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One estimate related to the periodic zeta-function

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Abstract. An estimate for the error term of the fourth moment of the periodic zeta-function is obtained.

Keywords: periodic zeta-function, Lerch zeta-function, approximate functional equation.

Let $s = \sigma + it$ be a comlex variable, $\lambda \in \mathbb{R}$. The periodic zeta-function $\zeta_\lambda(s)$ is defined, for $\sigma > 1$, by Dirichlet series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}}{m^s}.$$

If $\lambda \in \mathbb{Z}$, then $\zeta_\lambda(s)$ becomes the Riemann zeta-function $\zeta(s)$. Therefore, we suppose that $0 < \lambda < 1$. Let $L(\lambda, \alpha, s)$, $0 < \alpha \leqslant 1$, denote the Lerch zeta-function defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}.$$

If $0 < \lambda < 1$, then $L(\lambda, \alpha, s)$ is analytically continuable to an entire function [2]. From definitions of $\zeta_\lambda(s)$ and $L(\lambda, \alpha, s)$, we have that

$$\zeta_\lambda(s) = e^{2\pi i\lambda} L(\lambda, 1, s). \tag{1}$$

In [4], the asymptotics for the fourth power moment of the periodic zeta-function was considered and the following theorem was proved.

Theorem 1. *Suppose that λ is irrational, $0 < \lambda < 1$. Then, for $\frac{1}{2} < \sigma < 1$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T \left|\zeta_\lambda(\sigma + it)\right|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi\lambda(m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}}.$$

The aim of this note is to estimate the rate of convergence in Theorem 1.

Theorem 2. *Suppose that λ is irrational, $0 < \lambda < 1$, $\frac{1}{2} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$,*

$$\int_1^T \left|\zeta_\lambda(\sigma + it)\right|^4 dt$$
$$= T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi\lambda(m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}}\right) + O\big(T^{\frac{3}{2} - \sigma + \varepsilon}\big).$$

For the proof of Theorem 2, we will apply an approximate functional equation for the function $\zeta_\lambda(s)$. Let $[u]$ denote the integer part of u,

$$m(t)=\left[\sqrt{\frac{t}{2\pi}}-1\right],\qquad q(t)=\left[\sqrt{\frac{t}{2\pi}}\right],\qquad g(\lambda,t)=2\sqrt{\frac{t}{2\pi}}-m(t)-q(t)-\lambda-1,$$

$$\begin{aligned}f(\lambda,t)=&-\frac{t}{2\pi}\log\frac{t}{2\pi e}-\frac{7}{8}+\frac{1}{2}\bigl(1-\lambda^2\bigr)+m(t)-q(t)\\&+2\sqrt{\frac{t}{2\pi}}\bigl(q(t)-m(t)+\lambda-1\bigr)-\frac{1}{2}\bigl(q(t)+m(t)\bigr)-\lambda\bigl(1+q(t)-m(t)\bigr),\end{aligned}$$

and

$$\psi(z)=\frac{\cos\pi(\frac{z^2}{2}-z-\frac{1}{8})}{\cos\pi z}.$$

Lemma 1. *Let* $0<\lambda<1$, $0\leqslant\sigma\leqslant 1$ *and* $t\geqslant t_0>0$. *Then*

$$\begin{aligned}\zeta_\lambda(s)=&\sum_{1\leqslant m\leqslant m(t)}\frac{e^{2\pi i\lambda m}}{m^s}+\left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it}e^{it+\frac{\pi i}{4}}\sum_{0\leqslant m\leqslant q(t)}\frac{1}{(m+\lambda)^{1-s}}\\&+\left(\frac{t}{2\pi}\right)^{-\frac{\sigma}{2}}e^{\pi i f(\lambda,t)+2\pi i\lambda}\psi\bigl(g(\lambda,t)\bigr)+O\bigl(t^{\frac{\sigma}{2}-1}\bigr).\end{aligned}$$

Proof. The assertion of the lemma follows from an approximate functional equation for $\zeta_\lambda(s)$ [2] and equality (1).

Denote

$$S_1(s)=\sum_{1\leqslant m\leqslant m(t)}\frac{e^{2\pi i\lambda m}}{m^s}$$

and

$$S_2(s)=\left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it}e^{it+\frac{\pi i}{4}}\sum_{0\leqslant m\leqslant q(t)}\frac{1}{(m+\lambda)^{1-s}}.$$

Since the function $\psi(z)$ is bounded, we have by Lemma 1, that, for $\frac{1}{2}<\sigma<1$,

$$\zeta_\lambda(s)=S_1(s)+S_2(s)+O\bigl(t^{-\frac{1}{4}}\bigr).\tag{2}$$

Lemma 2. *Let* $\frac{1}{2}<\sigma<1$, *and* $T\to\infty$. *Then, for every* $\varepsilon>0$,

$$\begin{aligned}\int_1^T|S_1(\sigma+it)|^4\,dt=&\,T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}-2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}\right)\\&+O\bigl(T^{\frac{3}{2}-\sigma+\varepsilon}\bigr).\end{aligned}$$

Proof. From the definition of $S_1(s)$, we find that

$$\left|S_1(\sigma+it)\right|^4 = \sum_{m_1} \frac{e^{2\pi i\lambda m_1}}{m_1^{\sigma+it}} \sum_{m_2} \frac{e^{2\pi i\lambda n_1}}{n_1^{\sigma+it}} \sum_{m_2} \frac{e^{-2\pi i\lambda m_2}}{m_2^{\sigma-it}} \sum_{n_2} \frac{e^{-2\pi i\lambda n_2}}{n_2^{\sigma-it}}$$
$$= \sum_{m_1,n_1,m_2,n_2} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \left(\frac{m_2n_2}{m_1n_1}\right)^{it},$$

where in each sum we sum over $[1, m(t)]$. Let $T_1 = 2\pi \max((m_1+1)^2, (n_1+1)^2, (m_2+1)^2, (n_2+1)^2)$, then we have

$$\int_1^T \left|S_1(\sigma+it)\right|^4 dt = \int_1^T \sum_{m_1,n_1,m_2,n_2} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \left(\frac{m_2n_2}{m_1n_1}\right)^{it} dt$$
$$= \sum_{1\leqslant m_1,n_1,m_2,n_2\leqslant m(T)} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \int_{T_1}^T \left(\frac{m_2n_2}{m_1n_1}\right)^{it} dt$$
$$= \sum_{m_1n_1=m_2n_2}^{*} \frac{(T-T_1)e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1)^{2\sigma}}$$
$$+ O\biggl(\sum_{m_1n_1\neq m_2n_2}^{*} \frac{|\log \frac{m_2n_2}{m_1n_1}|^{-1}}{(m_1n_1m_2n_2)^\sigma}\biggr), \tag{3}$$

where the star $*$ means, that sum is taken over $m_1, n_1, m_2, n_2 \in [1, m(T)]$. Let $d(k) = \sum_{d/k} 1$, $k \in \mathbb{N}$, be the divisor function, and $N(k)$ is the number of solutions of the equation $m_1n_1 = m_2n_2 = k$. Then, we have that $N(k) = d^2(k)$, if $k \leqslant u$, $m_1, n_1, m_2, n_2 \leqslant u$, and $N(k) \leqslant d^2(k)$, if $k \geqslant u$, $m_1, n_1, m_2, n_2 \leqslant u$. It is well known [3] that, for $\sigma > \frac{1}{2}$,

$$\sum_{k=1}^{\infty} \frac{d^2(k)}{k^{2\sigma}} = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.$$

Also, $d(k) = O_\varepsilon(k^\varepsilon)$ with every $\varepsilon > 0$. Therefore

$$T \sum_{m_1n_1=m_2n_2}^{*} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1)^{2\sigma}}$$
$$= T \sum_{m_1n_1=m_2n_2\leqslant m(T)} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1)^{2\sigma}} + O\biggl(T \sum_{k\geqslant m(T)} \frac{d^2(k)}{k^{2\sigma}}\biggr)$$
$$= T \sum_{\substack{m_1n_1=m_2n_2\\ m_1+n_1=m_2+n_2}} \frac{1}{(m_1n_1)^{2\sigma}} + T \sum_{m_1n_1=m_2n_2} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1)^{2\sigma}} + O(T^{\frac{3}{2}-\sigma+\varepsilon})$$
$$= T \sum_{m_1n_1=m_2n_2} \frac{1}{(m_1n_1)^{2\sigma}} - T \sum_{m_1n_1=m_2n_2} \frac{1-\cos 2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}$$
$$+ iT \sum_{m_1n_1=m_2n_2} \frac{\sin 2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}} + O\bigl(T^{\frac{3}{2}-\sigma+\varepsilon}\bigr)$$

$$= T\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - T\sum_{m_1n_1=m_2n_2}\frac{1-\cos 2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}$$
$$+ iT\sum_{m_1n_1=m_2n_2}\frac{\sin 2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}} + O\big(T^{\frac{3}{2}-\sigma+\varepsilon}\big). \tag{4}$$

By the definition of T_1 and symmetry

$$\sum_{m_1n_1=m_2n_2}^{*}\frac{T_1}{(m_1n_1)^{2\sigma}} = O\Bigg(\sum_{m_1n_1=m_2n_2}^{*}\frac{m_1^2}{(m_1n_1m_2n_2)^{\sigma}}\Bigg) = O\Bigg(\sum_{m_1,n_1}^{*}\frac{m_1^2 d(m_1n_1)}{(m_1n_1)^{2\sigma}}\Bigg)$$
$$= O\Bigg(T^{\varepsilon}\sum_{m_1\leqslant m(T)}\frac{1}{m_1^{2\sigma-2}}\sum_{n_1\leqslant m(T)}\frac{1}{n_1^{2\sigma}}\Bigg) = O\big(T^{\frac{3}{2}-\sigma+\varepsilon}\big). \tag{5}$$

Using the estimate

$$\sum_{0<m}\sum_{<n\leqslant T}\frac{1}{m^{\sigma}n^{\sigma}\log\frac{n}{m}} = O\big(T^{2-2\sigma}\log T\big), \quad \frac{1}{2}\leqslant\sigma<1,$$

we find that

$$\sum_{m_1n_1\neq m_2n_2}^{*}\frac{|\log\frac{m_2n_2}{m_1n_1}|^{-1}}{(m_1n_1m_2n_2)^{\sigma}} = O\Bigg(\sum_{0<m<}\sum_{n\leqslant m^2(T)}\frac{d(m)d(n)}{(mn)^{\sigma}\log(\frac{n}{m})}\Bigg) = O\big(T^{2-2\sigma+\varepsilon}\big).$$

Thus, the lemma is a consequence of (3)–(5).

Now we deal with $S_2(s)$. We apply the following lemma [1].

Lemma 3. *Suppose that $u_1,\dots,u_r$ are complex numbers, $\lambda_1,\dots,\lambda_r$ are distinct real numbers, and $\delta_m = \min_{n\neq m}|\lambda_n-\lambda_m|$. Then*

$$\sum_{m=1}^{r}\sum_{n=1}^{r}u_m\bar{u}_n(\lambda_n-\lambda_m)^{-1} \ll \sum_{m=1}^{r}|u_m|^2\delta_m^{-1}.$$

Lemma 4. *Let $\frac{1}{2}<\sigma<1$, $T\to\infty$ and λ be irrational. Then, for every $\varepsilon>0$,*

$$\int_1^T\big|S_2(\sigma+it)\big|^4\,dt = O_\lambda\big(T^{2-2\sigma+\varepsilon}\big).$$

Proof. Clearly

$$Z(T) \stackrel{\text{def}}{=} \int_1^T\Bigg|\sum_{0\leqslant m\leqslant q(t)}\frac{1}{(m+\lambda)^{1-\sigma-it}}\Bigg|^4 dt$$
$$= O_\lambda(T) + \int_1^T\Bigg|\sum_{1\leqslant m\leqslant q(t)}\frac{1}{(m+\lambda)^{1-\sigma-it}}\Bigg|^4 dt. \tag{6}$$

As in the case of $S_1(s)$, the second term in the right-hand side of (6) is

$$Z_1(T) \stackrel{\text{def}}{=} \sum_{1\leqslant m_1,n_1,m_2,n_2\leqslant q(T)} \frac{1}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}} \times \int_{T_2}^{T} \left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)^{it} dt,$$

where $T_2 = 2\pi\max(m_1^2, n_1^2, m_2^2, n_2^2)$. Since λ is irrational, we have that $(m_1+\lambda)(n_1+\lambda) = (m_2+\lambda)(n_2+\lambda)$ if and only if $m_1n_1 = m_2n_2$ and $m_1+n_1 = m_2+n_2$. Therefore,

$$Z_1(T) = O\biggl(\sum_{m_1n_1=m_2n_2}^{*} \frac{T-T_2}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} + \sum_{(m_1+\lambda)(n_1+\lambda)\neq(m_2+\lambda)(n_2+\lambda)}^{*} \frac{|\log\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}|^{-1}}{(m_1n_1m_2n_2)^{1-\sigma}}\biggr). \tag{7}$$

The star $*$ means that the summing runs over $m_1, n_1, m_2, n_2 \in [1, q(T)]$. It is not difficult to see that

$$\sum_{m_1n_1=m_2n_2}^{*} \frac{T}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} = O\biggl(T\sum_{k\leqslant q^2(T)} \frac{d^2(k)}{k^{2-2\sigma}}\biggr) = O\bigl(T^{2\sigma+\varepsilon}\bigr) \tag{8}$$

and

$$\sum_{m_1n_1=m_2n_2}^{*} \frac{T_2}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} = O\biggl(\sum_{m_1,n_1\leqslant q(T)} \frac{m_1^2 d(m_1n_1)}{(m_1n_1)^{2-2\sigma}}\biggr) = O\bigl(T^{2\sigma+\varepsilon}\bigr). \tag{9}$$

Moreover, by Lemma 3,

$$\sum_{\substack{m_1n_1\neq m_2n_2\\(m_1+\lambda)(n_1+\lambda)\neq(m_2+\lambda)(n_2+\lambda)}}^{*} \frac{|\log\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}|^{-1}}{(m_1n_1m_2n_2)^{1-\sigma}} = O_\lambda\biggl(T^\varepsilon \sum_{1\leqslant m\leqslant q^2(T)} \frac{m}{m^{2-2\sigma}}\biggr) = O_\lambda\bigl(T^{2\sigma+\varepsilon}\bigr),$$

and

$$\sum_{\substack{m_1n_1= m_2n_2\\(m_1+\lambda)(n_1+\lambda)\neq(m_2+\lambda)(n_2+\lambda)}}^{*} \frac{|\log\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}|^{-1}}{(m_1n_1m_2n_2)^{1-\sigma}} = O_\lambda\biggl(\sum_{k\leqslant q^2(T)} \frac{kd^2(k)}{k^{2-2\sigma}}\biggr) = O\bigl(T^{2\sigma+\varepsilon}\bigr).$$

Two last estimates together with (6)–(9) prove the lemma.

Proof of Theorem 2. The estimate of the theorem easily follows from Lemmas 2 and 4, and the Cauchy–Schwarz inequality.

References

[1] A. Ivič. *The Riemann Zeta-Function*. John Wiley, New York, 1985.

[2] A. Laurinčikas and R. Garunkštis. *The Lerch Zeta-Function*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[3] E.C. Titchmarsh. *The Theory of the Riemann Zeta-Function*. Oxford University Press, 2 edition, 1986 (revised by D.R. Heath-Brown).

[4] D. Šiaučiūnas and A. Laurinčikas. On the fourth power moment of the function $\zeta_\lambda(s)$. *Integral Transforms and Special Functions*, 18:629–638, 2007.

REZIUMĖ

Periodines dzeta funkcijos ketvirtasis momentas

S. Černigova

Straipsnyje pateikiama asimptotinė formulė su liekamuoju nariu ketvirtajam periodinės dzeta funkcijos momentui.

Raktiniai žodžiai: periodinė dzeta funkcija, Lercho dzeta funkcija, artutinė funkcinė lygtis.