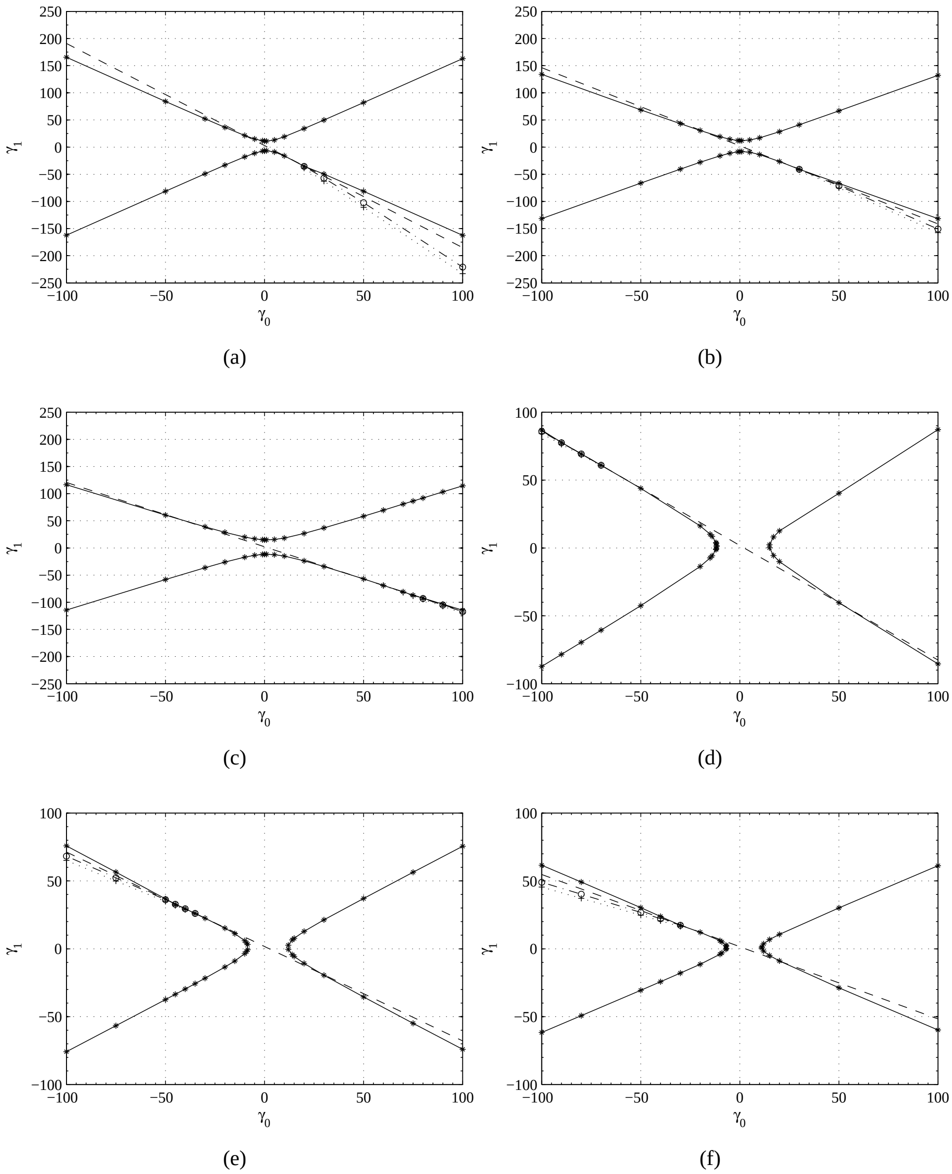

Fig. 1. The structure of the spectrum of the problem in Case 2: (a) $\alpha = -3/2$; (b) $\alpha = -1$; (c) $\alpha = -1/2$; (d) $\alpha = 1/2$; (e) $\alpha = 1$; (f) $\alpha = 3/2$. The considered problem has complex eigenvalues with real parts equal to zero, two negative eigenvalues, non-negative multiple eigenvalue or eigenvalue equal to zero when parameters γ_0 and γ_1 are located on dotted curves with crosses ($\cdot \cdot \cdot \cdot + \cdot \cdot \cdot \cdot$), dashed-dot curves with circles ($- \cdot - \circ - \cdot -$), solid curves with asterisks (——$*$——) and dashed curves (– – – – –), respectively.

Now let us generalize the results of numerical analysis of the problem in Case 2:
Subject to nonlocal boundary conditions parameters γ_0 *and* γ_1*, both real and complex numbers can be eigenvalues of the problem in Case* 2*. The problem has multiple non-negative eigenvalues when* γ_0 *and* γ_1 *belongs to the hyperbola-like curves. Parameters* γ_0 *and* γ_1*, when* $\lambda = 0$ *is an eigenvalue of the problem, are located on the straight line (see equation* (11)*). In addition, there exist such values of* γ_0 *and* γ_1 *that considered problem has one or two negative eigenvalues or complex eigenvalues with non-positive real parts.*

4 Conclusions

The results of numerical analysis allow us to make the following conclusions:

- We can investigate the general problem (9) in particular cases and formulate hypothesis about the qualitative structure of the spectrum of this problem when using the technique of numerical analysis.
- Despite the fact that in particular cases the qualitative structure of the spectrum is simple, in general case this structure can be rather complicated as well as the qualitative behaviour of eigenvalues: depending on the parameters γ_0 and γ_1 both real numbers (positive or non-positive) and complex numbers (with positive or non-positive real parts) can be eigenvalues of the problem.

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