On Duffing equation with random perturbations

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Abstract. We consider a family of particles with different initial states and/or velocities whose dynamics is described by a modified Duffing equation with random perturbations. Sufficient conditions ensuring almost identical sample paths of the particles after a long time are given.

Keywords: Duffing equation, noise-induced synchronization, random perturbations.

1 Introduction

We consider the Duffing equation with random perturbations defined by

$$\begin{cases} \mathrm{d}X_t = V_t\,\mathrm{d}t, \quad t > 0, \\ \mathrm{d}V_t = \big(X_t - X_t^3\big)\mathrm{d}t - \alpha V_t\,\mathrm{d}t, \quad t \in (\tau_i, \tau_{i+1}), \ \ i = 0, 1, 2, \dots, \\ V_{\tau_i} = \gamma V_{\tau_i -} + \xi_i, \quad i = 1, 2, \dots, \\ X_0 = x, \ \ V_0 = v. \end{cases} \tag{1}$$

Here $\alpha > 0$ (a demping parameter) and $\gamma \in [0, 1)$ are constants,

$$V_{\tau_i -} := \lim_{t \uparrow \tau_i} V_t,$$

$0 = \tau_0 < \tau_1 < \dots$ is a sequence of random times and $\xi_1, \xi_2, \dots$ is a sequence of random perturbations. The system (1) represents the motion of a particle with initial position x and velocity v, the velocity V_t of which is perturbed by random perturbations ξ_i at random times τ_i.

The aim of the paper is to discuss the following problem. Suppose, we have a family of particles with different initial positions and/or velocities evolving according to (1). What conditions ensure that the sample paths of particles are almost identical after a long time?

In this paper sufficient conditions are given under which the sample paths of modified system (1) are almost identical after a long time. The modification of (1) attaches to the rest states of the system $(-1,0), (0,0), (1,0)$ an additional attraction force which compels a path close to the rest state to jump into the latter. Such an attraction force is always present in computer simulations of the system dynamics due to a rounding error. Many aspects of the Duffing equations can be found in [8] (see also references therein).

This phenomenon observed in computer simulations of various systems was widely discussed in physical literature. In the paper [1] particular examples of one-dimensional maps and the Lorenz system, both in the chaotic region were presented, and numerical evidence showing that the addition of a common noise to different trajectories, which start from different initial positions, leads eventually to their perfect synchronization was given. When the synchronization occurs, the largest Lyapunov exponent becomes negative. In the article [2] the mechanism behind the connection between the transition to chaos of random dynamical systems and the synchronization of chaotic maps driven by an external common noise was studied. A two-dimensional random dynamical system and two coupled logistic maps driven by external common noise were analyzed. In the article [3] the noise-induced synchronization in Lorenz systems was investigated. It was found that a common noise can induce the synchronization even if its impact on the system is confined to a small region in the Lorenz phase space. In the article [4] in light of the LaSalle-type invariance principle for stochastic differential equations, chaos synchronization was investigated for a class of chaotic systems dissatisfying a globally Lipschitz condition with a noise perturbation. Sufficient criteria for both complete synchronization and generalized synchronization are rigorously established and thus successfully applied to realize chaos synchronization in the coupled unified chaotic systems. In the article [5] the analysis of transition from chaotic to nonchaotic behavior and synchronization in an ensemble of systems driven by identical random forces was presented. The synchronization phenomenon was investigated in an ensemble of particles moving with friction in a time-dependent potential and driven by an identical noise. The threshold values of the parameters for transition from chaotic to nonchaotic behavior were obtained and dependencies of the Lyapunov exponents and power spectral density of the current of the ensemble of particles on the nonlinearity of the systems and intensity of the driven force were analyzed. In [6], the noise-induced synchronization problem was considered for a system of particles perturbed by a Brownian motion.

2 Duffing equation

In this section, we discuss some properties of a solution of the Duffing equation. The Duffing equation

$$\begin{cases} \mathrm{d}x_t = v_t\,\mathrm{d}t, \\ \mathrm{d}v_t = \left(x_t - x_t^3\right)\mathrm{d}t - \alpha v_t\,\mathrm{d}t, \quad t > 0,\ \ \alpha > 0, \\ x_0 = x,\ \ v_0 = v \end{cases} \tag{2}$$

has a unique smooth solution $(x_t, v_t) = (x_t^{x,v}, v_t^{x,v}),\ t \geq 0$, for each initial state (x, v). There are three stationary states: $(-1, 0)$, $(1, 0)$ (stable) and $(0, 0)$ (unstable) with the following domains of attraction (see [8]):

$$A_- = \{(x,v) \in \mathbb{R}^2 \colon \lim_{t\to\infty} (x_t^{x,v}, v_t^{x,v}) = (-1,0)\},$$
$$A_0 = \{(x,v) \in \mathbb{R}^2 \colon \lim_{t\to\infty} (x_t^{x,v}, v_t^{x,v}) = (0,0)\},$$
$$A_+ = \{(x,v) \in \mathbb{R}^2 \colon \lim_{t\to\infty} (x_t^{x,v}, v_t^{x,v}) = (1,0)\}.$$

Let us

$$B_r(x,v) = \{(y,u) \in \mathbb{R}^2 \colon (x-y)^2 + (v-u)^2 \leq r^2\},$$
$$B_r = B_r(0,0),$$
$$\Gamma_r = B_r(-1,0) \cup B_r \cup B_r(1,0),$$
$$\tau_r^{x,v} = \inf\{t \geq 0 \colon (x_t^{x,v}, v_t^{x,v}) \in \Gamma_r\}.$$

The system (2) doesn't possess closed trajectories (except the three fixed points $(-1, 0)$, $(1, 0)$, $(0, 0)$) and, as a consequence, every trajectory converges to a rest state. The following assertion is a corollary of Theorem [9, p. 132].

Lemma 1. *For any $r > 0$ and $R > 0$ there is a constant T depending only on α, r, R such that*

$$\tau_r^{x,v} \leq T$$

for all $(x, v) \in B_R$.

Let us fix $(x, v) \in \mathbb{R}^2$ and introduce an auxiliary function

$$w_t = w_t^{x,v} = \frac{1}{2}(x_t^2 - 1)^2 + v_t^2, \quad t \geq 0,$$

where $(x_t, v_t) = (x_t^{x,v}, v_t^{x,v})$ is a solution of (2). The function w_t represents an "energy" of a particle whose dynamics is given by (2).

By the definition of w_t and (2),

$$\frac{\mathrm{d}}{\mathrm{d}t} w_t + 2\alpha v_t^2 = 0 \tag{3}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} w_t + 2\alpha w_t - \alpha(x_t^2 - 1)^2 = 0. \tag{4}$$

Therefore,

$$w_t + 2\alpha \int_{t_0}^{t} v_s^2 \,\mathrm{d}s = w_{t_0}, \quad \forall t_0 \in [0, t]. \tag{5}$$

Hence, for all $0 \le s \le t$

$$w_s \ge w_t. \tag{6}$$

Lemma 2. *For any $\varepsilon > 0$ there is a constant C_ε depending only on ε and α such that*

$$w_t \le \left(e^{-\frac{4}{3}\alpha t} + \varepsilon\right) w_0 + C_\varepsilon.$$

Proof. Let us introduce an auxiliary function

$$Q_t = x_t(2v_t + \alpha x_t).$$

Straightforward calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t + 2\left(x_t^2 - 1\right) = 2\left[v_t^2 - \left(x_t^2 - 1\right)^2\right].$$

Using (3) and (4) we have

$$3\frac{\mathrm{d}}{\mathrm{d}t} w_t + 4\alpha w_t = \frac{\mathrm{d}}{\mathrm{d}t} w_t + 2\alpha\left(x_t^2 - 1\right)^2 = 2\alpha\left[\left(x_t^2 - 1\right)^2 - v_t^2\right].$$

Therefore, w_t satisfies the equation

$$3\frac{\mathrm{d}}{\mathrm{d}t} w_t + 4\alpha w_t + 2\alpha\left(x_t^2 - 1\right) + \alpha\frac{\mathrm{d}}{\mathrm{d}t} Q_t = 0.$$

Hence

$$w_t = \psi_t w_0 - \frac{1}{3}\alpha \int_0^t \left[2(x_s^2 - 1) + \frac{\mathrm{d}}{\mathrm{d}s} Q_s\right] \psi_{t-s}\, \mathrm{d}s,$$

where $\psi_t = e^{-\frac{4}{3}\alpha t}$. Integrating by parts we have

$$\begin{aligned} w_t = \psi_t w_0 - \frac{2}{3}\alpha \int_0^t \left(x_s^2 - 1\right)\psi_{t-s}\, \mathrm{d}s - \frac{1}{3}\alpha Q_t \\ + \frac{1}{3}\alpha \psi_t Q_0 + \frac{4\alpha^2}{9} \int_0^t Q_s \psi_{t-s}\, \mathrm{d}s. \end{aligned} \tag{7}$$

Let us estimate the right-hand side of (7). By Young's inequality and (6), for every $\delta > 0$

$$\begin{aligned} Q_t &\le \delta v_t^2 + (1/\delta + \alpha) x_t^2 = \delta\left(v_t^2 + \frac{1}{2}\left(x_t^2 - 1\right)^2\right) - \frac{\delta}{2}\left(x_t^2 - 1\right)^2 + (\alpha + 1/\delta) x_t^2 \\ &\le \delta w_t + C(\delta, \alpha) \le \delta w_0 + C(\delta, \alpha), \end{aligned} \tag{8}$$

where

$$C(\delta, \alpha) = \sup_x \left\{ \left(\frac{1}{\delta} + \alpha \right) x^2 - \frac{\delta}{2} \big(x^2 - 1 \big)^2 \right\}$$

is a constant depending only on δ and α. Similarly, for any $\delta > 0$

$$-Q_t \leq -2x_t v_t \leq \delta v_t^2 + \frac{1}{\delta} x_t^2 \leq \delta w_t + C(\delta) \leq \delta w_0 + C(\delta),$$

where $C(\delta)$ is a constant depending only on δ. Using (8), we get

$$\int_0^t Q_s \psi_{t-s} \, \mathrm{d}s \leq \big(\delta w_0 + C(\delta, \alpha) \big) \int_0^t \psi_{t-s} \, \mathrm{d}s \leq \frac{3}{4\alpha} \big(\delta w_0 + C(\delta, \alpha) \big).$$

Obviously,

$$\int_0^t \big(1 - x_s^2 \big) \psi_{t-s} \, \mathrm{d}s \leq \int_0^t \psi_{t-s} \, \mathrm{d}s \leq \frac{3}{4\alpha}.$$

The equality (7), together with these estimates, implies the assertion of the lemma. □

3 Duffing equations with random perturbations

In this section, we consider the system (1). We denote

$$\begin{aligned}
A_+(x) &= \big\{ v \in \mathbb{R} \colon (x, v) \in A_+ \big\}, \\
A_-(x) &= \big\{ v \in \mathbb{R} \colon (x, v) \in A_- \big\}, \\
A_0(x) &= \big\{ v \in \mathbb{R} \colon (x, v) \in A_0 \big\}, \\
A_0^r(x) &= \big\{ v \in \mathbb{R} \colon \big(x_t^{x,v}, v_t^{x,v} \big) \in B_r \quad \text{for some} \ \ t \geq 0 \big\},
\end{aligned}$$

where $(x_t^{x,v}, v_t^{x,v})$ is a solution of the system (2).

Let

$$\begin{aligned}
A \ &= \big[A_+(-1) \cap A_+(0) \cap A_+(1) \big] \cup \big[A_-(-1) \cap A_-(0) \cap A_-(1) \big], \\
A_0^r &= A_0^r(-1) \cup A_0^r(0) \cup A_0^r(1).
\end{aligned}$$

We need the following assumptions on the sequence (τ_i, ξ_i), $i = 1, 2, \ldots$, of random perturbations defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

Assumption 1. (i) The random variables $\sigma_k = \tau_k - \tau_{k-1}$, $k = 1, 2, \ldots$, are mutually independent and identically distributed

$$\mathbb{P}\{\sigma_1 < \infty\} = 1;$$

(ii) There is $h > 0$ such that

$$\mathbb{P}\{\sigma_1 \geq h\} = 1;$$

(iii) For any $t > 0$

$$\mathbb{P}\{\sigma_1 \geq t\} > 0.$$

Assumption 2. (i) The random variables ξ_k, $k = 1, 2, \ldots$, are bounded, mutually independent, identically distributed and do not depend on random variables $\sigma_k, k = 1, 2, \ldots;$

(ii) There is $r > 0$ such that

$$\mathbb{P}\big\{\xi_1 \in A \setminus A_0^r\big\} > 0.$$

Remark 1. The set $A \setminus A_0^r \neq \emptyset$ with some $r > 0$ if there is a number a such that $\{(x, v) \in \mathbb{R}^2 \colon -1 \leq x \leq 1, v = a\} \cap A_0 = \emptyset$. This condition is satisfied if the number $|a|$ is sufficiently large (see. Fig. 1).

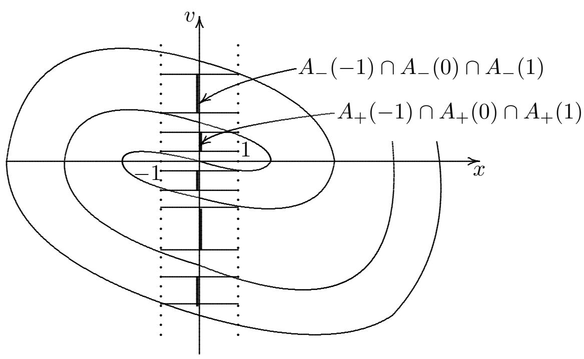

Fig. 1. The phase portrait of the separatrices.

Let $(X_t, V_t) = (X_t^{x,v}, V_t^{x,v})$ be a solution to (1). As in Section 2, let us introduce an auxiliary function

$$W_t = W_t^{x,v} = \frac{1}{2}\big(X_t^2 - 1\big)^2 + V_t^2, \quad t \geq 0.$$

Lemma 3. *Let* $\alpha > 0, \gamma \in [0, 1)$ *and let Assumptions* 1, 2 *be satisfied. Then for every* $R > 0$ *there is a constant* C *such that for all* $(x, v) \in B_R$

$$\sup_{t \geq 0} W_t^{x,v} \leq C \quad \textit{a.s.}$$

Proof. Let us fix $(x,v)\in B_R$. Due to (6), $W_t=W_t^{x,v}$ is a decreasing function on each time interval $[\tau_i,\tau_{i+1})$. Therefore, it can achieve maximal values only at times τ_i, i.e.

$$\sup_{t\geq 0} W_t=\sup_{i=0,1,\dots} W_{\tau_i}.$$

Obviously,

$$W_{\tau_i}=W_{\tau_i}-W_{\tau_i-}+W_{\tau_i-}=(\gamma V_{\tau_i-}+\xi_i)^2-V_{\tau_i-}^2+W_{\tau_i-}, i=1,2,\dots. \quad (9)$$

According to Lemma 2 and Assumption 1, for any $\varepsilon>0$ there is a constant C_ε such that

$$W_{\tau_i-}\leq \big(e^{-\frac{4}{3}\alpha\sigma_i}+\varepsilon\big)W_{\tau_{i-1}}+C_\varepsilon\leq \big(e^{-\frac{4}{3}\alpha h}+\varepsilon\big)W_{\tau_{i-1}}+C_\varepsilon \quad \text{a.s.}, \quad (10)$$

where $h>0$ is the number from Assumption 1(ii). Let us fix $\varepsilon>0$ such that

$$\kappa=e^{-\frac{4}{3}\alpha h}+\varepsilon<1.$$

By Young's inequality

$$2\gamma V_{\tau_i-}\xi_i\leq \big(1-\gamma^2\big)V_{\tau_i-}^2+\frac{\gamma^2}{1-\gamma^2}\,\xi_i^2.$$

Therefore,

$$(\gamma V_{\tau_i-}+\xi_i)^2-V_{\tau_i-}^2\leq \frac{1}{1-\gamma^2}\,\xi_i^2. \quad (11)$$

According to Assumption 2 and (9)–(11), there is a constant C such that

$$W_{\tau_i}\leq \kappa W_{\tau_{i-1}}+C \quad \text{a.s.},\ i=1,2,\dots.$$

Therefore, for each $n=1,2,\dots$

$$W_{\tau_n}\leq W_0\kappa^n+C\sum_{i=0}^{n}\kappa^i\leq W_0+\frac{1}{1-\kappa}C \quad \text{a.s.},$$

and the assertion follows.

Convention 1. We change the dynamics of the system (1) as follows. Fix $r>0$. The state $(X_{\tau_{i+1}},V_{\tau_{i+1}-})$, $i=0,1,2,\dots$, is replaced by the state $(-1,0)$ (respectively, $(0,0),(1,0)$) if at a time $t\in[\tau_i,\tau_{i+1})$ the state $(X_t^{x,v},V_t^{x,v})$ belongs to the set $B_r(-1,0)$ (respectively, $B_r,\ B_r(1,0)$). Otherwise, the dynamics of the system remains the same as given by (1).

The convention attaches to the states $(-1,0)$, $(0,0)$, $(1,0)$ an additional attraction force. Such an attraction force is always present in computer simulations of the system dynamics due to a rounding error.

Remark 2. As can be easily seen, Convention 1 does not change the assertion of Lemma 3.

Theorem 1. *Let $\alpha > 0, \gamma \in [0,1), R > 0$ and let Assumptions* 1 *and* 2 *be satisfied. Let $(\tilde{X}_t^{x,v}, \tilde{V}_t^{x,v}), t \geq 0$ be a solution to* (1) *the dynamics of which is changed according to Convention 1 with the number r satisfying Assumption* 2(ii). *Then*

$$\lim_{t\to\infty} \mathbb{P}\left\{ \sup_{(x,v)\in B_R} \tilde{X}_t^{x,v} \neq \inf_{(x,v)\in B_R} \tilde{X}_t^{x,v} \right\} = 0.$$

Proof. According to Lemmas 1 and 3, Convention 1 and Remark 2, there is a constant T such that

$$\left(\tilde{X}_{\tau_i}^{x,v}, \tilde{V}_{\tau_i -}^{x,v}\right) = (-1,0), (0,0) \text{ or } (1,0)$$

for each $(x,v) \in B_R$ if $\sigma_i = \tau_i - \tau_{i-1} > T$.

Let us introduce the random variables

$$\eta_i = \begin{cases} 1, & \sigma_i > T, \\ 0, & \sigma_i \leq T, \quad i = 1,2,\ldots, \end{cases}$$

and

$$k_m^* = \inf\{k \geq 1\colon \eta_k = 1, \eta_{k+1} = 1, \ldots, \eta_{k+m} = 1\}, \quad m = 1,2,\ldots$$

According to Assumption 2, there is a constant $\mu > 0$ such that

$$\mathbb{P}\{\eta_i = 1\} = \mathbb{P}\{\sigma_i > T\} \geq \mu, \quad i = 1,2,\ldots.$$

Since $\eta_i, i = 1,2,\ldots$, are mutually independent and identically distributed random variables, we have (see [7, Ch. XIII, Sect. 7]).

$$\lim_{N\to\infty} \mathbb{P}\{k_m^* \leq N\} = 1 \tag{12}$$

for each $m = 1,2,\ldots$.

According to our assumptions,

$$\left(\widetilde{X}_{\tau_{k_m^*+i}}^{x,v}, \widetilde{V}_{\tau_{k_m^*+i}-}^{x,v}\right) = (-1,0), (0,0) \quad \text{or} \quad (1,0)$$

for any $i = 0,1,\ldots,m$ and $(x,v) \in B_R$. In addition, if $\xi_{\tau_{k_m^*+i}} \in A \setminus A_0^r$ for some $i = 0,1,\ldots,m-1$, then

$$\sup_{(x,v)\in B_R} \widetilde{X}_{\tau_{k_m^*+i+1}}^{x,v} = \inf_{(x,v)\in B_R} \tilde{X}_{\tau_{k_m^*+i+1}}^{x,v} = -1 \text{ or } 1.$$

Therefore,

$$\left\{t \geqslant \tau_{k_m^*+m}\right\} \cap \left\{ \bigcup_{i=0}^{m-1} \left\{\xi_{k_m^*+i} \in A \setminus A_0^r\right\} \right\} \subset \left\{ \sup_{(x,v)\in B_R} \widetilde{X}_t^{x,v} = \inf_{(x,v)\in B_R} \widetilde{X}_t^{x,v} \right\}. \tag{13}$$

Obviously, for each $N = 1, 2, \ldots$,

$$\mathbb{P}\{t < \tau_{k_m^*+m}\} \leqslant \mathbb{P}\{t < \tau_{N+m}\} + \mathbb{P}\{k_m^* > N\}. \tag{14}$$

By Assumption 2 and (12),

$$\begin{aligned}
&\mathbb{P}\Bigg\{\bigcap_{i=0}^{m-1}\big\{\xi_{k_m^*+i} \notin A \setminus A_0^r\big\}\Bigg\} \\
&\quad = \sum_{k=1}^{\infty} \mathbb{P}\big\{\xi_{k_m^*} \notin A \setminus A_0^r, \ldots, \xi_{k_m^*+m-1} \notin A \setminus A_0^r, k_m^* = k\big\} \\
&\quad = \sum_{k=1}^{\infty} \mathbb{P}\big\{\xi_k \notin A \setminus A_0^r, \ldots, \xi_{k+m-1} \notin A \setminus A_0^r\big\}\mathbb{P}\{k_m^* = k\} \\
&\quad = \sum_{k=1}^{\infty}\prod_{i=0}^{m-1} \mathbb{P}\big\{\xi_{k+i} \notin A \setminus A_0^r\big\}\mathbb{P}\{k_m^* = k\} \\
&\quad = (1-\rho)^m,
\end{aligned} \tag{15}$$

where $\rho = \mathbb{P}\{\xi_1 \in A \setminus A_0^r\} > 0$.

Finally, using (13)–(15), we have

$$\begin{aligned}
&\mathbb{P}\Bigg\{\sup_{(x,v)\in B_R} \widetilde{X}_t^{x,v} \inf_{(x,v)\in B_R} \widetilde{X}_t^{x,v}\Bigg\} \\
&\quad \leqslant \mathbb{P}\big\{t < \tau_{k_m^*+m}\big\} + \mathbb{P}\Bigg\{\bigcap_{i=0}^{m-1}\big\{\xi_{k_m^*+i} \notin A \setminus A_0^r\big\}\Bigg\} \\
&\quad \leqslant \mathbb{P}\big\{t < \tau_{N+m}\big\} + \mathbb{P}\big\{k_m^* > N\big\} + (1-\rho)^m.
\end{aligned} \tag{16}$$

Due to (12), the last two terms in the right-hand side of (16) can be made arbitrarily small choosing sufficiently large m and N. By Assumption 1(i), $\mathbb{P}\{t < \tau_{N+m}\} \to 0$ as $t \to \infty$. Therefore, passing on to the limit in (16) as $t \to \infty$, we get the assertion of the theorem. □

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