Investigation of the spectrum for the Sturm–Liouville problem with one integral boundary condition

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Abstract. In this paper, the Sturm–Liouville problem with one classical first type boundary condition and other nonlocal integral boundary conditions of two cases is investigated. We analyze how complex eigenvalues of these problems depend on the parameters of nonlocal integral boundary conditions. Some new results are given on complex spectra of these problems. Many results are presented as graphs of complex characteristic functions.

Keywords: Sturm–Liouville problem, nonlocal integral boundary conditions, complex eigenvalues.

1 Introduction

Problems with an integral *Nonlocal Boundary Condition* (NBC) arise in various fields of mathematical physics, biology, biotechnology, etc. Nowadays the investigation of problems with various types of NBCs is an actual problem. One of the originators of such problems was J. Cannon. He introduced nonlocal integral boundary conditions [1]. L.I. Kamynin began to investigate parabolic equations with nonlocal integral boundary conditions [2]. The problems with integral NBCs were investigated in many papers, such as [3,4] (NBCs for hyperbolic equations), [5,6] (NBCs for elliptic equations), etc. Integral BCs are the special case of a more general nonlocal BC for stationary BVP [7–9].

Investigation of the spectrum (and complex part of the spectrum particularly) of differential equations with NBCs is quite a new, but important, area related to the problems in this field. The eigenvalue problems, investigation of the spectra, analysis of nonnegative solutions and similar problems for the operators with NBCs of Bitsadze–Samarskii or of integral-type are given in the papers [10–14]. Complex eigenvalues for differential operators with NBCs are less investigated than the real case. Some results of these eigenvalues for a problem with one Samarskii–Bitsadze NBC are published [14, 15].

In this paper, we analyze a complex eigenvalue problem for a stationary differential operator with two cases of nonlocal integral NBC. We investigate how the complex eigenvalues of these problems depend on the parameters γ and ξ of the nonlocal integral boundary conditions. As the theoretical investigation of the complex spectrum is a very difficult problem, we present the results of modelling and computational analysis and illustrate the existing situation in graphs. Zeroes, poles and critical points of the characteristic function are important for investigating complex eigenvalues. Real eigenvalues of the Sturm–Liouville problem are also important. So, in Section 2, we formulate the problem and present the earlier obtained main results on characteristic functions, zeroes, poles, and critical points in all cases. In Section 3, a short review of real eigenvalues properties of the analyzed problem is given. These results are wider discussed in the previous papers [16–18] and they are useful for investigating complex eigenvalues.

2 Sturm–Liouville problem with integral type NBC

Let us consider a Sturm–Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

and another nonlocal integral boundary condition:

$$u(1) = \gamma \int\limits_{\xi}^{1} u(t)\,\mathrm{d}t \quad \text{(Case 1)}, \tag{3_1}$$

$$u(1) = \gamma \int\limits_{0}^{\xi} u(t)\,\mathrm{d}t \quad \text{(Case 2)}, \tag{3_2}$$

with parameters $\gamma \in \mathbb{C}$ and $\xi \in [0,1]$. Also we analyze the Sturm–Liouville problem (1) with the boundary condition

$$u'(0) = 0 \tag{4}$$

on the left side, and with nonlocal integral boundary conditions (3) on the right side of the interval. We enumerate these cases from Case $1'$ to Case $2'$.

In Case 1, $1'$ for $\xi = 0$ and Case 2, $2'$ for $\xi = 1$ we have the same integral NBC. In the general case, the eigenvalues $\lambda \in \mathbb{C}$ and eigenfunctions $u(t)$ are the complex functions. For $\gamma = \infty$, we get NBC:

$$\int\limits_{\xi}^{1} u(t)\,\mathrm{d}t = 0, \quad 0 \leqslant \xi < 1, \qquad \int\limits_{0}^{\xi} u(t)\,\mathrm{d}t = 0, \quad 0 < \xi \leqslant 1. \tag{$5_{1,2}$}$$

Note that the index in the number of a formula (for example in formula (3)) denotes the case. If there is no index, then the rule (or results) holds on in all the cases of NBCs. If we write two indexes in the number of formulae, as in (5), then the first part of this formula is related to Case 1 and the second part is related to Case 2. If we write one index, then the formula is related to one case.

Remark 1 (**classical case**). If $\gamma = 0$ or $\xi = 1$ in problem (1), (2), (3_1) or problem (1), (4), (3_1) and $\gamma = 0$ or $\xi = 0$ in problem (1),(2), (3_2) or problem (1), (4), (3_2), we have the problem with the classical boundary conditions and their eigenvalues and eigenfunctions are well known [16]:

$$\lambda_k = k^2\pi^2, \quad u_k(t) = \sin(k\pi t), \quad k \in \mathbb{N} := \{1, 2, \dots\}, \tag{$6_{1,2}$}$$

$$\lambda_k = \left(k - \frac{1}{2}\right)^2 \pi^2, \quad u_k(t) = \cos\left(\left(k - \frac{1}{2}\right)\pi t\right), \quad k \in \mathbb{N}. \tag{$6_{1',2'}$}$$

If $\lambda = 0$, then the function $u(t) = ct$ satisfies problem (1)–(2) and the function $u(t) = c$ satisfies problem (1), (4). By substituting these solutions into NBCs, we derive that there exists a nontrivial solution ($c \neq 0$) if

$$1 - \gamma\frac{1 - \xi^2}{2} = 0, \qquad 1 - \gamma\frac{\xi^2}{2} = 0, \tag{$7_{1,2}$}$$

$$1 - \gamma(1 - \xi) = 0, \qquad 1 - \gamma\xi = 0. \tag{$7_{1',2'}$}$$

Lemma 1. *The eigenvalue* $\lambda = 0$ *exists if, and only if*

$$\gamma = \frac{2}{1 - \xi^2}, \qquad \gamma = \frac{2}{\xi^2}, \tag{$8_{1,2}$}$$

$$\gamma = \frac{1}{1 - \xi}, \qquad \gamma = \frac{1}{\xi}. \tag{$8_{1',2'}$}$$

In the general case, if $\lambda \neq 0$ and eigenvalues $\lambda = q^2$, then the solution of problem (1)–(2) is $u = c\sin(qt)$ and the solution of problem (1), (4) is $u(t) = \cos(qt)$. In both cases ($q = 0$ and $q \neq 0$), it is one formula for the nontrivial solutions $u = c\sin(qt)/q = c\sinh(-iqt)/q$ of BC (2) and $u = c\cos(qt) = c\cosh(-iqt)$ of BC (4), where

$$q \in \mathbb{C}_q := \{q \in \mathbb{C}\colon \operatorname{Re} q > 0 \text{ or } \operatorname{Re} q = 0,\ \operatorname{Im}\ q > 0 \text{ or } q = 0\}.$$

If $q \in \mathbb{C}_q$ then $\lambda = q^2 \in \mathbb{C}_\lambda = \mathbb{C}$ and vice versa. Moreover, we have a bijection between $\mathbb{C}_q$ and $\mathbb{C}$.

Let us return to problems (1)–(3) and (1),(3), (4) and consider that $0 < \xi < 1$ and $\gamma \in \mathbb{C}$. If $\lambda \neq 0$, the NBC is satisfied and there exists a nontrivial solution (eigenfunction) if q is the root of the equation

$$f(q) := 2\gamma\frac{\sin((1+\xi)q/2)\sin((1-\xi)q/2)}{q^2} - \frac{\sin q}{q} = 0, \tag{9_1}$$

$$f(q) := 2\gamma\frac{\sin^2((\xi q)/2)}{q^2} - \frac{\sin q}{q} = 0, \tag{9_2}$$

$$f(q) := 2\gamma\frac{\cos((1+\xi)q/2)\sin((1-\xi)q/2)}{q} - \cos q = 0, \tag{$9_{1'}$}$$

$$f(q) := \gamma\frac{\sin(\xi q)}{q} - \cos q = 0. \tag{$9_{2'}$}$$

We define a *constant eigenvalue* as the eigenvalue $\lambda = q^2$ that does not depend on the parameter $\gamma \in \mathbb{C}$. For any constant eigenvalue, we define the *constant eigenvalue point* $q \in \mathbb{C}_q$ and the *constant eigenvalue γ-value point* $(q, \gamma) \in \mathbb{C}_q \times \mathbb{C}$, respectively [16, 17].

For a constant eigenvalue, the set of γ-value points in $\mathbb{C}_q \times \mathbb{C}$ is a vertical line. We name other eigenvalues as *nonconstant*. We get all the nonconstant eigenvalue points q as the roots of the systems:

$$\begin{cases} \sin q = 0, \\ \cos(\xi q) - \cos q = 0, \end{cases} \qquad \begin{cases} \sin q = 0, \\ 1 - \cos(\xi q) = 0, \end{cases} \tag{$10_{1,2}$}$$

$$\begin{cases} \cos q = 0, \\ \sin q - \sin(\xi q) = 0, \end{cases} \qquad \begin{cases} \cos q = 0, \\ \sin(\xi q) = 0. \end{cases} \tag{$10_{1',2'}$}$$

Constant eigenvalues exist only for rational $\xi = r = m/n \in (0, 1)$, those eigenvalues are equal to $\lambda_k = c_k^2$, $k \in \mathbb{N}$, where constant eigenvalue points c_k are given by formulae shown in Table 1. In Table 1 and further in the paper, we used the sets $\mathbb{N}_o := \{1, 3, 5, \ldots\}$, $\mathbb{N}_e := \{2, 4, 6, \ldots\}$ and $\mathbb{N}_l := \{lk\colon k \in \mathbb{N}\}$, $l \in \mathbb{N}$.

Table 1. Constant eigenvalue points c_k, $k \in \mathbb{N}$.

Case	$n-m \in \mathbb{N}_e$	$n-m \in \mathbb{N}_o$	$m \in \mathbb{N}_e$	$m \in \mathbb{N}_o$
Case 1	$n\pi k$	$2n\pi k$	–	–
Case 1′	$n\pi(k-1/2)$	$2n\pi(k-1/2)$	–	–
Case 2	–	–	$n\pi k$	$2n\pi k$
Case 2′	–	–	$n\pi(k-1/2)$	$2n\pi(k-1/2)$

All nonconstant eigenvalues are γ-points of the meromorphic function (the complex characteristic function)

$$\gamma(q) := \frac{q\sin q}{\cos(\xi q) - \cos q} = \frac{q\sin q}{2\sin((1+\xi)q/2)\sin((1-\xi)q/2)}, \tag{11_1}$$

$$\gamma(q) := \frac{q\sin q}{1 - \cos(\xi q)} = \frac{q\sin q}{2\sin^2(\xi q/2)}, \tag{11_2}$$

$$\gamma(q) := \frac{q\cos q}{\sin q - \sin(\xi q)} = \frac{q\cos q}{2\cos((1+\xi)q/2)\sin((1-\xi)q/2)}, \tag{$11_{1'}$}$$

$$\gamma(q) := \frac{q\cos q}{\sin(\xi q)} \tag{$11_{2'}$}$$

for $\xi \notin \mathbb{Q}$. So, we can find the eigenvalues $\lambda = q^2$ in two ways: as constant eigenvalues from (10) (only for rational ξ); as nonconstant eigenvalues, using the complex characteristic function (11).

For the investigation of constant eigenvalues as well as for the analysis of complex eigenvalues, zero and pole points of the characteristic function are important.

Proposition 1. *Zero points z of the functions $\gamma(q)$ are of the first order. These positive zeroes are equal to*:

$$z_k := k\pi, \quad k \in \mathbb{N}, \tag{$12_{1,2}$}$$

$$z_k := (k-1/2)\pi, \quad k \in \mathbb{N}. \tag{$12_{1',2'}$}$$

Proposition 2. *Points $p_k = 2\pi k/\xi, k \in \mathbb{N}$ are poles of the second order for the function $\gamma(q)$ in Case 2 and there are no first order poles in this case. Other poles are of the first order, and they are equal to*:

$$p_k := \frac{2\pi k}{1+\xi}, \quad k \in \mathbb{N} \qquad \textit{and} \qquad \tilde{p}_l := \frac{2\pi l}{1-\xi}, \quad l \in \mathbb{N}, \tag{13_1}$$

$$p_k := \frac{2\pi(k-1/2)}{1+\xi}, \quad k \in \mathbb{N} \qquad \textit{and} \qquad \tilde{p}_l := \frac{2\pi l}{1-\xi}, \quad l \in \mathbb{N}, \tag{$13_{1'}$}$$

$$p_k := \frac{\pi k}{\xi}, \quad k \in \mathbb{N}. \tag{$13_{2'}$}$$

If $\xi = r = m/n \in \mathbb{Q}$, then a part of zeroes z_j of the function $\gamma(q)$ are coincident with the poles p_k or $\tilde{p}_l$. If the pair (n,m) is coprime numbers, then pairs $(m+n, m-n)$, $(m+n, n)$, $(m-n, n)$ are also coprime numbers in Case 1, 1′. We have two families of poles: $p_k = \pi k/(n-m)$, $k \in \mathbb{N}_{2n}$, and $\tilde{p}_l = (\pi l/(n+m))$, $l \in \mathbb{N}_{2n}$. The poles from the first family coincide with the poles from the second family at the points $q_j = \pi j$, $j \in \mathbb{N}_n$, $n-m \in \mathbb{N}_e$ or $q_j = \pi j$, $j \in \mathbb{N}_{2n}$, $n-m \in \mathbb{N}_o$. These points are zeroes of the sinus function as well, therefore they are coincident in the case of constant eigenvalues. Thus, in Case 1 all the points p_k, $k \in \mathbb{N}_{2n}$ or $\tilde{p}_l$, $l \in \mathbb{N}_{2n}$ are poles of the first order.

Points $p_k = \pi k/m$, $k \in \mathbb{N}_{2n}$ are poles of Case 2. They are poles of the second order, except $k \in \mathbb{N}_{mn}$, $m \in \mathbb{N}_e$ and $k \in \mathbb{N}_{2mn}$, $m \in \mathbb{N}_o$ (coincident with the case of constant eigenvalues), which are the first order poles. When $m = 1$ and $m = 2$, there are no poles of the second order.

In Case 1′, we also have two families of poles $p_k = \pi k/(n-m)$, $k \in \mathbb{N}_{2n}$ and $p_l = \pi(l-1/2)/(n+m)$, $l \in \mathbb{N}_{2n}$. All the positive poles of this problem are of the first order. These poles of two families are coincident with constant eigenvalue points $c_k = n\pi(k-1/2)$, $n-m \in \mathbb{N}_e$, $k \in \mathbb{N}$.

In Case 2′, the points $p_k = \pi k/m$, $k \in \mathbb{N}_n$ are poles of the first order. If these pole points are coincident with zeroes of the cosine function at the points $z_k = \pi(k-1/2)n$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, we have constant eigenvalue points.

3 Real eigenvalues of the Sturm–Liouville problem

If we take q only in the rays $q = x \geqslant 0$, $q = -ix$, $x \leqslant 0$ instead of $q \in \mathbb{C}_q$, we get positive eigenvalues in case the ray $q = x > 0$, and we get negative eigenvalues in the ray $q = -x$, $x < 0$. The point $q = x = 0$ corresponds to $\lambda = 0$. So, in this case, for complex functions (11) the *real characteristic functions* are:

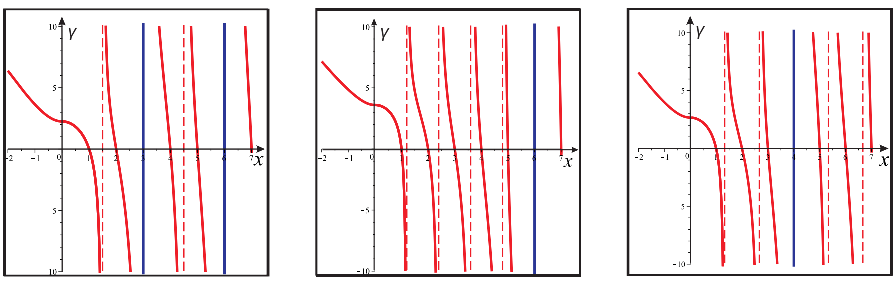
$$\gamma(x) := \begin{cases} \dfrac{x \sinh x}{2\sinh((1+\xi)x/2)\sinh((1-\xi)x/2)}, & x \leqslant 0, \\ \dfrac{x \sin x}{2\sin((1+\xi)x/2)\sin((1-\xi)x/2)}, & x \geqslant 0, \end{cases} \tag{14_1}$$

$$\gamma(x) := \begin{cases} \dfrac{x \sinh x}{2\sinh^2(\xi x/2)}, & x \leqslant 0, \\ \dfrac{x \sin x}{2\sin^2(\xi x/2)}, & x \geqslant 0, \end{cases} \tag{14_2}$$

$$\gamma(x) := \begin{cases} \dfrac{x \cosh x}{2\cosh((1+\xi)x/2)\sinh((1-\xi)x/2)}, & x \leqslant 0, \\ \dfrac{x \cos x}{2\cos((1+\xi)x/2)\sin((1-\xi)x/2)}, & x \geqslant 0, \end{cases} \tag{$14_{1'}$}$$

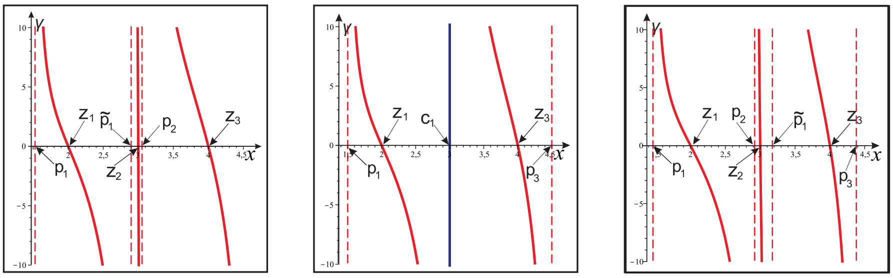
$$\gamma(x) := \begin{cases} \dfrac{x \cosh x}{\sinh(\xi x)}, & x \leqslant 0, \\ \dfrac{x \cos x}{\sin(\xi x)}, & x \geqslant 0. \end{cases} \tag{$14_{2'}$}$$

Those functions are useful for the investigation of real negative, zero, and positive eigenvalues. The graphs of these real characteristic functions for some parameter ξ values are presented in Figs. 1, 2, and 3. Note that the x-axis is scaled π times and $x = 1$ is really $x = \pi$ in all figures. The vertical solid lines correspond to constant eigenvalues, vertical dashed lines cross the x-axis at the points of poles. For some cases, the vertical line of the constant eigenvalue is coincident with the vertical asymptotic line at the point of a pole. More properties of the real characteristic function and real spectrum in each case are investigated in [17].



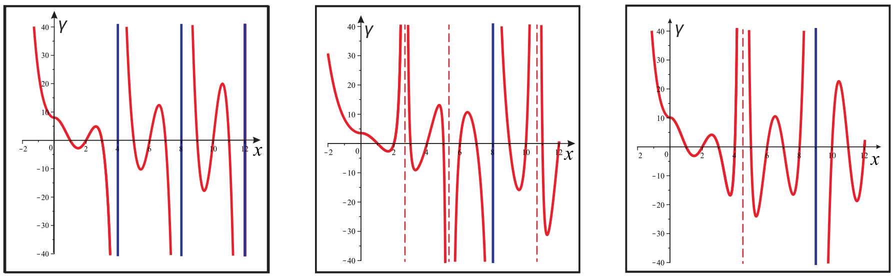
$\xi = 1/3$ $\xi = 2/3$ $\xi = 1/2$

Fig. 1. Real function $\gamma(\pi x)$ for various ξ in Case 1.



$\xi = 0.31$ $\xi = 1/3$ $\xi = 0.37$

Fig. 2. Real characteristic function $\gamma(\pi x)$ in the neighborhood of the constant eigenvalue point in Case 1.



$\xi = 1/2$ $\xi = 3/4$ $\xi = 4/9$

Fig. 3. Real function $\gamma(\pi x)$ for various ξ in Case 2.

3.1 The spectra in Cases 1, 1′

The spectra for problems (1), (2), (3_1) and (1), (4), (3_1) lie on the real axis as shown in papers [16, 17].

The function $\gamma(x)$ is a monotone decreasing function in each interval (α, β), where α and β are the points of the first order poles. In Fig. 1, we see constant eigenvalue points and poles in Case 1 where the value of the parameter ξ is changing. For example, if $\xi = \xi_c = 1/3$ in Fig. 1, we have two constant eigenvalues. In this case two poles are coincident with constant eigenvalue point. If we change $\xi < \xi_c$ or $\xi > \xi_c$, then poles are moving from each other. Such a situation is shown in Fig. 2. We have the same situation with the spectrum in Case $1'$. So, if the poles p_k and $\tilde{p}_l$ move toward the zero point z_r, then a part of the graph of the characteristic function, that was in $(\tilde{p}_l, p_k)$, becomes a vertical line, i.e., we have a constant eigenvalue point $c_s = p_k = \tilde{p}_l = z_r$ for $\xi = \xi_c$. For $\xi > \xi_c$ we have the interval $(p_k, \tilde{p}_l)$, i.e., the poles change places with each other.

4 Complex eigenvalues of the Sturm–Liouville problem

In the recent scientific literature there are many papers, in which real eigenvalues of the Sturm–Liouville problem are analysed. However, a complex spectrum of this problem is considerably less investigated [14, 15].

It is important to investigate complex eigenvalues of the Sturm-Liouville problems (1), (2), (3) and (1), (4), (3) with $\gamma \in \mathbb{R}$. The restriction $\operatorname{Im}\gamma(q) = 0$ of the complex characteristic function (11) is called as a *complex-real characteristic function*. Those restricted functions $\gamma(q) : \mathcal{N} \to \mathbb{R}$ are defined on some subset (net): $\mathcal{N} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q\colon \operatorname{Im}\gamma(q) = 0\}$. In the general case, the subset $\mathcal{N}$ is a union of curves in the complex domain $\mathbb{C}_q$.

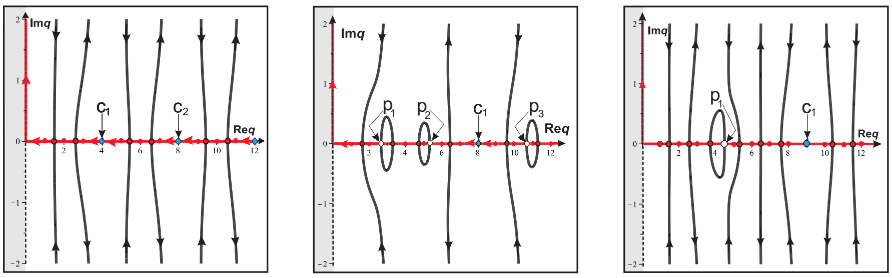
The poles of the function $\gamma(q)$ are eigenvalues of the problems (1)–(3) and (1), (4), (3) in the case $\gamma = \infty$. All zeros and poles of the meromorphic function $\gamma(q)$ lie on the positive part of the real axis. From (11) and from the properties of sine and cosine functions, we have that all zeros of this function are real numbers $q = k\pi, k \in \mathbb{N}$ in Cases 1, 2 and $q = (k - 1/2)\pi, k \in \mathbb{N}$ in Cases $1'$,$2'$. So, only positive zeroes and poles exist in $\mathbb{C}_q$.

4.1 Dynamics of complex eigenvalues in Case 2

In this case, the spectrum of complex eigenvalues is more complicated. By changing the value of the parameter ξ we get various types of the domain $\mathcal{N}$, defined in Section 4.

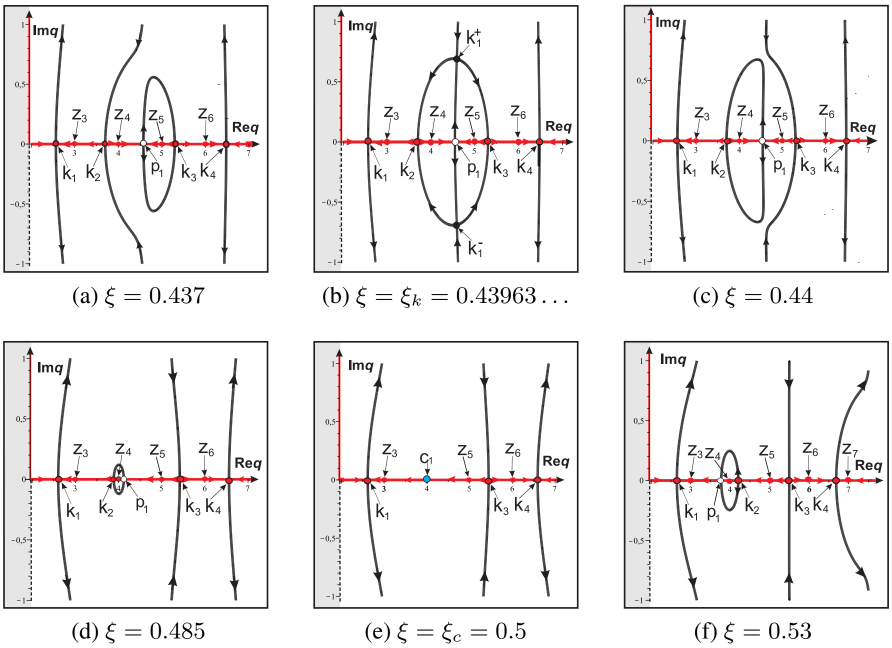
We can see a qualitative view of dependence of complex eigenvalue curves on the parameter ξ in Fig. 4. Note that the $\operatorname{Re} q$-axis and $\operatorname{Im} q$-axis in $\mathbb{C}_q$ are scaled π times. In Case 2, there are two types of bifurcation. The first type is where two different complex curves join at a critical point. We get the second type by changing the value of the parameter ξ, so that zero and pole points of the characteristic function become coincident with the critical points (in which constant eigenvalues exist) and the loop type curves disappear.

Fig. 5 shows, how the domain $\mathcal{N}$ is changing dependent on the parameter ξ value near to $\xi_k = 0.43963\ldots$ (we call it a critical point in the complex part of $\mathbb{C}_q$) and ξ_c (constant eigenvalue point) points. One complex eigenvalue curve makes a loop. In this example, the value of ξ is increasing from 0.437 to 0.53. When $\xi \lesssim \xi_k$, two complex eigenvalues become close, and when $\xi = \xi_k$, those different curves join each other at the critical points k_1^+ and k_1^- (see Fig. 5(a), (b), (c)). Next, when $\xi \gtrsim \xi_0$, those loop type curves are changing places with each other. The order of poles does not change in this bifurcation. Zero is inside the loop. As $\xi \in (\xi_k; \xi_c)$, the loop tightens and intersects the real axis at the pole and critical points. When zero and pole consist with the critical point, we have a “collapse”, i.e., a constant eigenvalue point. We can see 3D view of complex-real characteristic functions in Fig. 6.



$\xi = 1/2$ $\qquad$ $\xi = 3/4$ $\qquad$ $\xi = 4/9$

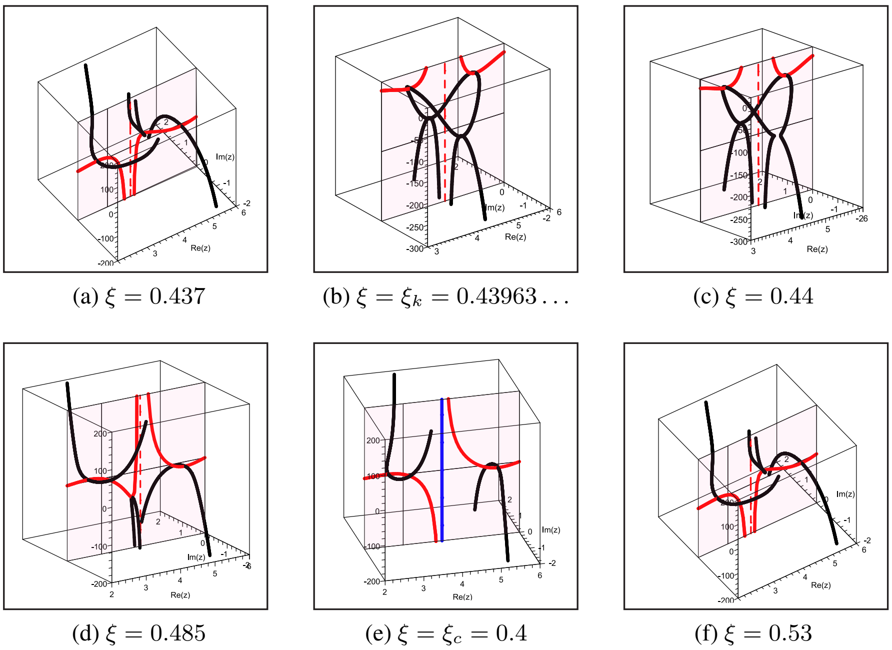
Fig. 4. Domain $\mathcal{N}$ with various ξ for the complex-real function $\gamma(\pi q)$ in Case 2.



(a) $\xi = 0.437$ (b) $\xi = \xi_k = 0.43963\ldots$ (c) $\xi = 0.44$

(d) $\xi = 0.485$ (e) $\xi = \xi_c = 0.5$ (f) $\xi = 0.53$

Fig. 5. Domain $\mathcal{N}$ with various ξ for the complex-real function $\gamma(\pi q)$ in Case 2.



(a) $\xi = 0.437$ (b) $\xi = \xi_k = 0.43963\ldots$ (c) $\xi = 0.44$

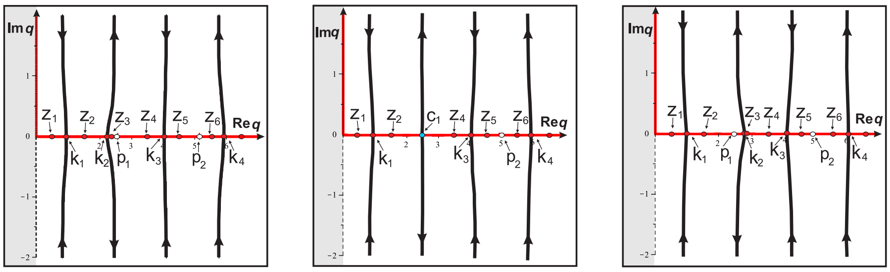
(d) $\xi = 0.485$ (e) $\xi = \xi_c = 0.4$ (f) $\xi = 0.53$

Fig. 6. Complex-real function $\gamma(\pi q)$ for various ξ in Case 2.

4.2 Dynamics of complex eigenvalues in Case 2′

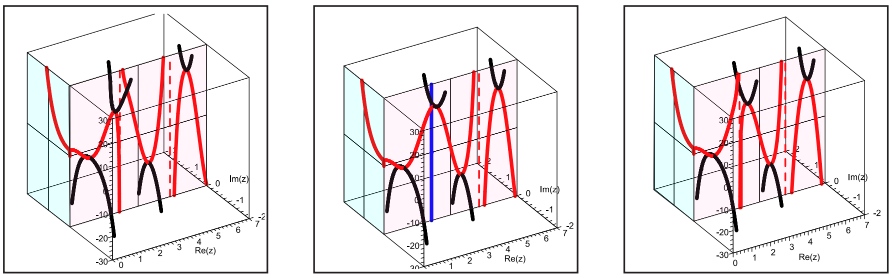
The complex spectrum in Case 2′ is not so complicated as in Case 2. In Fig. 7 it is shown how the spectrum of complex eigenvalues is approaching to the constant eigenvalue point ξ_c.

If $\xi < \xi_c = 2/5$, the pole moves toward zero from right side and the curve of the complex eigenvalue moves toward zero from the left. When the pole and zero meet, we have a constant eigenvalue point. If ξ is growing, the pole moves to the left. We can see 3D view of complex-real characteristic functions in Fig. 8, too.



$\xi = 0.39$ $\xi = \xi_c = 0.4$ $\xi = 0.41$

Fig. 7. Domain $\mathcal{N}$ with various ξ for the complex-real function $\gamma(\pi q)$ in Case 2′.



$\xi = 0.39$ $\xi = \xi_c = 0.4$ $\xi = 0.41$

Fig. 8. Complex-real function $\gamma(\pi q)$ for various ξ in Case 2′.

5 Conclusions

In this paper, the complex spectrum of the Sturm–Liouville problem with the classical or first type boundary condition on the left side of the interval and nonlocal integral boundary condition of two types on the right side of the interval was investigated.

- All eigenvalues of the Sturm–Liouville problem in Case 1 and Case 1′ are real. Complex eigenvalues do not exist for any values of NBC parameter ξ.
- In Case 2 there are two types of bifurcation: when two different complex curves join at the critical point; a loop type curve disappears, when the zero and pole points of

the characteristic function become coincident with the critical point, i.e., points, at which constant eigenvalues exist.

- In Case $2'$ the curves of complex and constant eigenvalues intersect at one point, where we have pole and zero. The dynamic view in this case is much simpler than in Case 2.

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