

# Separable Solutions to an Interacting Human Communities Model

V. Skakauskas

*Department of Mathematics, Vilnius University*  
Naugarduko 24, 2600 Vilnius, Lithuania  
E-mail: [vladas.skakauskas@maf.vu.lt](mailto:vladas.skakauskas@maf.vu.lt)

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**Abstract.** A model for age-structured human communities is studied taking into account a religion factor. The model describes dynamics of interacting religions which tolerate pairs with different confessions, parents can choose a religion not necessarily their own for their offsprings, but it is forbidden to change a confession for any individual. In the case of stationary vital rates, the existence of separable solutions is studied.

**Key words:** population dynamics, age-structured population, religion factor, human community model.

## 1 Introduction

Two-sex population models are of great importance for genetics (see e.g. Svirezhev and Passekov [12] and references therein), demography and epidemiology, in particular for modelling sexually transmitted diseases (see e.g. references in Haderler [2], Prüss and Schappacher [4]). Both random mating (without formation of permanent male-female couples) and monogamous marriage models (see Frederickson [1], Hoppensteadt [3], Staroverov [11], Haderler [2], Skakauskas [9] and references therein) are usually used. The most general sex-age-structured population deterministic model taking into account marriages has been proposed by Hoppensteadt [3] and Staroverov [11], and consists of a system of three integro-differential equations for the densities of single (unmarried) females, single males, and pairs. Haderler [2] simplified this model by introducing a maturation period into the mating law.

The Staroverov-Haderler model has been generalized by Skakauskas [7, 8, 10] taking into account a religion factor and child care. In ecology an individual can be characterized by age and sex. In genetics it can be done by age, sex and a genotype parameter, and in epidemiology age, sex and disease parameters are used. In [7], we extended the list of essential parameters characterizing an individual by adding a religion factor which is very important for pair formation

of some human communities. Two different population dynamics models were presented in [7]. The first model tolerates the religion change for the sake of marriage. The second one forbids any confession change of individuals but lets parents to choose a religion not necessarily their own for their newborns.

In the present paper, we consider the second model and study its separable solutions. The plan for this paper is as follows. In Section 2, the model for interacting religion communities is described. In Section 3, conditions for the existence or nonexistence of separable solutions are given. In Section 4, we examine separable solutions in the case of constant vital rates. In Section 5, two lemmas are proved. The concluding remarks are given in Section 6.

## 2 The Mathematical Model

In this section we describe the model of human communities taking into account interacting religions which forbid any change of the confession for their individuals, but let parents to choose the religion not necessary their own for their newborns [7]. The model consists of the following equations

$$\partial_t u_i^k + \partial_{\tau_k} u_i^k = -\nu_i^k u_i^k - L_i^k + S_i^k, \quad t > 0, \quad \tau_k \in Q^k, \quad k = 1, 2, \tag{2.1}$$

$$\partial_t u_{ij}^3 + \sum_{k=1}^3 \partial_{\tau_k} u_{ij}^3 = -(\nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij}) u_{ij}^3, \quad t > 0, \quad (\tau_1, \tau_2, \tau_3) \in Q^3, \tag{2.2}$$

$$L_i^1 = \begin{cases} 0, & \tau_1 < \tau, \\ \sum_{s=1}^n \int_{\tau}^{\infty} f_{is} d\tau_2, & \tau_1 > \tau, \end{cases} \quad L_i^2 = \begin{cases} 0, & \tau_2 < \tau, \\ \sum_{s=1}^n \int_{\tau}^{\infty} f_{si} d\tau_1, & \tau_2 > \tau, \end{cases} \tag{2.3}$$

$$S_i^1 = \begin{cases} 0, & \tau_1 < \tau, \\ \int_0^{\tau_1 - \tau} d\tau_3 \int_{\tau_3 + \tau}^{\infty} \sum_{s=i}^n (\nu_{is}^2 + \sigma_{is}) u_{is}^3 d\tau_2, & \tau_1 > \tau, \end{cases} \tag{2.4}$$

$$S_i^2 = \begin{cases} 0, & \tau_2 < \tau, \\ \int_0^{\tau_2 - \tau} d\tau_3 \int_{\tau_3 + \tau}^{\infty} \sum_{s=i}^n (\nu_{si}^1 + \sigma_{si}) u_{si}^3 d\tau_1, & \tau_2 > \tau \end{cases} \tag{2.5}$$

subject to the conditions

$$u_i^k \Big|_{\tau_i=0} = \int_0^{\infty} d\tau_3 \int_{\tau_3 + \tau}^{\infty} y d\tau_1 \int_{\tau_3 + \tau}^{\infty} \sum_{s,j=1}^n u_{sj}^3 b_{sj}^k \Omega_{sj,i} d\tau_2, \quad t \geq 0, \quad k = 1, 2, \tag{2.6}$$

$$f_{ij} := u_{ij}^3 \Big|_{\tau_3=0} = 2m_{ij} u_i^1 u_j^2 / \sum_{k=1}^2 \sum_{s=1}^n \int_{\tau}^{\infty} u_s^k d\xi, \quad t \geq 0, \quad \tau_1, \tau_2 \in (\tau, \infty), \tag{2.7}$$

$$u_i^k \Big|_{t=0} = u_i^{k0}, \tau_1 \in (0, \infty), k = 1, 2, \quad u_{ij}^3 \Big|_{t=0} = u_{ij}^{30}, (\tau_1, \tau_2, \tau_3) \in Q^3, \quad (2.8)$$

$$[u_i^k \Big|_{\tau_k=\tau}] = 0, t > 0, k = 1, 2, \quad \sum_{s=1}^n \Omega_{ij,s} = 1, t > 0, (\tau_1, \tau_2, \tau_3) \in \bar{Q}^3.$$

Here  $i, j = 1, \dots, n$ , where  $n$  is the number of interacting religions,  $Q^k = (0, \tau) \cup (\tau, \infty)$ ,  $k = 1, 2$ ,  $Q^3 = \{(\tau_1, \tau_2, \tau_3) : \tau_1 \in (\tau_3 + \tau, \infty), \tau_2 \in (\tau_3 + \tau, \infty), \tau_3 \in (0, \infty)\}$ ,  $u_i^1(t, \tau_1)$  is the density at time  $t$  of single (unmarried) males of age  $\tau_1$  and of the  $i$ th religion,  $u_i^2(t, \tau_2)$  is the density at time  $t$  of single (unmarried) females of age  $\tau_2$  and of the  $i$ th religion,  $u_{ij}^3(t, \tau_1, \tau_2, \tau_3)$  is the density at time  $t$  of a pair which is formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion, and which have existed for  $\tau_3$  units of time,  $\nu_i^1(t, \tau_1)$  denotes the death rate at time  $t$  of single males of age  $\tau_1$  and of the  $i$ th religion,  $\nu_i^2(t, \tau_2)$  is the death rate at time  $t$  of single females of age  $\tau_2$  and of the  $i$ th religion;  $\nu_{ij}^1(t, \tau_1, \tau_2, \tau_3)$  denotes the death rate at time  $t$  of males from a pair formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion, and which have existed for  $\tau_3$  units of time,  $\nu_{ij}^2(t, \tau_1, \tau_2, \tau_3)$  is the death rate at time  $t$  of females from a pair formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion, and which have existed for  $\tau_3$  units of time,  $\sigma_{ij}(t, \tau_1, \tau_2, \tau_3)$  is the divorce rate at time  $t$  of pairs formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion, and which have existed for  $\tau_3$  units of time,  $b_{ij}^1(t, \tau_1, \tau_2, \tau_3)$  denotes the birth rate at time  $t$  of males produced by a pair formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion, and which have existed for  $\tau_3$  units of time,  $b_{ij}^2(t, \tau_1, \tau_2, \tau_3)$  is the birth rate at time  $t$  of females produced by a pair formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion, and which have existed for  $\tau_3$  units of time,  $L_i^k(t, \tau_k)$ ,  $k = 1, 2$  are the single individuals of age  $\tau_k$  and of the  $i$ th religion loss rate caused by the marriage,  $S_i^k(t, \tau_k)$ ,  $k = 1, 2$  are the single individuals of age  $\tau_1$  and of the  $i$ th religion gain rate caused by the death of the pair partner and divorce of pairs,  $f_{ij}(t, \tau_1, \tau_2)$  denotes the formation rate of pairs consisting of a male of age  $\tau_1$  and of the  $i$ th religion and a female of  $\tau_2$  age and of the  $j$ th religion,  $\Omega_{s,j,i}(t, \tau_1, \tau_2, \tau_3)$  is the probability that a pair, formed of a male of age  $\tau_1$  and of the  $i$ th religion and a female of age  $\tau_2$  and of the  $j$ th religion and which have existed for  $\tau_3$  units of time, will choose the  $i$ th religion for its newborn produced at time  $t$ ,  $u_i^{k0}(\tau_k)$ ,  $k = 1, 2$ ;  $u_{ij}^{30}(\tau_1, \tau_2, \tau_3)$  are the initial distributions,  $[u^k(t, \tau)]$  denotes the jump discontinuity of  $u_i^k$  at the line  $\tau_k = \tau$ ,  $k = 1, 2$ .

### 3 Separable Solutions

In this section, we examine system (2.1)–(2.8) with vital rates  $\nu_i^k$ ,  $\nu_{ij}^k$ ,  $\sigma_{ij}$ ,  $b_i^k$ ,  $m_{ij}$ , and  $\Omega_{s,j,i}$  independent of  $t$  and look for solutions of the form

$$\begin{cases} u_i^k(t, \tau_k) = U_i^k(\tau_k) \exp\{\lambda t\}, u_i^{k0} = U_i^k, \\ u_{ij}^3(t, \tau_1, \tau_2, \tau_3) = U_{ij}^3(\tau_1, \tau_2, \tau_3) \exp\{\lambda t\}, u_{ij}^{30} = U_{ij}^3, \end{cases} \quad (3.1)$$

where  $U_i^k$ ,  $U_{ij}^3$ , and the constant  $\lambda$  are to be determined. We substitute Eq. (3.1) into (2.1)–(2.7) to obtain

$$\begin{cases} \frac{dU_i^k}{d\tau_k} = -(\lambda + \nu_i^k)U_i^k, \tau_k \in (0, \tau], \\ U_i^k(0) = \int_0^\infty d\tau_3 \int_{\tau_3+\tau}^\infty d\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^n U_{sj}^3 b_{sj}^k \Omega_{sj,i} d\tau_2, \quad k = 1, 2, \end{cases} \tag{3.2}$$

$$\begin{aligned} \frac{dU_i^1}{d\tau_1} &= -\left(\lambda + \nu_i^1 + \frac{2}{f} \sum_{s=1}^n \int_\tau^\infty m_{is} U_s^2 d\tau_2\right) U_i^1 + \int_0^{\tau_1-\tau} d\tau_3 \int_{\tau_3+\tau}^\infty \sum_{s=i}^n (\nu_{is}^2 + \sigma_{is}) U_{is}^3 d\tau_2, \\ \tau_1 > \tau, [U_i^1(\tau)] &= 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{dU_i^2}{d\tau_2} &= -\left(\lambda + \nu_i^2 + \frac{2}{f} \sum_{s=1}^n \int_\tau^\infty m_{si} U_s^1 d\tau_1\right) U_i^2 + \int_0^{\tau_2-\tau} d\tau_3 \int_{\tau_3+\tau}^\infty \sum_{s=i}^n (\nu_{si}^1 + \sigma_{si}) U_{si}^3 d\tau_2, \\ \tau_2 > \tau, [U_i^2(\tau)] &= 0, \end{aligned} \tag{3.4}$$

$$\begin{cases} \sum_{\alpha=1}^3 \partial_{\tau_k} U_{ij}^3 = -(\lambda + \nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij}) U_{ij}^3, \quad (\tau_1, \tau_2, \tau_3) \in Q^3, \\ U_{ij}^3|_{\tau_1=0} = \frac{2m_{ij} U_i^1 U_j^2}{f}, \end{cases} \tag{3.5}$$

where  $f = \sum_{k=1}^2 \sum_{\alpha=1}^n \int_\tau^\infty U_\alpha^k(\tau_k) d\tau_k$ . Set

$$U_i^k = f w_i^k, U_{ij}^3 = f w_{ij}^3. \tag{3.6}$$

Then

$$\sum_{k=1}^2 \sum_{\alpha=1}^n \int_\tau^\infty w_\alpha^k(\tau_k) d\tau_k = 1. \tag{3.7}$$

Formal integration of Eq. (3.5) gives

$$w_{ij}^3(\tau_1, \tau_2, \tau_3) = w_i^1(\tau_1 - \tau_3) w_j^2(\tau_2 - \tau_3) K_{ij}(\tau_1 - \tau_3, \tau_2 - \tau_3, \tau_3; \lambda), \tag{3.8}$$

where

$$\begin{aligned} &K_{ij}(\tau_1 - \tau_3, \tau_2 - \tau_3, \tau_3; \lambda) \\ &= 2m_{ij}(\tau_1 - \tau_3, \tau_2 - \tau_3) \exp \left\{ - \int_0^{\tau_3} (\tilde{\nu}_{ij}(\xi + \tau_1 - \tau_3, \xi + \tau_2 - \tau_3, \xi) + \lambda) d\xi \right\} \\ &= K_{ij}(\tau_1 - \tau_3, \tau_2 - \tau_3, \tau_3; 0) \exp(-\lambda\tau_3), \quad \tilde{\nu}_{ij} = \nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij}. \end{aligned}$$

Inserting (3.6) and (3.8) into (3.3), (3.4), and ((3.2))<sub>2</sub> we get

$$\begin{aligned} \frac{dw_i^1}{d\tau_1} &= -\left(\lambda + \nu_i^1 + 2 \sum_{s=1}^n \int_{\tau}^{\infty} m_{is} w_s^2 d\tau_2\right) w_i^1 + \int_0^{\tau_1 - \tau} d\tau_3 \int_{\tau_3 + \tau}^{\infty} \sum_{s=i}^n (\nu_{is}^2 + \sigma_{is}) w_s^1 \\ &\times (\tau_1 - \tau_3) w_s^2 (\tau_2 - \tau_3) K_{is}(\tau_1 - \tau_3, \tau_2 - \tau_3, \tau_3; \lambda) d\tau_2, \quad \tau_1 > \tau, [w_i^1(\tau)] = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \frac{dw_i^2}{d\tau_2} &= -\left(\lambda + \nu_i^2 + 2 \sum_{s=1}^n \int_{\tau}^{\infty} m_{si} w_s^1 d\tau_1\right) w_i^2 + \int_0^{\tau_2 - \tau} d\tau_3 \int_{\tau_3 + \tau}^{\infty} \sum_{s=i}^n (\nu_{si}^1 + \sigma_{si}) w_s^1 \\ &\times (\tau_1 - \tau_3) w_i^2 (\tau_2 - \tau_3) K_{si}(\tau_1 - \tau_3, \tau_2 - \tau_3, \tau_3; \lambda) d\tau_1, \quad \tau_2 > \tau, [w_i^2(\tau)] = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} w_i^k(0) &= \int_0^{\infty} d\tau_3 \int_{\tau_3 + \tau}^{\infty} d\tau_1 \int_{\tau_3 + \tau}^{\infty} \sum_{s,j=1}^n (b_{sj}^k \Omega_{sj,i})|_{(\tau_1, \tau_2, \tau_3)} w_s^1(\tau_1 - \tau_3) w_j^2(\tau_2 - \tau_3) \\ &\times K_{sj}(\tau_1 - \tau_3, \tau_2 - \tau_3, \tau_3; \lambda) d\tau_1, \quad k = 1, 2, \end{aligned} \tag{3.11}$$

Set  $\bar{w}^1 = (w_1^1, \dots, w_n^1)$ ,  $\bar{w}^2 = (w_1^2, \dots, w_n^2)$ ,

$$r_i^1(\tau_1, \bar{w}^2) = 2 \sum_{s=1}^n \int_{\tau}^{\infty} m_{is}(\tau_1, \tau_2) w_s^2(\tau_2) d\tau_2,$$

$$r_i^2(\tau_2, \bar{w}^1) = 2 \sum_{s=1}^n \int_{\tau}^{\infty} m_{si}(\tau_1, \tau_2) w_s^1(\tau_1) d\tau_1,$$

$$\begin{aligned} R_i^1(\tau_1, x; \bar{w}^2, \lambda) &= \int_{\tau}^{\infty} \sum_{s=1}^n w_s^2(y) K_{is}(x, y, \tau_1 - x; \lambda) \\ &\times (\nu_{is}^2 + \sigma_{is})|_{(\tau_1, y + \tau_1 - x, \tau_1 - x)} dy = R_i^1(\tau_1, x; \bar{w}^2, 0) \exp\{-\lambda(\tau_1 - x)\}, \end{aligned}$$

$$\begin{aligned} R_i^2(x, \tau_2; \bar{w}^1, \lambda) &= \int_{\tau}^{\infty} \sum_{s=1}^n w_s^1(y) K_{si}(y, x, \tau_2 - x; \lambda) \\ &\times (\nu_{si}^1 + \sigma_{si})|_{(y + \tau_2 - x, \tau_2, \tau_2 - x)} dy = R_i^2(x, \tau_2; \bar{w}^1, 0) \exp\{-\lambda(\tau_2 - x)\}, \end{aligned}$$

$$\kappa_{sj,i}^k(x, y; \lambda) = \int_0^{\infty} K_{sj}(x, y, \tau_3; \lambda) (b_{sj}^k \Omega_{sj,i})|_{(x + \tau_3, y + \tau_3, \tau_3)} d\tau_3.$$

This allows us to rewrite Eqs. (3.9)–(3.11) in the form

$$\begin{aligned} \frac{dw_i^1}{d\tau_1} &= -(\lambda + \nu_i^1 + r_i^1(\tau_1; \bar{w}^2))w_i^1 + \int_{\tau}^{\tau_1} w_i^1(x)R_i^1(\tau_1, x; \bar{w}^2, \lambda) dx, \\ w_i^1|_{\tau_1=\tau} &= w_i^1(\tau), \end{aligned} \tag{3.12}$$

$$\begin{aligned} \frac{dw_i^2}{d\tau_1} &= -(\lambda + \nu_i^2 + r_i^2(\tau_2; \bar{w}^1))w_i^2 + \int_{\tau}^{\tau_2} w_i^2(x)R_i^2(x, \tau_2; \bar{w}^1, \lambda) dx, \\ w_i^2|_{\tau_2=\tau} &= w_i^2(\tau), \end{aligned} \tag{3.13}$$

$$w_i^k(0) = \int_{\tau}^{\infty} dx \int_{\tau}^{\infty} \sum_{s,j=1}^n w_s^1(x)w_j^2(y)\kappa_{s,j,i}^k(x, y; \lambda) dy, \quad k = 1, 2. \tag{3.14}$$

From Eqs. (3.2) and (3.6)<sub>1</sub> it follows that

$$w_i^k(\tau_k) = w_i^k(0)f_i^k(\tau_k; \lambda), \quad f_i^k(\tau_k; \lambda) = \exp\left\{-\int_0^{\tau_k} (\lambda + \nu_i^k(x)) dx\right\}.$$

Hence,

$$w_i^k(\tau) = w_i^k(0)f_i^k(\tau; \lambda). \tag{3.15}$$

It remains to solve Eqs. (3.12)–(3.15) and (3.7). Therefore, separable solutions (3.1) of Eqs. (2.1)–(2.7) correspond to solutions  $(\bar{w}^1, \bar{w}^2, \lambda)$  of problem (3.12)–(3.15) and (3.7). To treat the existence or nonexistence of separable solutions we follow the papers [10, 9] in which we used the Prüss and Schappacher [4] method applied for the investigation of separable solutions to the Staroverov [11] model with the Keyfitz-Hadeler pair formation function [5]. According to this method, we have to reformulate system (3.12)–(3.15) and then apply Schauder’s fixed-point principle [6]. For this reformulation we choose the space

$$X = \left\{(\bar{w}^1, \bar{w}^2) : w_s^k \in L^1_+(\tau, \infty), \quad k = 1, 2; \quad s = 1, \dots, n\right\}$$

and set

$$D = \left\{(\bar{w}^1, \bar{w}^2) : (\bar{w}^1, \bar{w}^2) \in X, \quad \sum_{k=1}^2 \sum_{s=1}^n \|w_s^k\| = 1\right\},$$

where  $\|\cdot\| := \|\cdot\|_{L^1(\tau, \infty)}$ . Obviously,  $D$  is closed, bounded, and convex. We shall construct the operator  $F^\lambda : D \rightarrow D$  with at least one fixed point in  $D$ . To do this, we linearize system (3.12) and (3.13) by prescribing  $(\bar{w}^1, \bar{w}^2) \in D$  involved in  $r_i^1(\tau_1; \bar{w}^2), r_i^2(\tau_2; \bar{w}^1), R_i^1(\tau_1, x; \bar{w}^2, \lambda)$ , and  $R_i^2(x, \tau_2; \bar{w}^1, \lambda), i = 1, \dots, n$ . Then, for given  $(\bar{w}^1, \bar{w}^2)$  and a real  $\lambda$ , we define

$$\begin{aligned} l_i^1(\tau_1; \lambda) &= \lambda + \nu_i^1(\tau_1) + r_i^1(\tau_1; \bar{w}^2), \quad l_i^2(\tau_2; \lambda) = \lambda + \nu_i^2(\tau_2) + r_i^2(\tau_2; \bar{w}^1), \\ g_i^1(\tau_1, x; \lambda) &= R_i^1(\tau_1, x; \bar{w}^2, \lambda), \quad g_i^2(x, \tau_2; \lambda) = R_i^2(x, \tau_2; \bar{w}^1, \lambda) \end{aligned}$$

with  $i = 1, \dots, n$  and rewrite Eqs. (3.12) and (3.13) as follows:

$$\frac{dw_i^1}{d\tau_1} = -l_i^1(\tau_1; \lambda)w_i^1 + \int_{\tau}^{\tau_1} w_i^1(x)g_i^1(\tau_1, x; \lambda)dx, \quad w_i^1(\tau) = w_i^1(0)f_i^1(\tau; \lambda), \quad (3.16)$$

$$\frac{dw_i^2}{d\tau_2} = -l_i^2(\tau_2; \lambda)w_i^2 + \int_{\tau}^{\tau_1} w_i^2(x)g_i^2(x, \tau_2; \lambda)dx, \quad w_i^2(\tau) = w_i^2(0)f_i^2(\tau; \lambda) \quad (3.17)$$

with  $w_i^k(0)$  defined by Eq. (3.14). Letting

$$w_i^k(\tau_k) = w_i^k(\tau)z_i^k(\tau_k) \quad (3.18)$$

with  $k = 1, 2, i = 1, \dots, n$ , and  $w_i^k(\tau)$  defined by (3.15) from Eqs. (3.16) and (3.17) we get

$$\frac{dz_i^1}{d\tau_1} = -l_i^1(\tau_1; \lambda)z_i^1 + \int_{\tau}^{\tau_1} z_i^1(x)g_i^1(\tau_1, x; \lambda) dx, \quad z_i^1(\tau) = 1, \quad (3.19)$$

$$\frac{dz_i^2}{d\tau_2} = -l_i^2(\tau_2; \lambda)z_i^2 + \int_{\tau}^{\tau_1} z_i^2(x)g_i^2(x, \tau_2; \lambda) dx, \quad z_i^2(\tau) = 1. \quad (3.20)$$

Note that these equations can be examined separately and that

$$\begin{aligned} z_i^1 &= z_i^1(\tau_1; \bar{w}^2, \lambda) = z_i^1(\tau_1; \bar{w}^2, 0) \exp \{ -\lambda(\tau_1 - \tau) \}, \\ z_i^2 &= z_i^2(\tau_2; \bar{w}^1, \lambda) = z_i^2(\tau_2; \bar{w}^1, 0) \exp \{ -\lambda(\tau_2 - \tau) \}. \end{aligned}$$

By formal integration of Eqs. (3.19) and (3.20) we derive

$$\begin{aligned} z_i^1(\tau_1) &= \exp \left\{ - \int_{\tau}^{\tau_1} l_i^1(\xi; \lambda) d\xi \right\} \\ &+ \int_{\tau}^{\tau_1} \exp \left\{ - \int_{\eta}^{\tau_1} l_i^1(\xi; \lambda) d\xi \right\} \left( \int_{\tau}^{\eta} z_i^1(x)g_i^1(\eta, x; \lambda) dx \right) d\eta, \end{aligned} \quad (3.21)$$

$$\begin{aligned} z_i^2(\tau_2) &= \exp \left\{ - \int_{\tau}^{\tau_2} l_i^2(\xi; \lambda) d\xi \right\} \\ &+ \int_{\tau}^{\tau_2} \exp \left\{ - \int_{\eta}^{\tau_2} l_i^2(\xi; \lambda) d\xi \right\} \left( \int_{\tau}^{\eta} z_i^2(x)g_i^2(x, \eta; \lambda) dx \right) d\eta, \end{aligned} \quad (3.22)$$

which can be written in the form

$$z_i^1(\tau_1) = \exp \left\{ - \int_{\tau}^{\tau_1} l_i^1(\xi; \lambda) d\xi \right\} + \int_{\tau}^{\tau_1} z_i^1(x)G_i^1(\tau_1, x; \lambda) dx, \quad (3.23)$$

$$z_i^2(\tau_2) = \exp \left\{ - \int_{\tau}^{\tau_2} l_i^2(\xi; \lambda) d\xi \right\} + \int_{\tau}^{\tau_2} z_i^2(x) G_i^2(x, \tau_2; \lambda) dx \tag{3.24}$$

with

$$G_i^1(\tau_1, x; \lambda) = \int_x^{\tau_1} g_i^1(\eta, x; \lambda) \exp \left\{ - \int_{\eta}^{\tau_1} l_i^1(\xi; \lambda) d\xi \right\} d\eta,$$

$$G_i^2(x, \tau_2; \lambda) = \int_x^{\tau_2} g_i^2(x, \eta; \lambda) \exp \left\{ - \int_{\eta}^{\tau_2} l_i^2(\xi; \lambda) d\xi \right\} d\eta.$$

Set  $\bar{z}^k = (z_1^k, \dots, z_n^k)$ ,

$$m^* = \max_{i,s} \sup_{[\tau, \infty)^2} m_{is}, \quad \nu_* = \min \left( \min_{k,i} \inf_{Q^k} \nu_i^k, \min_{k,i,s} \inf_{Q^3} \nu_{is}^k \right),$$

$$z_{i*}^k(\tau_k, \lambda) = \exp \left\{ - \int_{\tau}^{\tau_k} (\lambda + \nu_i^k(x) + 2m^*) dx \right\}, \quad z^{k*}(\tau_k, \lambda) = \exp \left\{ -(\lambda + \nu_*)(\tau_k - \tau) \right\},$$

with  $\lambda > -\nu_*$ ,

$$\tilde{D} = \left\{ (\bar{z}^1, \bar{z}^2) : z_i^k \in C^1([\tau, \infty)), z_{i*}^k \leq z_i^k \leq z^{k*}, \left\| \frac{dz_i^k}{d\tau_k} \right\| \leq 1 + \frac{4m^*}{\lambda + \nu_*}, k = 1, 2 \right\}.$$

**Lemma 1.** *Let  $\nu_*$  and  $\sigma_{is}$  be positive,  $m_{is}$  be positive and bounded, and let  $\nu_i^k \in C^0([\tau, \infty))$ ,  $m_{is} \in C^0([\tau, \infty)^2)$ ,  $\nu_{ij}^k$  and  $\sigma_{ij} \in C^0(Q^3)$ ,  $(\bar{w}^1, \bar{w}^2) \in D$ . Then, for a real  $\lambda > -\nu_*$ , the following assertions are true:*

(i) *Eqs. (3.23) and (3.24) have a unique positive solution  $z_i^1(\tau_1; w^2, \lambda)$  and  $z_i^2(\tau_2; w^1, \lambda)$  such that*

$$z_i^k \in C^0([\tau, \infty)) \cap C^1(\tau, \infty), \quad z_{i*}^k \leq z_i^k \leq z^{k*}, \quad \left\| \frac{dz_i^k}{d\tau_k} \right\| \leq 1 + \frac{4m^*}{\lambda + \nu_*},$$

that is  $(\bar{z}^1, \bar{z}^2) \in \tilde{D}$ .

(ii) *The operator  $\tilde{F}^\lambda : (\bar{w}^1, \bar{w}^2) \rightarrow (\bar{z}^1(\cdot, \bar{w}^2, \lambda), \bar{z}^2(\cdot, \bar{w}^1, \lambda))$  from  $D$  to  $X$  is completely continuous (compact), moreover  $\tilde{F}^\lambda(D) \subset \tilde{D} \subset X$ .*

The proof of Lemma 1 is given in Section 4.

Inserting (3.18) with  $w_i^k(\tau)$  defined by (3.15) into (3.14), we get the system

$$w_i^k(0) = \sum_{s,j=1}^n p_{sj,i}^k(\bar{w}^1, \bar{w}^2, \lambda) w_s^1(0) w_j^2(0) \tag{3.25}$$

with

$$p_{sj,i}^k(\bar{w}^1, \bar{w}^2, \lambda) = f_s^1(\tau; \lambda) f_j^2(\tau; \lambda) \times \int_{\tau}^{\infty} dx \int_{\tau}^{\infty} \kappa_{sj,i}^k(x, y; \lambda) z_s^1(x, \bar{w}^2, \lambda) z_j^2(y, \bar{w}^1, \lambda) dy.$$

Examination of the solution to system (3.25) and its behaviour as  $\lambda \rightarrow \infty$  in general case of vital rates is a complicated problem. Therefore, we restrict ourselves to the consideration of the following two cases:

1. The vital functions  $m_{is}$ ,  $\nu_{is}^k$ ,  $\nu_i^k$ , and  $\sigma_{is}$  satisfy conditions of Lemma 1 and do not depend on the confession number, functions  $\nu_{is}^k$  and  $\nu_i^k$  increase as ages tend to  $\infty$ ,  $b_{ij}^k = \beta_{ij}^k b(\tau_1, \tau_2, \tau_3)$  with positive constants  $\beta_{ij}^k$  and positive bounded  $b \in C^0(\bar{Q}^3)$ , all  $\Omega_{sj,i}$  are nonnegative constants;
2. All vital rates are continuous and bounded with positive lower bounds,  $n = 2$ , and sex ratio of newborns is a constant, i.e.,  $b_{ij}^2 = \gamma b_{ij}^1$ .

**Case 1.** Definition of  $f_i^k$ ,  $r_i^k$ ,  $l_i^k$ ,  $R_i^k$ ,  $K_{ij}$ , and  $G_i^k$  and Eqs. (3.23) and (3.24) show that  $z_i^k$  is also independent of the confession number. Letting  $\nu^k = \nu_i^k$ ,  $\bar{\nu}^k = \nu_{ij}^k$ ,  $\sigma = \sigma_{ij}$ ,  $z^k = z_i^k$ ,  $K = K_{ij}$ , and  $f^k(\tau, \lambda) = f_i^k(\tau, \lambda)$  we rewrite Eqs. (3.25) as follows

$$w_i^k(0) = \sum_{s,j=1}^n w_s^1(0)w_j^2(0)a_{sj,i}^k q(\bar{w}^1, \bar{w}^2, \lambda), \quad a_{sj,i}^k = \beta_{sj}^k \Omega_{sj,i}, \quad (3.26)$$

$$q(\bar{w}^1, \bar{w}^2, \lambda) = f^1(\tau, \lambda)f^2(\tau, \lambda) \int_{\tau}^{\infty} z^1(x, \bar{w}^2, \lambda)dx \int_0^{\infty} z^2(x, \bar{w}^1, \lambda)dy \\ \times \int_0^{\infty} \exp\{-\lambda\tau_3\}K(x + \tau_3, y + \tau_3, \tau_3, 0)b(x + \tau_3, y + \tau_3, \tau_3)d\tau_3 \quad (3.27)$$

We examine system (3.26) in two cases: (1<sub>1</sub>) all  $\Omega_{sj,i} > 0$  and (1<sub>2</sub>) all  $\Omega_{sj,i} \geq 0$ . In the first case the following statement is true.

**Lemma 2.** Let  $\Omega_{sj,i}$  and  $\beta_{ij}^k$  be positive constants and  $b \in C^0(\bar{Q}^3)$  be positive and bounded. Then under the conditions of Lemma 1 system (3.26) has a positive solution  $(\bar{w}^1(0), \bar{w}^2(0))$  with

$$w_i^k(0) = \frac{h_i^k}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \quad h_i^k = \phi_i^k / \sum_{s,j=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1 \quad i = 1, \dots, n, \quad (3.28)$$

where  $\phi_1^1 = 1$ , while  $(\phi_2^1, \dots, \phi_n^1, \phi_1^2, \dots, \phi_n^2)$  is at least one positive solution of the system

$$\phi_i^k = \sum_{s,j=1}^n \phi_s^1 \phi_j^2 a_{sj,i}^k / \sum_{s,j=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1, \quad i = 2, \dots, n.$$

The proof of Lemma 2 is given Section 5.

Next we consider the case (1<sub>2</sub>). More precisely, we examine a usual case for some confessions, where

$$\Omega_{ss,s} = 1, \\ \Omega_{sj,k} = 0, \quad s, j \neq k, \\ \Omega_{sj,s} + \Omega_{sj,j} = 1, \quad \Omega_{sj,s} > 0, \quad \Omega_{sj,j} > 0, \quad s \neq j. \quad (3.29)$$

In this case, system (3.26) reduces to

$$\frac{w_i^k(0)}{q(\bar{w}^1, \bar{w}^2, \lambda)} = w_i^1(0)w_i^2(0)a_{ii,i}^k + \sum_{j \neq i} (w_i^1(0)w_j^2(0)a_{ij,i}^k + w_j^1(0)w_i^2(0)a_{ji,i}^k). \tag{3.30}$$

We examine this system only in the case where the sex ratio of newborns is constant, i.e.,  $\beta_{ij}^2 = \gamma\beta_{ij}^1$ . It is easy to see that

$$w_i^k(0) = \frac{h_i^k}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \quad h_i^1 = \frac{\tilde{w}_i}{\gamma}, \quad h_i^2 = \tilde{w}_i, \quad i = 1, 2, \dots, n, \tag{3.31}$$

where  $(\tilde{w}_1, \dots, \tilde{w}_n)$  is a positive solution of the system

$$1 = \tilde{w}_i a_{ii,i}^1 + \sum_{j \neq i} \tilde{w}_j (a_{ij,i}^1 + a_{ji,i}^1). \tag{3.32}$$

The solution of this system can be written as

$$\tilde{w}_i = \frac{\Delta_i}{\Delta}, \quad i = 1, \dots, n \tag{3.33}$$

and all  $\tilde{w}_i > 0$ , if  $\text{sign } \Delta_i = \text{sign } \Delta$ . Here  $\Delta$  and  $\Delta_i$  are the determinants

$$\left| \begin{array}{cccccc} a_{11,1}^1 & a_{12,1}^1 + a_{21,1}^1 & \dots & a_{1j,1}^1 + a_{j1,1}^1 & \dots & a_{1n,1}^1 + a_{n1,1}^1 \\ a_{21,2}^1 + a_{12,2}^1 & a_{22,2}^1 & \dots & a_{2j,2}^1 + a_{j2,2}^1 & \dots & a_{2n,2}^1 + a_{n2,2}^1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1,n}^1 + a_{1n,n}^1 & a_{n2,n}^1 + a_{2n,n}^1 & \dots & a_{nj,n}^1 + a_{jn,n}^1 & \dots & a_{nn,n}^1 \end{array} \right|, \tag{3.34}$$

$$\left| \begin{array}{cccccc} a_{11,1}^1 & a_{12,1}^1 + a_{21,1}^1 & \dots & 1 & \dots & a_{1n,1}^1 + a_{n1,1}^1 \\ a_{21,2}^1 + a_{12,2}^1 & a_{22,2}^1 & \dots & 1 & \dots & a_{2n,2}^1 + a_{n2,2}^1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1,n}^1 + a_{1n,n}^1 & a_{n2,n}^1 + a_{2n,n}^1 & \dots & 1 & \dots & a_{nn,n}^1 \end{array} \right| \tag{3.35}$$

with the  $i$ th column  $(1, 1, \dots, 1)^T$ , respectively. If  $n = 2$ , then

$$\left\{ \begin{array}{l} \tilde{w}_1 = \frac{a_{22,2}^1 - a_{12,1}^1 - a_{21,1}^1}{a_{11,1}^1 a_{22,2}^1 \left(1 - \frac{a_{12,1}^1 + a_{21,1}^1}{a_{22,2}^1} \frac{a_{21,2}^1 + a_{12,2}^1}{a_{11,1}^1}\right)}, \\ \tilde{w}_2 = \frac{a_{11,1}^1 - a_{12,2}^1 - a_{21,2}^1}{a_{11,1}^1 a_{22,2}^1 \left(1 - \frac{a_{12,1}^1 + a_{21,1}^1}{a_{22,2}^1} \frac{a_{21,2}^1 + a_{12,2}^1}{a_{11,1}^1}\right)} \end{array} \right. \tag{3.36}$$

which are positive provided that

$$\text{sign}(a_{22,2}^1 - a_{12,1}^1 - a_{21,1}^1) = \text{sign}(a_{11,1}^1 - a_{12,2}^1 - a_{21,2}^1).$$

If  $a_{22,2}^1 = a_{12,1}^1 + a_{21,1}^1$  and  $a_{11,1}^1 = a_{12,2}^1 + a_{21,2}^1$ , that is if  $\beta_{22}^1 = \beta_{12}^1 \Omega_{12,1} + \beta_{21}^1 \Omega_{21,1}$  and  $\beta_{11}^1 = \beta_{21}^1 \Omega_{21,2} + \beta_{12}^1 \Omega_{12,2}$ , then from (3.36) we get only one equation for  $\tilde{w}_1$  and  $\tilde{w}_2$ ,

$$\tilde{w}_1 \beta_{11}^1 + \tilde{w}_2 \beta_{22}^1 = 1. \tag{3.37}$$

Hence, either  $\tilde{w}_1 < (\beta_{11}^1)^{-1}$  or  $\tilde{w}_2 < (\beta_{22}^1)^{-1}$  is a free positive constant. Obviously,  $\beta_{11}^1 + \beta_{22}^1 = \beta_{12}^1 + \beta_{21}^1$ .

**Lemma 3.** *Let hypotheses (3.29) hold,  $\beta_{ij}^2 = \gamma\beta_{ij}^1$  with  $\beta_{ij}^k = \text{const} > 0$ . Assume that  $b \in C^0(\bar{Q}^3)$  is positive and bounded and sign of determinants (3.34) and (3.35) is the same. Then under conditions of Lemma 1 system (3.30) has unique positive solution (3.31) with  $\tilde{w}_i$  defined by Eq. (3.33) if  $n > 2$ , by Eq. (3.36) if  $n = 2$  and  $\text{sign } \Delta_1 = \text{sign } \Delta_2$ , and by Eq. (3.37) if  $n = 2$  and  $\Delta_1 = \Delta_2 = 0$ . In the last case either  $\tilde{w}_1 < (\beta_{11}^1)^{-1}$  or  $\tilde{w}_2 < (\beta_{22}^1)^{-1}$  is a free positive constant.*

**Case 2.** Let us set

$$\begin{aligned} \nu^* &= \max_{k,i,j} (\sup_{Q^k} \nu_i^k, \sup_{Q^3} \nu_{ij}^k), & \sigma^* &= \max_{i,j} \sup_{Q^3} \sigma_{ij}, & m^* &= \max_{i,j} \sup_{(\tau,\infty)^2} m_{ij}, \\ b^* &= \max_{k,i,j} \sup_{Q^3} b_{ij}^k, & \Omega^* &= \max_{k,i,j} \sup_{Q^3} \Omega_{ki,j}, & b_*^1 &= \min_{i,j} \inf_{Q^3} b_{ij}^1, \\ m_* &= \min_{i,j} \inf_{(\tau,\infty)^2} m_{ij}, & \Omega_* &= \min_{k,i,j} \inf_{Q^3} \Omega_{ki,j}. \end{aligned}$$

By the transform  $w_i^2(0) = \gamma w_i^1(0)$ ,  $w_i^1(0) = \tilde{w}_i/\gamma$  system (3.25) can be written in the form

$$\tilde{w}_i = (\tilde{w}_1)^2 p_{11,i}^1 + \tilde{w}_1 \tilde{w}_2 (p_{12,i}^1 + p_{21,i}^1) + (\tilde{w}_2)^2 p_{22,i}^1, \quad i = 1, 2.$$

Then by substitution  $\tilde{w}_2 = \phi_2 \tilde{w}_1$  we reduce this system to

$$\begin{aligned} \tilde{w}_1 &= (p_{11,1}^1 + \phi_2(p_{12,1}^1 + p_{21,1}^1) + (\phi_2)^2 p_{22,1}^1)^{-1}, \\ \phi_2 &= \frac{p_{11,2}^1 + \phi_2(p_{12,2}^1 + p_{21,2}^1) + (\phi_2)^2 p_{22,2}^1}{p_{11,1}^1 + \phi_2(p_{12,1}^1 + p_{21,1}^1) + (\phi_2)^2 p_{22,1}^1}. \end{aligned}$$

Since all  $p_{sk,i}^1$  are continuous in  $(\bar{w}^1, \bar{w}^2, \lambda)$  and, for  $\phi_2 > 0$ , function  $\xi_1(\phi_2) := \phi_2(p_{11,1}^1 + \phi_2(p_{12,1}^1 + p_{21,1}^1) + (\phi_2)^2 p_{22,1}^1)$  grows faster than  $\xi_2(\phi_2) := p_{11,2}^1 + \phi_2(p_{12,2}^1 + p_{21,2}^1) + (\phi_2)^2 p_{22,2}^1$  and  $\xi_1(0) < \xi_2(0)$ , this cubic equation has only one positive continuous in  $(\bar{w}^1, \bar{w}^2, \lambda)$  solution  $\phi_2 = \phi(\bar{w}^1, \bar{w}^2, \lambda) \in [p_*/p^*, p^*/p_*]$ , where by the definition of  $p_{ij,k}^1$  and because of Lemma 1 we have

$$\begin{aligned} p_*(\lambda) &= \frac{2m_* b_*^1 \Omega_* \exp\{-2\tau(\lambda + \nu^*)\}}{(2\nu^* + \sigma^* + \lambda)(\nu^* + 2m^* + \lambda)^2} \leq \min_{i,j,k} p_{ij,k}^1(\lambda), \\ p^*(\lambda) &= \frac{2m^* b^1 \Omega^* \exp\{-2\tau(\lambda + \nu_*)\}}{(2\nu_* + \lambda)(\nu_* + \lambda)^2} \geq \max_{i,j,k} p_{ij,k}^1(\lambda). \end{aligned}$$

Then it follows that

$$\begin{cases} \tilde{w}_1 = \frac{1}{q(\bar{w}^1, \bar{w}^2, \lambda)}, & \tilde{w}_2 = \frac{\phi(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \\ q(\bar{w}^1, \bar{w}^2, \lambda) = p_{11,1}^1 + \phi(p_{12,1}^1 + p_{21,1}^1) + \phi^2 p_{22,1}^1 \end{cases} \quad (3.38)$$

$$w_i^k(0) = h_i^k(\bar{w}^1, \bar{w}^2, \lambda)/q(\bar{w}^1, \bar{w}^2, \lambda), \quad (3.39)$$

where

$$h_1^1 = 1/\gamma, \quad h_2^1 = \phi(\bar{w}^1, \bar{w}^2, \lambda)/\gamma, \quad h_1^2 = 1, \quad h_2^2 = \phi(\bar{w}^1, \bar{w}^2, \lambda).$$

Thus the function

$$\frac{p_*(\bar{\lambda})}{p^*(\bar{\lambda})} \leq \frac{p_*(\lambda)}{p^*(\lambda)} = \frac{m_* b_*^1 \Omega_*}{m^* b^{1*} \Omega^*} \exp\{-2\tau(\nu^* - \nu_*)\} \left(\frac{\lambda + \nu_*}{\lambda + \nu^* + 2m^*}\right)^2 \frac{\lambda + 2\nu_*}{\lambda + \sigma^* + 2\nu^*}$$

$$\rightarrow \frac{m_* b_*^1 \Omega_*}{m^* b^{1*} \Omega^*} \exp\{-2\tau(\nu^* - \nu_*)\}$$

grows monotonically as  $\lambda \rightarrow \infty$ , where  $\bar{\lambda} = \epsilon - \nu_*$ ,  $\epsilon > 0$ . This shows that, for  $\lambda \in (\bar{\lambda}, \infty)$ , all  $h_i^k(\bar{w}^1, \bar{w}^2, \lambda)$  have positive lower and finite upper bounds

$$h_* = \frac{p_*(\bar{\lambda})}{p^*(\bar{\lambda})} \min(1, 1/\gamma), \quad h^* = \frac{p^*(\bar{\lambda})}{p_*(\bar{\lambda})} \max(1, \gamma). \tag{3.40}$$

**Lemma 4.** *Under the conditions of Lemma 1 and Case 2 the system (3.25) has only one positive solution defined by Eqs. (3.38) and (3.39).*

Now we construct the operator  $F^\lambda : D \rightarrow D$ . From (3.15), (3.18), (3.28), (3.31), and (3.39) we have

$$w_i^k(\tau) = \frac{f_i^k(\tau, \lambda) h_i^k(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda)}.$$

From here and by Eq. (3.18) we conclude that  $(\bar{w}^1, \bar{w}^2)$  must be a solution of the system

$$w_i^k = \frac{z_i^k f_i^k(\tau, \lambda) h_i^k(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda)}, \quad i = 1, \dots, n, \quad k = 1, 2. \tag{3.41}$$

Set

$$M_i^k(\bar{w}^1, \bar{w}^2, \lambda) = \frac{z_i^k(\cdot, \bar{w}^s, \lambda) f_i^k(\tau, \lambda) h_i^k(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda) \tilde{\rho}(\lambda)},$$

$$\bar{M}^k(\bar{w}^1, \bar{w}^2, \lambda) = (M_1^k(\bar{w}^1, \bar{w}^2, \lambda), \dots, M_n^k(\bar{w}^1, \bar{w}^2, \lambda))$$

with

$$\tilde{\rho}(\lambda) = \rho(\bar{w}^1, \bar{w}^2, \lambda) = \frac{\sum_{k=1}^2 \sum_{i=1}^n \|z_i^k(\cdot, \bar{w}^s, \lambda)\| f_i^k(\tau, \lambda) h_i^k(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda)} \tag{3.42}$$

and define the operator

$$F^\lambda : (\bar{w}^1, \bar{w}^2) \rightarrow (\bar{M}^1(\bar{w}^1, \bar{w}^2, \lambda), \bar{M}^2(\bar{w}^1, \bar{w}^2, \lambda)).$$

Here  $s = 1, 2$ ,  $s \neq k$  and  $\lambda > \bar{\lambda}$  is a free parameter. Obviously,  $F^\lambda : D \rightarrow D$ . Now we prove that  $F^\lambda$  is continuous in  $D$ . By Lemma 1 operator  $\tilde{F}^\lambda : (\bar{w}^1, \bar{w}^2) \rightarrow (\tilde{z}^1(\cdot, \bar{w}^2, \lambda), \tilde{z}^2(\cdot, \bar{w}^1, \lambda))$  for fixed  $\lambda > \bar{\lambda}$  is completely continuous in  $D$  and  $\tilde{F}^\lambda(D) \subset \tilde{D}$ . Then, for  $\lambda > \bar{\lambda}$ , the map  $(\bar{w}^1, \bar{w}^2) \rightarrow q(\bar{w}^1, \bar{w}^2, \lambda)$  is also continuous and  $q(\bar{w}^1, \bar{w}^2, \lambda) > 0$ . Thus, for  $\lambda > \bar{\lambda}$  we have that  $\tilde{\rho}(\lambda) > 0$  and the map  $(\bar{w}^1, \bar{w}^2) \rightarrow \rho(\bar{w}^1, \bar{w}^2, \lambda)$  is continuous, as well. All  $h_i^k$  are positive

and the map  $(\bar{w}^1, \bar{w}^2) \rightarrow h_i^k(\bar{w}^1, \bar{w}^2, \lambda)$  for fixed  $\lambda, k$  and  $i$  is also continuous. Therefore,  $F^\lambda$  is completely continuous and the Schauder fixed point principle yields the existence of at least one fixed point of  $F^\lambda$  in  $D$ . This means that the system

$$w_i^k = \frac{z_i^k(\cdot, \bar{w}^s, \lambda) f_i^k(\tau, \lambda) h_i^k(\bar{w}^1, \bar{w}^2, \lambda)}{q(\bar{w}^1, \bar{w}^2, \lambda) \rho(\bar{w}^1, \bar{w}^2, \lambda)}, \quad i = 1, \dots, n, \quad k, s = 1, 2, \quad s \neq k$$

with  $\lambda \geq \bar{\lambda}$  has at least one solution  $(\bar{w}^1(\cdot, \lambda), \bar{w}^2(\cdot, \lambda)) \in D$ . If there exists  $\tilde{\lambda} > \bar{\lambda}$  such that  $\tilde{\rho}(\tilde{\lambda}) = 1$ , then this system coincides with (3.41) for  $\lambda = \tilde{\lambda}$  and the function  $(\bar{w}^1(\cdot, \tilde{\lambda}), \bar{w}^2(\cdot, \tilde{\lambda}))$  represents a solution of (3.41).

Now we get conditions for the existence or nonexistence of  $\tilde{\lambda}$ . Set

$$h_* = \min_{k,i} h_i^k > 0, \quad h^* = \max_{k,i} h_i^k \text{ in case 1,}$$

$$p^1(\lambda) = \sup_{y \in [\tau, \infty)} \int_{\tau}^{\infty} z^{1*}(x, \lambda) dx \int_0^{\infty} \exp\{-\lambda \tau_3\} b(x + \tau_3, y + \tau_3, \tau_3) \\ \times K(x + \tau_3, y + \tau_3, \tau_3, 0) d\tau_3,$$

$$p^2(\lambda) = \sup_{x \in [\tau, \infty)} \int_{\tau}^{\infty} z^{2*}(y, \lambda) dy \int_0^{\infty} \exp\{-\lambda \tau_3\} b(x + \tau_3, y + \tau_3, \tau_3) \\ \times K(x + \tau_3, y + \tau_3, \tau_3, 0) d\tau_3,$$

$$\tilde{\rho}_*(\lambda) = \sum_{i=1}^n h_i^1 (f^2(\tau, \lambda) p^2(\lambda))^{-1} + \sum_{i=1}^n h_i^2 (f^1(\tau, \lambda) p^1(\lambda))^{-1} \text{ in case 1,}$$

$$\tilde{\rho}_*(\lambda) = \frac{2h_*}{m^* b^{1*} \Omega^*} \left( \frac{p_*}{p_* + p^*} \right)^2 (\lambda + \nu^*)^2 \frac{\lambda + \nu_*}{\lambda + \nu^* + 2m^*} \text{ in case 2,}$$

$$\tilde{\rho}^*(\lambda) = \sum_{i=1}^n \sum_{k=1}^2 h_i^k f_i^k(\tau, \lambda) \|z^{k*}\| / q_*(\lambda) \text{ in case 1,}$$

$$\tilde{\rho}^*(\lambda) = \sum_{i=1}^2 \sum_{k=1}^2 h^* f_i^k(\tau, \lambda) \|z^{k*}\| / q_*(\lambda) \text{ in case 2,}$$

$$q_*(\lambda) = f^1(\tau, \lambda) f^2(\tau, \lambda) \int_{\tau}^{\infty} z_*^1(x, \lambda) dx \int_{\tau}^{\infty} z_*^2(y, \lambda) dy \int_0^{\infty} \exp\{-\lambda \tau_3\} \\ \times b(x + \tau_3, y + \tau_3, \tau_3) K(x + \tau_3, y + \tau_3, \tau_3, 0) d\tau_3 \text{ in case 1,}$$

$$q_*(\lambda) = p_{11,1}^1 + \frac{p_*}{p^*} (p_{12,1}^1 + p_{21,1}^1) + \left( \frac{p_*}{p^*} \right)^2 p_{22,1}^1 \text{ in case 2,}$$

$$q^*(\lambda) = f^1(\tau, \lambda) f^2(\tau, \lambda) \int_{\tau}^{\infty} \int_0^{\infty} \exp\{-\lambda \tau_3\} z^{1*}(x, \lambda) dx \int_{\tau}^{\infty} z^{2*}(y, \lambda) dy \\ \times b(x + \tau_3, y + \tau_3, \tau_3) K(x + \tau_3, y + \tau_3, \tau_3, 0) d\tau_3 \text{ in case 1,}$$

$$q^*(\lambda) = p_{11,1}^{1*} + \frac{p_*}{p^*} (p_{12,1}^{1*} + p_{21,1}^{1*}) + \left( \frac{p_*}{p^*} \right)^2 p_{22,1}^{1*} \text{ in case 2.}$$

Here

$$p_{s_j, i^*}^1 = f_s^1(\tau, \lambda) f_j^2(\tau, \lambda) \int_{\tau}^{\infty} z_{s^*}^1(x, \lambda) dx \int_{\tau}^{\infty} z_{j^*}^2(y, \lambda) \kappa_{s_j, 1}^1(x, y, \lambda) dy,$$

$$p_{s_j, i^*}^{1*} = f_s^1(\tau, \lambda) f_j^2(\tau, \lambda) \int_{\tau}^{\infty} z_s^{1*}(x, \lambda) dx \int_{\tau}^{\infty} z_j^{2*}(y, \lambda) \kappa_{s_j, 1}^1(x, y, \lambda) dy.$$

In Case 2, constants  $h_*$  and  $h^*$  are defined by (3.40). Equations (3.27), (3.38), and (3.39) show that  $\tilde{\rho}(\lambda) \geq \tilde{\rho}_*(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ . If  $\tilde{\rho}(\bar{\lambda}) < 1$ , then there exists  $\tilde{\lambda} > \bar{\lambda}$ , such that  $\tilde{\rho}(\tilde{\lambda}) = 1$ , since  $\tilde{\rho} \in C^0(\bar{\lambda}, \infty)$ . If  $\min_{\lambda \in [\bar{\lambda}, \lambda_1]} \tilde{\rho}(\lambda) > 1$ , where  $\lambda_1$  is a unique root of  $\tilde{\rho}_*(\lambda) = 1$ , then there is no such  $\tilde{\lambda} > \bar{\lambda}$  that  $\tilde{\rho}(\tilde{\lambda}) = 1$

**Theorem 1.** *Let conditions of one of Lemmas 2, 3, and 4 be satisfied. Then the following assertions are true:*

- If  $\tilde{\rho}(\bar{\lambda}) \leq 1$ , then system (3.7), (3.9), (3.10) has at least one positive solution  $(\bar{w}^1(\tau_1, \lambda), \bar{w}^2(\tau_2, \lambda))$  with  $\lambda \geq \bar{\lambda}$ , such that  $w_i^k \in C^0(\bar{Q}^k) \cap C^1(Q^k) \cap L^1(0, \infty)$ ,

$$z_*^k(\tau_k, \tilde{\lambda}) f^k(\tau, \tilde{\lambda}) h_*/q^*(\tilde{\lambda}) \leq w_i^k(\tau_k, \lambda) \leq z^{k*}(\tau_k, \tilde{\lambda}) f^k(\tau, \tilde{\lambda}) h^*/q_*,$$

where  $k = 1, 2, i = 1, \dots, n$ . If conditions of Lemma 3 are satisfied and  $n = 2, \Delta_1 = \Delta_2 = 0$ , then this solution depends on one positive parameter (see (3.37)).

- If  $\min_{\lambda \in [\bar{\lambda}, \lambda_1]} \tilde{\rho}(\lambda) > 1$  with a unique root,  $\lambda_1$  of  $\tilde{\rho}_*(\lambda) = 1$ , then system (3.7), (3.9), (3.10) has no positive solutions for  $\lambda \geq \bar{\lambda}$ .

Obviously, conditions of this theorem are satisfied if the inequalities  $\tilde{\rho}^*(\bar{\lambda}) < 1$  and  $\tilde{\rho}_*(\bar{\lambda}) > 1$  are satisfied, respectively.

**Theorem 2.** *Let conditions of Theorem 1 be satisfied and let in addition  $m_{is} \in C^1([\tau, \infty)^2)$ ,  $\sigma_{is}$  and  $\nu_{is}^k \in C^{1,1,0}(\bar{Q}^3)$ . Then the following assertions are true:*

- Under condition  $\tilde{\rho}(\bar{\lambda}) \leq 1$  system (2.1)–(2.7) has at least one class of positive solutions of type (3.1) which depends on parameter  $f$  and posses the properties:

$$U_i^k \in C^0(\bar{Q}^k) \cap C^1(Q^k), \quad k = 1, 2, \quad i = 1, \dots, n, \quad U_{ij}^3 \in C^0(\bar{Q}^3) \cap C^1(Q^3),$$

and  $\lambda = \tilde{\lambda} \geq \bar{\lambda}$ . Under the conditions of Lemma 3 and  $n = 2, \Delta_1 = \Delta_2 = 0$  this class depends on two parameters (see (3.37)).

- Under condition  $\min_{\lambda \in [\bar{\lambda}, \lambda_1]} \tilde{\rho}(\lambda) > 1$  system (2.1)–(2.7) has no nontrivial separable solutions of type (3.1) for  $\lambda \geq \bar{\lambda}$ .

Definition of  $q$  and  $\rho$  shows that separable solutions exist if pairs produce sufficiently large number of offsprings.

### 4 The Case of Constant Vital Rates

In this section, we consider the case where vital rates satisfy the following conditions:

(H)  $\nu_i^k$  and  $\nu_{sj}^k \forall k, i, s,$  and  $j$  are positive constants and do not depend on the sex and confession number,  $b_{sj}^k, \forall k, s, j$  are positive constants and do not depend on  $k, m_{sj}$  and  $\sigma_{sj}, \forall s$  and  $j$  are positive constants and do not depend on the confession number,  $\Omega_{sj,i}$  are a positive constants.

Then  $w_i^k$  is also independent of  $k$ . Set

$$\sigma = \sigma_{sj}, b_{sj} = b_{sj}^k, \nu = \nu_i^k = \nu_{sj}^k, m = m_{sj}, w_i = w_i^k.$$

Equations (3.12), (3.13), and (3.14) can be written in the form

$$\begin{aligned} \frac{dw_i}{d\tau} &= -(\nu + m + \lambda)w_i + m(\nu + \sigma) \exp\{-(2\nu + \sigma + \lambda)\tau_1\} \\ &\times \int_{\tau}^{\tau_1} \exp\{(2\nu + \sigma + \lambda)x\}w_i(x) dx, \quad \tau_1 > \tau, \end{aligned} \tag{4.1}$$

$$w_i(\tau) = w_i(0) \exp\{-(\nu + \lambda)\tau\}, \tag{4.2}$$

$$\begin{aligned} w_i(0) &= \frac{m}{2\nu + \sigma + \lambda} \sum_{s,j=1}^n b_{sj} \Omega_{sj,i} \int_{\tau}^{\infty} w_s(x) dx \int_{\tau}^{\infty} w_j(y) dy, \\ \lambda + 2\nu + \sigma &> 0. \end{aligned} \tag{4.3}$$

Letting  $z_i(\tau_1) = w_i(\tau_1) \exp\{(2\nu + \sigma + \lambda)\tau_1\}$ , we get the integro-differential equation

$$z_i' - (\nu + \sigma - m)z_i - m(\nu + \sigma) \int_{\tau}^{\tau_1} z_i(x) dx = 0,$$

which by Eq. (4.2) reduces to

$$\begin{aligned} z_i'' + (m - \nu - \sigma)z_i' - m(\nu + \sigma)z_i &= 0, \quad \tau_1 > \tau = 0, \\ z_i(\tau) = w_i(0) \exp\{(\nu + \sigma)\tau\}, \quad z_i'(\tau) &= (\nu + \sigma - m)z_i(\tau). \end{aligned}$$

This problem has the solution

$$z_i(\tau_1) = \frac{w_i(0)}{\nu + \sigma + m} ((\nu + \sigma) \exp\{(\nu + \sigma)\tau_1\} + m \exp\{-m\tau_1 + (\nu + \sigma + m)\tau\}).$$

Thus

$$\frac{w_i(\tau_1)}{w_i(0)} = \frac{\exp\{-(\nu + \lambda)\tau_1\}}{\nu + \sigma + m} (\nu + \sigma + m \exp\{-(\nu + \sigma + m)(\tau_1 - \tau)\}). \tag{4.4}$$

Then

$$\begin{aligned} \int_{\tau}^{\infty} w_i(x) dx &= w_i(0)A(\lambda), \\ A(\lambda) &= \frac{\exp\{-(\nu + \lambda)\tau\}}{\nu + \sigma + m} \left( \frac{\nu + \sigma}{\nu + \lambda} + \frac{m}{2\nu + \sigma + m + \lambda} \right), \quad \lambda + \nu > 0. \end{aligned} \tag{4.5}$$

Now from (4.3) it follows that

$$w_i(0) = \frac{mA(\lambda)^2}{2\nu + \sigma + \lambda} \sum_{s,j=1}^n b_{sj} \Omega_{sj,i} w_s(0) w_j(0). \tag{4.6}$$

System (4.6) is of the form (3.26) with  $q = \frac{mA(\lambda)^2}{2\nu + \sigma + \lambda}$  and  $a_{sj,i}^k = b_{sj} \Omega_{sj,i}$  and therefore has at least one positive solution

$$w_i(0) = \frac{h_i(2\nu + \sigma + \lambda)}{mA(\lambda)^2}, \quad h_i = \frac{\phi_i}{\sum_{s,j=1}^n \phi_s \phi_j b_{sj} \Omega_{sj,1}}, \quad i = 1, \dots, n; \quad \phi_1 = 1. \tag{4.7}$$

By using  $2 \sum_{i=1}^n \int_{\tau}^{\infty} w_i(x) dx = 1$  and (4.5)–(4.7) we get the equation for  $\lambda$ :

$$2 \sum_{i=1}^n h_i(2\nu + \sigma + \lambda)/m = A(\lambda), \tag{4.8}$$

which has a root

$$\begin{aligned} -\nu < \tilde{\lambda} < 0, & \quad \text{if} \quad 2 \sum_{i=1}^n h_i(2\nu + \sigma)/m > A(0) := \frac{(2\nu + \sigma) \exp\{-\nu\tau\}}{\nu(2\nu + \sigma + m)}, \\ \tilde{\lambda} = 0, & \quad \text{if} \quad 2 \sum_{i=1}^n h_i(2\nu + \sigma)/m = A(0), \\ \tilde{\lambda} > 0, & \quad \text{if} \quad 2 \sum_{i=1}^n h_i(2\nu + \sigma)/m < A(0). \end{aligned}$$

**Theorem 3.** *Under the hypothesis (H) system (3.2)–(3.7) has the unique solution of the form (4.4) where  $(w_1(0), \dots, w_n(0))$  is defined by Eqs. (4.7) with at least one set  $(h_1, \dots, h_n)$  of positive constants  $h_i, i = 1, \dots, n$ . For each set  $(h_1, \dots, h_n)$ , the unique  $\tilde{\lambda}$  is determined by Eq. (4.8).*

### 5 Proofs of Lemmas 1 and 2

*Proof of Lemma 1.* We first prove that  $G_i^1(\tau_1, x, \lambda) \leq 2m^* \|\bar{w}^2\|$  and is continuous for sufficiently large  $\lambda$ . For  $\lambda > -\nu_*$  and  $\bar{w}^2 \in L^1_+(\tau, \infty)$ , by definition we get

$$\begin{aligned} G_i^1(\tau_1, x, \lambda) &= \int_x^{\tau_1} R_i^1(\eta, x, \bar{w}^2, \lambda) \exp \left\{ - \int_{\eta}^{\tau_1} l_i^1(\xi, \lambda) d\xi \right\} d\eta \leq \int_x^{\tau_1} R_i^1(\eta, x, \bar{w}^2, \lambda) d\eta \\ &= \int_x^{\tau_1} d\eta \int_{\tau}^{\infty} \sum_{s=1}^n w_s^2(y) K_{is}(x, y, \eta - x, \lambda) (\nu_{is}^2 + \sigma_{is}) \Big|_{(\eta, y+\eta-x, \eta-x)} dy \\ &= \int_{\tau}^{\infty} \sum_{s=1}^n w_s^2(y) dy \int_x^{\tau_1} K_{is}(x, y, \eta - x, \lambda) (\nu_{is}^2 + \sigma_{is}) \Big|_{(\eta, y+\eta-x, \eta-x)} d\eta \end{aligned}$$

$$\begin{aligned} &\leq \int_{\tau}^{\infty} \sum_{s=1}^n 2m_{is}(x, y)w_s^2(y) dy \int_x^{\tau_1} \exp\left\{-\int_0^{\eta-x} (\nu_{is}^2 + \sigma_{is})\Big|_{(\xi+x, \xi+y, \xi)} d\xi\right\} \\ &\times (\nu_{is}^2 + \sigma_{is})\Big|_{(\eta, y+\eta-x, \eta-x)} d\eta \leq \sum_{s=1}^n \int_{\tau}^{\infty} w_s^2(y)2m_{is}(x, y) dy \leq 2m^* \|\bar{w}^2\|. \end{aligned}$$

Hence,

$$G_i^1(\tau_1, x, \lambda) \leq \int_x^{\tau_1} g_i^1(\eta, x, \lambda)d\eta \leq 2m^* \|\bar{w}^2\| \tag{5.1}$$

for  $\lambda > -\nu_*$  and  $\bar{w}^2 \in L^1_+(\tau, \infty)$ , and

$$G_i^2(x, \tau_2, \lambda) \leq \int_x^{\tau_2} g_i^2(x, \eta, \lambda)d\eta \leq 2m * \|\bar{w}^1\| \tag{5.2}$$

for  $\lambda > -\nu_*$  and  $\bar{w}^1 \in L^1_+(\tau, \infty)$ . It is easy to see that, due to hypotheses of Lemma 1,  $G_i^1(\tau_1, x, \lambda)$ ,  $l_i^1(\tau_1, \lambda)$ ,  $G_i^2(x, \tau_2, \lambda)$ ,  $l_i^2(\tau_2, \lambda)$ , and  $l_i^2(\tau_2, \lambda)$  are continuous functions. Therefore, Volterra equations (3.23) and (3.24) have a unique positive global solution. Obviously, it satisfies Eqs. (3.19) and (3.20) and  $z_i^k \in C^0([\tau, \infty)) \cap C^1((\tau, \infty))$  for  $\lambda > -\nu_*$ .

Now we prove inequalities  $z_{i*}^k \leq z_i^k \leq z_i^{k*}$ . Let  $(\bar{w}^1, \bar{w}^1) \in D$ . The lower estimate of  $z_i^k$  follows easily from Eqs. (3.25) and (3.26). It remains to prove the upper estimate. Define

$$\begin{aligned} Z_i^1(\tau_1) &= z_i^1(\tau_1) + \int_{\tau}^{\tau_1} z_i^1(x)dx \int_{\tau}^{\infty} \sum_{s=1}^n w_s^2(y)K_{is}(x, y, \tau_1 - x, \lambda) dy, \quad Z_i^1(\tau) = 1, \\ Z_i^2(\tau_2) &= z_i^2(\tau_2) + \int_{\tau}^{\tau_2} z_i^2(x)dx \int_{\tau}^{\infty} \sum_{s=1}^n w_s^1(y)K_{si}(y, x, \tau_2 - x, \lambda) dy, \quad Z_i^2(\tau) = 1. \end{aligned}$$

Taking into account Eqs. (3.19) and (3.20) and the definition of  $K_{is}$ ,  $r_i^k$ ,  $l_i^k$ ,  $R_i^k$ , and  $g_i^k$  we get

$$\begin{aligned} \frac{dZ_i^1}{d\tau_1} &= \frac{dz_i^1}{d\tau_1} + z_i^1 \int_{\tau}^{\infty} \sum_{s=1}^n w_s^2(y)K_{is}(\tau_1, y, 0, \lambda) dy \\ &- \int_{\tau}^{\tau_1} z_i^1(x) dx \int_{\tau}^{\infty} \sum_{s=1}^n w^2(y)K_{is}(x, y, \tau_1-x, \lambda)(\lambda + \nu_{is}^1 + \nu_{is}^2 + \sigma_{is})\Big|_{(\tau_1, \tau_1+y-x, \tau_1-x)} dy \\ &= -(\lambda + \nu_i^1 + r_i^1)z_i^1 + \int_{\tau}^{\tau_1} z_i^1(x)g_i^1(\tau_1, x, \lambda) dx + z_i^1 r_i^1(\tau_1, \bar{w}^2) \end{aligned}$$

$$\begin{aligned}
 & - \int_{\tau}^{\tau_1} z_i^1(x) dx \int_{\tau}^{\infty} \sum_{s=1}^n w^2(y) K_{is}(x, y, \tau_1 - x, \lambda) (\lambda + \nu_{is}^1(\tau_1, \tau_1 + y - x, \tau_1 - x)) dy \\
 & - \int_{\tau}^{\tau_1} dz_i^1(x) R_i^1(\tau_1, x, \bar{w}^2, \lambda) dx = -(\lambda + \nu_i^1) z_i^1 \\
 & - \int_{\tau}^{\tau_1} z_i^1(x) dx \int_{\tau}^{\infty} \sum_{s=1}^n w^2(y) K_{is}(x, y, \tau_1 - x, \lambda) (\lambda + \nu_{is}^1(\tau_1, \tau_1 + y - x, \tau_1 - x)) dy \\
 & \leq -(\lambda + \nu_*) Z_i^1, \quad Z_i^1(\tau) = 1.
 \end{aligned}$$

Hence,  $z_i^1 \leq Z_i^1 \leq \exp\{-(\lambda + \nu_*)(\tau_1 - \tau)\}$  and similarly  $z_i^2 \leq Z_i^2 \leq \exp\{-(\lambda + \nu_*)(\tau_2 - \tau)\}$ .

Now we find the upper estimate for  $\|dz_i^k/d\tau_k\|$ . Define

$$\Pi_i^k(\tau_k, \lambda) = \exp \left\{ - \int_{\tau}^{\tau_k} l_i^k(x, \lambda) dx \right\}.$$

Since

$$\frac{d\Pi_i^k(\tau_k, \lambda)}{d\tau_k} = -l_i^k(\tau_k, \lambda) \Pi_i^k(\tau_k, \lambda) < 0 \quad \text{for } \lambda > -\nu_*,$$

from (3.23) it follows that

$$\begin{aligned}
 \left| \frac{dz_i^1}{d\tau_1} \right| & \leq - \frac{d\Pi_i^1}{d\tau_1} \left\{ 1 + \int_{\tau}^{\tau_1} (\Pi_i^1(\eta, \lambda))^{-1} d\eta \int_{\tau}^{\eta} z_i^1(x) g_i^1(\eta, x, \lambda) dx \right\} \\
 & \quad + \int_{\tau}^{\tau_1} z_i^1(x) g_i^1(\tau_1, x, \lambda) dx.
 \end{aligned}$$

Then we get

$$\begin{aligned}
 \left\| \frac{dz_i^1}{d\tau_1} \right\| & \leq 1 - \int_{\tau}^{\infty} \frac{d\Pi_i^1}{d\tau_1} d\tau_1 \int_{\tau}^{\tau_1} (\Pi_i^1(\eta, \lambda))^{-1} d\eta \int_{\tau}^{\eta} z_i^1(x) g_i^1(\eta, x, \lambda) dx \\
 & \quad + \int_{\tau}^{\infty} d\tau_1 \int_{\tau}^{\tau_1} z_i^1(x) g_i^1(\tau_1, x, \lambda) dx = 1 + \int_{\tau}^{\infty} d\tau_1 \int_{\tau}^{\tau_1} z_i^1(x) g_i^1(\tau_1, x, \lambda) dx \\
 & \quad - \int_{\tau}^{\infty} (\Pi_i^1(\eta, \lambda))^{-1} \int_{\eta}^{\infty} \frac{d\Pi_i^1(\tau_1, \lambda)}{d\tau_1} d\tau_1 \int_{\tau}^{\eta} z_i^1(x) g_i^1(\eta, x, \lambda) dx \\
 & = 1 + 2 \int_{\tau}^{\infty} d\tau_1 \int_{\tau}^{\tau_1} z_i^1(x) g_i^1(\tau_1, x, \lambda) dx \\
 & = 1 + 2 \int_{\tau}^{\infty} z_i^1(x) dx \int_x^{\infty} g_i^1(\tau_1, x, \lambda) d\tau_1, \quad \lambda > -\nu_*.
 \end{aligned}$$

Similarly,

$$\left\| \frac{dz_i^2}{d\tau_2} \right\| \leq 1 + 2 \int_{\tau}^{\infty} z_i^2(x) dx \int_x^{\infty} g_i^2(x, \tau_2, \lambda) d\tau_2, \quad \lambda > -\nu_*.$$

From here by estimates (5.1) and (5.2) we get

$$\left\| \frac{dz_i^j}{d\tau_1} \right\| \leq 1 + 4m^* \|z_i^j\| \leq 1 + \frac{4m^*}{\lambda + \nu_*}, \quad j = 1, 2,$$

since  $(\bar{w}^1, \bar{w}^2) \in D$  and  $\lambda > -\nu_*$ . Therefore, by the Riesz [6] criterion set  $\tilde{D}$  is compact.

It remains to prove the continuity of the operator  $\tilde{F}^\lambda$  in  $D$ . Let  $(\bar{w}^{11}, \bar{w}^{21})$  and  $(\bar{w}^{12}, \bar{w}^{22}) \in D$ . Let corresponding pairs be  $(\bar{z}^1(\cdot, \bar{w}^{21}), \bar{z}^2(\cdot, \bar{w}^{11}))$  and  $(\bar{z}^1(\cdot, \bar{w}^{22}), \bar{z}^2(\cdot, \bar{w}^{12}))$ . Set

$$\Delta \bar{w}^k = \bar{w}^{k2} - \bar{w}^{k1}, \quad \Delta z_i^k = z_i^k(\cdot, \bar{w}^{22}, \cdot) - z_i^k(\cdot, \bar{w}^{21}, \cdot), \quad k = 1, 2.$$

From Eqs. (3.21) and (3.22) it follows that

$$\begin{aligned} \frac{d\Delta z_i^1}{d\tau_1} &= -(\lambda + \nu_i^1(\tau_1) + r_i^1(\tau_1, \bar{w}^{22}))\Delta z_i^1 - z_i^{11}r_i^1(\tau_1, \Delta \bar{w}^2) \\ &+ \int_{\tau}^{\tau_1} \Delta z_i^1(x)R_i^1(\tau_1, x, \bar{w}^{22}, \lambda)dx + \int_{\tau}^{\tau_1} z_i^{11}(x)R_i^1(\tau_1, x, \Delta \bar{w}^2, \lambda)dx, \quad \Delta z_i^1(\tau) = 0, \\ \frac{d\Delta z_i^2}{d\tau_2} &= -(\lambda + \nu_i^2(\tau_2) + r_i^2(\tau_2, \bar{w}^{12}))\Delta z_i^2 - z_i^{21}r_i^2(\tau_2, \Delta \bar{w}^1) \\ &+ \int_{\tau}^{\tau_2} \Delta z_i^2(x)R_i^2(x, \tau_2, \bar{w}^{12}, \lambda)dx + \int_{\tau}^{\tau_2} z_i^{21}(x)R_i^2(x, \tau_2, \Delta \bar{w}^1, \lambda)dx, \quad \Delta z_i^2(\tau) = 0. \end{aligned}$$

Hence

$$\begin{aligned} \Delta z_i^1 &= \int_{\tau}^{\tau_1} \exp\left\{-\int_{\eta}^{\tau_1} (\lambda + \nu_i^1(y) + r_i^1(y, \bar{w}^{22}))dy\right\} \left\{ \int_{\tau}^{\eta} [\Delta z_i^1(x)R_i^1(\eta, x, \bar{w}^{22}, \lambda) \right. \\ &\quad \left. + z_i^{11}(x)R_i^1(\eta, x, \Delta \bar{w}^2, \lambda)] dx - z_i^{11}(\eta)r_i^1(\eta, \Delta \bar{w}^2) \right\} d\eta \end{aligned}$$

or

$$\begin{aligned} \Delta z_i^1 &= \int_{\tau}^{\tau_1} dx \int_x^{\tau_1} \exp\left\{-\int_{\eta}^{\tau_1} (\lambda + \nu_i^1(y) + r_i^1(y, \bar{w}^{22}))dy\right\} \\ &\quad \times [\Delta z_i^1(x)R_i^1(\eta, x, \bar{w}^{22}, \lambda) + z_i^{11}(x)R_i^1(\eta, x, \Delta \bar{w}^2, \lambda)] d\eta \\ &\quad - \int_{\tau}^{\tau_1} \exp\left\{-\int_{\eta}^{\tau_1} (\lambda + \nu_i^1(y) + r_i^1(y, \bar{w}^{22}))\right\} z_i^{11}(\eta)r_i^1(\eta, \Delta \bar{w}^2) d\eta. \end{aligned}$$

From here by using Eqs. (5.1) and (3.15), for  $\lambda > -\nu_*$ , we get the inequality

$$|\Delta z_i^1| \leq 2m^* \int_{\tau}^{\tau_1} |\Delta z_i^1(x)| dx + 2m^* \|\Delta \bar{w}^2\| \frac{1 + e^{-1}}{\lambda + \nu_*},$$

which shows that

$$|\Delta z_i^1| \leq \frac{2m^*(1 + e^{-1})}{\lambda + \nu_*} \|\bar{w}^2\| \exp\{2m^*(\tau_1 - \tau)\}$$

and similarly,

$$|\Delta z_i^2| \leq \frac{2m^*(1 + e^{-1})}{\lambda + \nu_*} \|\bar{w}^1\| \exp\{2m^*(\tau_2 - \tau)\}.$$

But  $|\Delta z_i^1| \leq 2z_i^{1*} = 2 \exp\{-(\lambda + \nu_*)(\tau_1 - \tau)\}$  for  $\lambda > -\nu_*$ . Hence,

$$\begin{aligned} \|\Delta z_i^1\| &\leq \int_{\tau}^{\infty} |\Delta z_i^1| d\tau_1 \leq \frac{2m^*(1 + e^{-1})}{\lambda + \nu_*} \|\bar{w}^2\| \int_{\tau}^{\tilde{\tau}_*(\lambda)} \exp\{2m^*(\tau_1 - \tau)\} d\tau_1 \\ &+ 2 \int_{\tilde{\tau}_*(\lambda)}^{\infty} \exp\{2m^*(\tau_1 - \tau)\} d\tau_1 = c_1(\lambda) \|\Delta \bar{w}^2\| + c_2(\lambda), \end{aligned}$$

where

$$\begin{aligned} c_1(\lambda) &= \frac{1 + e^{-1}}{\lambda + \nu_*} (\exp\{2m^*(\tilde{\tau}_*(\lambda) - 1)\} - 1) \leq c_1(\bar{\lambda}), \\ c_2(\lambda) &= \frac{2}{\lambda + \nu_*} \exp\{-(\lambda + \nu_*)(\tilde{\tau}_*(\lambda) - \tau)\} \leq c_2(\bar{\lambda}), \quad \lambda > \bar{\lambda}. \end{aligned}$$

Choose  $\tilde{\tau}_*(\lambda)$  and  $\|\Delta \bar{w}^2\|$  such that  $c_2(\lambda) < \epsilon_1/2$  for  $\lambda \geq \bar{\lambda}$  and  $\|\Delta \bar{w}^2\| < \epsilon_1/2c_1(\bar{\lambda})$ ,  $\epsilon_1 > 0$ . Then  $\|\Delta z_i^2\| \leq \epsilon_1$  for  $\lambda \geq \bar{\lambda}$ . Similarly,  $\|\Delta z_i^1\| \leq \epsilon_1$  for  $\lambda \geq \bar{\lambda}$ . Therefore, operator  $\tilde{F}^\lambda$  is continuous in  $D$  for  $\lambda \geq \bar{\lambda}$ . Thus, we conclude that operator  $\tilde{F}^\lambda$  is completely continuous. The proof of Lemma 1 is completed.

*Proof of Lemma 2.* Letting  $w_i^k(0) = w_1^1(0)\phi_i^k$ , from Eq. (3.26) we get

$$w_1^1(0) = \left\{ \sum_{i,j} \phi_s^1 \phi_j^2 a_{s,j,1}^1 q(\bar{w}^1, \bar{w}^2, \lambda) \right\}^{-1}, \tag{5.3}$$

$$\phi_1^1 = 1, \quad \phi_i^k = \frac{\sum_{s,j=1}^n \phi_s^1 \phi_j^2 a_{s,j,i}^k}{\sum_{s,j=1}^n \phi_s^1 \phi_j^2 a_{s,j,1}^1}, \quad i = 2, \dots, n, \quad k = 1, 2. \tag{5.4}$$

We look for positive  $\phi_s^1, \phi_j^2 \forall s, j$ . We consider the case  $a_{s,j,i}^k > 0$ , i.e.  $\Omega_{s,j,i} > 0 \forall s, j, k$ . It is evident that  $\phi_i^k \in (\phi_*^k, \phi^{k*})$ ,  $i = 2, \dots, n$ , where

$$\phi_*^k = \min_{s,j,i} a_{s,j,i}^k / \max_{s,j,i} a_{s,j,1}^k, \quad \phi^{k*} = \max_{s,j,i} a_{s,j,i}^k / \min_{s,j,i} a_{s,j,1}^k.$$

From (5.4) with  $k = 1$  and  $i = 2$  we get the quadratic equation for  $\phi_2^1$ ,

$$\phi_2^1 \left( \phi_2^1 \sum_{j=1}^n \phi_j^2 a_{2j,1}^1 + \sum_{s \neq 2} \sum_{j=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1 \right) = \phi_2^1 \sum_{j=1}^n \phi_j^2 a_{2j,2}^1 + \sum_{s \neq 2} \sum_{j=1}^n \phi_s^1 \phi_j^2 a_{sj,2}^1,$$

which has a unique positive continuous solution  $\phi_2^1 = \Phi_2^1(\phi_3^1, \dots, \phi_n^2)$ . Letting

$$y_3^1(\phi_3^1; \Phi_2^1, \phi_4^1, \dots, \phi_n^2) = (\phi_3^1)^2 \sum_{j=1}^n \phi_j^2 a_{3j,1}^1 + \phi_3^1 \sum_{s \neq 3} \sum_{j=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1 \Big|_{\phi_2^1 = \Phi_2^1(\phi_3^1, \dots, \phi_n^2)},$$

$$z_3^1(\phi_3^1; \Phi_2^1, \phi_4^1, \dots, \phi_n^2) = \phi_3^1 \sum_{j=1}^n \phi_j^2 a_{3j,3}^1 + \sum_{s \neq 3} \sum_{j=1}^n \phi_s^1 \phi_j^2 a_{sj,3}^1 \Big|_{\phi_2^1 = \Phi_2^1(\phi_3^1, \dots, \phi_n^2)}$$

from (5.4) we derive the equation for  $\phi_3^1$ :

$$y_3^1(\phi_3^1; \Phi_2^1(\phi_3^1, \phi_4^1, \dots, \phi_n^2), \phi_4^1, \dots, \phi_n^2) = z_3^1(\phi_3^1; \Phi_2^1(\phi_3^1, \dots, \phi_n^2), \phi_4^1, \dots, \phi_n^2). \tag{5.5}$$

Since

$$y_3^1(\phi_3^1; \phi_*^1, \phi_4^1, \dots, \phi_n^2) \leq y_3^1(\phi_3^1; \Phi_2^1(\phi_3^1, \dots, \phi_n^2), \phi_4^1, \dots, \phi_n^2) \leq y_3^1(\phi_3^1; \phi^{1*}, \phi_4^1, \dots, \phi_n^2),$$

$$z_3^1(\phi_3^1; \phi_*^1, \phi_4^1, \dots, \phi_n^2) \leq z_3^1(\phi_3^1; \Phi_2^1(\phi_3^1, \dots, \phi_n^2), \phi_4^1, \dots, \phi_n^2) \leq z_3^1(\phi_3^1; \phi^{1*}, \phi_4^1, \dots, \phi_n^2),$$

and  $\Phi_2^1(\phi_3^1, \dots, \phi_n^2)$  is continuous, the graph analysis shows that Eq. (5.5) has at least one positive continuous solution  $\phi_3^1 = \Phi_3^1(\phi_4^1, \dots, \phi_n^2)$ . Therefore,

$$\phi_2^1 = \Phi_2^1(\Phi_3^1(\phi_4^1, \dots, \phi_n^2), \phi_4^1, \dots, \phi_n^2).$$

We proceed this argument getting from (5.4) with  $i = n$  the equation for  $\phi_n^1$ ,

$$\phi_n^1 \left( \phi_n^1 \sum_{j=1}^n \phi_j^2 a_{nj,1}^1 + \sum_{s=1}^{n-1} \sum_{j=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1 \right) = \phi_n^1 \sum_{j=1}^n \phi_j^2 a_{nj,n}^1 + \sum_{s=1}^{n-1} \sum_{j=1}^n \phi_s^1 \phi_j^2 a_{sj,n}^1,$$

where  $\phi_2^1, \phi_3^1, \dots, \phi_{n-1}^1$  depend only on  $\phi_n^1, \phi_1^2, \dots, \phi_n^2$  and are known continuous functions. By the argument applied to Eq.(5.5) we prove that this equation has at least one positive continuous solution  $\phi_n^1 = \Phi_n^1(\phi_1^2, \dots, \phi_n^2)$ . Knowing  $\phi_n^1$  we determine  $\phi_2^1, \dots, \phi_{n-1}^1$  as functions of  $\phi_1^2, \dots, \phi_n^2$ .

Now we consider (5.4) for  $k = 2$  and  $i = 1$  getting the equation for  $\phi_2^2$ :

$$\phi_2^2 \left( \phi_2^2 \sum_{s=1}^n \phi_s^1 a_{s1,1}^1 + \sum_{j=2}^n \sum_{s=1}^n \phi_s^1 \phi_j^2 a_{sj,1}^1 \right) = \phi_2^2 \sum_{s=1}^n \phi_s^1 a_{s1,1}^2 + \sum_{s=1}^n \sum_{j=2}^n \phi_s^1 \phi_j^2 a_{sj,1}^2, \tag{5.6}$$

where  $\phi_2^1, \dots, \phi_n^1$  are known functions of  $\phi_1^2, \dots, \phi_n^2$ . By the argument similar to that applied to (5.5) we prove that Eq. (5.6) has at least one positive continuous solution  $\phi_2^2 = \Phi_2^2(\phi_1^2, \dots, \phi_n^2)$ . This allows us to determine  $\phi_2^1, \dots, \phi_n^1$  as functions of  $\phi_1^2, \dots, \phi_n^2$ .

We proceed this argument getting, for  $i = n$ , the equation

$$\phi_n^2 \left( \phi_n^2 \sum_{s=1}^n \phi_s^1 a_{sn,1}^1 + \sum_{s=1}^n \sum_{j=1}^{n-1} \phi_s^1 \phi_j^2 a_{sj,1}^1 \right) = \phi_n^2 \sum_{s=1}^n \phi_s^1 a_{sn,n}^2 + \sum_{j=1}^{n-1} \sum_{s=1}^n \phi_s^1 \phi_j^2 a_{sj,n}^2, \tag{5.7}$$

where  $\phi_2^1, \dots, \phi_n^1, \phi_1^2, \phi_{n-1}^2$  are known continuous functions of  $\phi_n^2$ . By using the argument applied to Eq.(5.4) we prove that Eq.(5.7) has at least one positive continuous solution  $\phi_n^2$ . This allows us to determine  $\phi_2^1, \dots, \phi_{n-1}^2$ .

Thus, when  $a_{s_j,i}^k > 0, \forall s, j, k$ , system (3.26) has a positive solution

$$(w_1^1(0), w^1(0)\phi_2^1, \dots, w_1^1(0)\phi_n^1, w_1^1(0)\phi_1^2, \dots, w_1^1(0)\phi_n^2)$$

with at least one positive set  $(\phi_2^1, \dots, \phi_n^1, \phi_1^2, \dots, \phi_n^2)$  independent of  $\lambda$  and  $w_1^1(0)$  defined by Eq. (5.3). The proof of Lemma 2 is completed.

## 6 Concluding Remarks

The existence and non-existence of separable solutions are studied to age-sex – religion structured human communities model which forbid any confession change for individuals but let parents choose a religion not necessary their own for their offsprings. The proof of the existence of separable solutions is based on the Schauder fixed-point principle which does not let to prove the uniqueness.

The main problem in study of separable solutions to this model is to prove the existence theorem to system (3.25) and examine the asymptotic behaviour of its solution as  $\lambda$  tends to  $\infty$ . Therefore we have studied only some particular cases of this system. Examination of the general case is an open problem.

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