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# Совместная дискретная универсальность для $L$-функций из класса Сельберга и периодических дзета-функций Гурвица

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**Аннотация**

Класс Сельберга $\mathcal{S}$ составляют ряды Дирихле

$$\mathcal{L}(s)=\sum_{m=1}^{\infty}\frac{a(m)}{m^s},\quad s=\sigma+it,$$

коффициенты которых при всяком $\varepsilon>0$ удовлетворяют оценке $a(m)\ll_\varepsilon m^\varepsilon$; существует целое $k\geqslant 0$ такое, что $(s-1)^k\mathcal{L}(s)$ является целой функцией конечного порядка; для $\mathcal{L}$ имеет место функциональное уравнение, связывающее $s$ и $1-s$, и эйлерово произведение по простым числам. Штойдинг пополнил класс $\mathcal{S}$ условием

$$\lim_{x\to\infty}\left(\sum_{p\leqslant x}1\right)^{-1}\sum_{p\leqslant x}|a(p)|^2=\kappa>0,$$

где $p$ означает простые числа. Полученный класс обозначается через $\widetilde{\mathcal{S}}$.

Пусть $\alpha$, $0<\alpha\leqslant 1$, – фиксированный параметр, а $\mathfrak{a}=\{a_m: m\in\mathbb{N}_0\}$ – периодическая последовательность комплексных чисел. Другой объект статьи – периодическая дзета-функция Гурвица $\zeta(s,\alpha;\mathfrak{a})$ при $\sigma>1$ определяется рядом Дирихле

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s},$$

и мероморфно продолжается на всю комлексную плоскость.

В статье расматривается дискретная универсальность набора

$$\left(\mathcal{L}(\widetilde{s}),\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})\right),$$

где $\mathcal{L}(\widetilde{s})\in\widetilde{S}$, а $\zeta(s,\alpha_j;\mathfrak{a}_{jl_j})$ – периодические дзета-функции Гурвица, т. е., одновременное приближение набора широкого класса аналитических функций

$$\left(f(\widetilde{s}),f_{11}(s),\dots,f_{1l_1}(s),\dots,f_{r1}(s),\dots,f_{rl_r}(s)\right)$$

набором сдвигов

$$\big(\mathcal{L}(\widetilde{s}+ikh), \zeta(s+ikh_1, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s+ikh_1, \alpha_1; \mathfrak{a}_{1l_1}), \dots,\\ \zeta(s+ikh_r, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s+ikh_r, \alpha_r; \mathfrak{a}_{rl_r})\big),$$

где $h, h_1, \dots, h_r$ – положительные числа. При этом требуется линейная независимость над полем рациональных чисел для множества

$$\left\{(h\log p : p \in \mathbb{P}), (h_j \log(m+\alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r), 2\pi\right\},$$

где $\mathbb{P}$ – множество всех простых чисел.

*Ключевые слова:* Дзета-функция Гурвица, класс Сельберга, периодическая дзета-функция Гурвица, ряды Дирихле, слабая сходимость, универсальность.

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# Joint discrete universality for $L$-functions from the Selberg class and periodic Hurwitz zeta-functions

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**Abstract**

The Selberg class $\mathcal{S}$ contains Dirichlet series

$$\mathcal{L}(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad s = \sigma + it,$$

such that, for every $\varepsilon > 0$, $a(m) \ll_\varepsilon m^\varepsilon$; there exists an integer $k \geqslant 0$ such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order; the functions $\mathcal{L}$ satisfy a functional equation connecting $s$ with $1-s$, and have a product representation over prime numbers. Steuding introduced a subclass $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ with additional condition

$$\lim_{x\to\infty} \Bigg(\sum_{p\leqslant x} 1\Bigg)^{-1} \sum_{p\leqslant x} |a(p)|^2 = \kappa > 0,$$

where $p$ runs prime numbers.

Let $\alpha$, $0 < \alpha \leqslant 1$, be a fixed parameter, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers. The second object of the paper is the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ which is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and is meromorphically continued to the whole complex plane.

The paper is devoted to the discrete universality of the collection

$$(\mathcal{L}(\widetilde{s}), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})),$$

where $\mathcal{L}(\widetilde{s}) \in \widetilde{S}$, and $\zeta(s, \alpha_j; \mathfrak{a}_{jl_j})$ are periodic Hurwitz zeta-functions, i. e., to the simultaneous approximation of a collection

$$(f(\widetilde{s}), f_{11}(s), \dots, f_{1l_1}(s), \dots, f_{r1}(s), \dots, f_{rl_r}(s))$$

of analytic functions from a wide class by a collection of shifts

$$\big(\mathcal{L}(\widetilde{s} + ikh), \zeta(s + ikh_1, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s + ikh_1, \alpha_1; \mathfrak{a}_{1l_1}), \dots,$$
$$\zeta(s + ikh_r, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s + ikh_r, \alpha_r; \mathfrak{a}_{rl_r})\big),$$

where $h, h_1, \dots, h_r$ are positive numbers, is considered. For this, the linear independence over the field of rational numbers for the set

$$\{(h \log p : p \in \mathbb{P}), (h_j \log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r), 2\pi\},$$

where $\mathbb{P}$ denotes the set of all prime numbers, is applied.

*Keywords:* Dirichlet series, Hurwitz zeta-function, periodic Hurwitz zeta-function, Selberg class, universality, weak convergence.

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***In honor of Professor Antanas Laurinčikas on the occasion of his 70th birthday***

## 1. Introduction

After a pioneer Voronin's work [27], it is known that some zeta and $L$-functions are universal in the sense that their shifts approximate a wide class of analytic functions. Also, this universality property was extended to collections of zeta-functions simultaneously approximating a given collections of analytic functions. In other words, some zeta and $L$-functions are jointly universal in the approximation sense. The first joint universality theorem was obtained also by Voronin. In [28], investigating the joint functional independence of Dirichlet $L$-functions, he first obtained in a not explicit form their joint universality, see also [10], [11]. A very interesting is the so-called mixed joint universality of zeta and $L$-functions. In this case, a collection of analytic functions is approximated by the collection of zeta and $L$-functions consisting of functions having and having no Euler's product over primes. This type of universality was proposed by Mishou in [19] who proved a mixed joint universality theorem for the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, and the Hurwitz zeta-function

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},\quad \sigma>1,$$

with transcendental parameter $\alpha$, $0<\alpha\leqslant 1$. Let $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D$ with connected complements, by $H(K)$ with $K\in\mathcal{K}$ the class of continuous functions on $K$ that are analytic in the interior of $K$, and by $H_0(K)$ with $K\in\mathcal{K}$ the subclass of $H(K)$ of non-vanishing functions on $K$. Then the Mishou theorem is the following statement.

Theorem 1. *Suppose that $\alpha$ is transcendental. Let $K_1,K_2\in\mathcal{K}$, $f_1(s)\in H_0(K_1)$ and $f_2(s)\in H(K_2)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\ \sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\right\}>0.$$

In [7], Theorem 1 was extended for zeta-functions with periodic coefficients. Let $\mathfrak{a}=\{a_m:m\in\mathbb{N}\}$ and $\mathfrak{b}=\{b_m:m\in\mathbb{N}_0\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, be a periodic sequences of complex numbers with minimal periods $q_1\in\mathbb{N}$ and $q_2\in\mathbb{N}$, respectively. Then the periodic zeta-function $\zeta(s;\mathfrak{a})$ and periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$, $0<\alpha\leqslant 1$, are defined, for $\sigma>1$, by

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}\quad\text{and}\quad\zeta(s,\alpha;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^s},$$

and can be continued meromorphically to the whole complex plane with possible simple pole at the point $s=1$. The results of [14] were generalized for collections consisting from $r_1$ periodic zeta-functions with multiplicative coefficients and $r_2$ periodic Hurwitz zeta-functions with algebraically independent over $\mathbb{Q}$ parameters $\alpha_1,\dots,\alpha_{r_2}$. More general results were obtained in the theses of K. Janulis [6] and S. Račkauskienė [22].

The above mentioned universality results for zeta-functions are of continuous type, $\tau$ in shifts $\zeta(s+i\tau;\mathfrak{a})$ and $\zeta(s+i\tau,\alpha;\mathfrak{b})$ can take arbitrary real values. Reich in [23] proposed an another type of universality when $\tau$ takes values from a certain discrete set. He used the set $\{kh:k\in\mathbb{N}_0\}$ with fixed $h>0$. The Reich theorem in the case of Riemann zeta-function is of the following form. In the sequel, $\#A$ denotes the cardinality of the set $A$, and $N$ runs over non-negative integers.

Theorem 2. *Suppose that $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$ and $h>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon\right\}>0.$$

Theorem 2 independently by an another method was also proved in [1].

The first discrete version of Theorem 1 was obtained in [3]. Define the set

$$L(\mathbb{P},\alpha,h,\pi)=\left\{(\log p:p\in\mathbb{P}),(\log(m+\alpha):m\in\mathbb{N}_0),\frac{2\pi}{h}\right\}.$$

Then the main result of [3] is the following theorem.

Theorem 3. *Suppose that the set $L(\mathbb{P},\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K_1}|\zeta(s+ikh)-f_1(s)|<\varepsilon,\ \sup_{s\in K_2}|\zeta(s+ikh,\alpha)-f_2(s)|<\varepsilon\right\}>0.$$

In [4], Theorem 3 was generalized for shifts $\zeta(s+ikh_1)$ and $\zeta(s+ikh_2,\alpha)$ by using the linear independence over $\mathbb{Q}$ of the set

$$L(\mathbb{P},\alpha,h_1,h_2,\pi)=\{(h_1\log p:p\in\mathbb{P}),(h_2\log(m+\alpha):m\in\mathbb{N}_0),2\pi\}.$$

An analogue of Theorem 3 for the functions $\zeta(s;\mathfrak{a})$ with multiplicative coefficients and $\zeta(s,\alpha;\mathfrak{b})$ was proved in [14]. Finally, in [15], the results of [14] were extended for a wide collections consisting from periodic and periodic Hurwitz zeta-functions.

The aim of this paper is discrete universality theorems for $L$-functions from the Selberg class and periodic Hurwitz zeta-functions.

The Selberg class $\mathcal{S}$ was introduced in [24], and consists of Dirichlet series

$$\mathcal{L}(s)=\sum_{m=1}^{\infty}\frac{a(m)}{m^s}$$

that satisfy the following axioms:

(i) (Ramanujan conjecture). For every $\varepsilon>0$, the estimate $a(m)\ll_\varepsilon m^\varepsilon$ takes place.

(ii) (analytic continuation). There exists $r\in\mathbb{N}_0$ such that $(s-1)^r\mathcal{L}(s)$ is an entire function of finite order.

(iii) (functional equation). The functional equation

$$\Lambda_{\mathcal{L}(s)}=w\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},\quad \Lambda_{\mathcal{L}}(s)=\mathcal{L}(s)Q^s\prod_{j=1}^{f}\Gamma(\lambda_j s+\mu_j),$$

with $Q_j,\lambda_j\in\mathbb{R}$ and $\mu_j,w\in\mathbb{C}$, $\mathrm{Re}\mu_j\geqslant 0$ and $|w|=1$ is satisfied for all $s$.

(iv) (Euler product). The product representation over primes

$$\mathcal{L}(s)=\prod_p\mathcal{L}_p(s),$$

where

$$\log\mathcal{L}_p(s)=\sum_{l=1}^{\infty}\frac{b(p^k)}{p^{ls}}$$

with $b(p^l)\ll p^{l\theta}$ with some $\theta<\frac{1}{2}$, is valid.

It is well known that the majority of classical zeta and $L$-functions are elements of the class $\mathcal{S}$. The first universality results for $L$-functions from the Selberg class were obtained by J. Steuding in [25] and [26]. The most general universality theorem for the above $L$-functions is given in [21]. In this theorem, an additional condition that

$$\lim_{x\to\infty}\left(\sum_{p\leqslant x}1\right)^{-1}\sum_{p\leqslant x}|a(p)|^2=\kappa>0 \tag{1}$$

is required. Moreover, for $\mathcal{L}\in\mathcal{S}$, let

$$d_{\mathcal{L}}=2\sum_{j=1}^{f}\lambda_j,$$

and

$$\sigma_{\mathcal{L}} = \max\left\{\frac{1}{2}, 1 - \frac{1}{d_{\mathcal{L}}}\right\}, \quad D = D_{\mathcal{L}} = \{s \in \mathbb{C} : \sigma_{\mathcal{L}} < \sigma < 1\}.$$

Denote by $\mathcal{K}_{\mathcal{L}}$ the class of compact subsets of $D_{\mathcal{L}}$ with connected complements, and by $H_{0\mathcal{L}}$ with $K \in \mathcal{K}_{\mathcal{L}}$ the class of continuous non-vanishing functions on $K$ that are analytic in the interior of $K$. The main result of the paper [21] is the following theorem. Denote the class $\mathcal{S}$ with condition (1) by $\widetilde{\mathcal{S}}$.

THEOREM 4. *Suppose that $\mathcal{L} \in \widetilde{\mathcal{S}}$. Let $K \in \mathcal{K}_{\mathcal{L}}$ and $f(s) \in H_{0\mathcal{L}}(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\mathcal{L}(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Joint universality theorems of $L$-functions from the class $\widetilde{\mathcal{S}}$ and periodic Hurwitz zeta-functions were proved in [8], [9] and [17].

The discrete version of Theorem 4 was given in [16].

THEOREM 5. *Under hypotheses of Theorem* 4, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\mathcal{L}(s + ikh) - f(s)| < \varepsilon\right\} > 0.$$

The aim of this paper is to obtain joint discrete universality for $L$-functions in the class $\widetilde{\mathcal{S}}$ and periodic Hurwitz zeta-functions. Such a theorem for a pair $(\mathcal{L}(s), \zeta(s, \alpha; \mathfrak{a}))$ was obtained in [10].

For $h > 0$ define define the set

$$L(\mathbb{P}; \alpha_1, \ldots, \alpha_r; h, \pi) = \left\{(\log p : p \in \mathbb{P}), (\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \ldots, r), \frac{2\pi}{h}\right\}.$$

THEOREM 6. *Suppose that the set $L(\mathbb{P}; \alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$ and $\mathcal{L} \in \widetilde{\mathcal{S}}$. Let $K \in \mathcal{K}_{\mathcal{L}}$, $K_1, \ldots, K_r \in \mathcal{K}$, and $f(s) \in H_{0\mathcal{L}}(K)$, $f_1(s) \in H(K_1), \ldots, f_r(s) \in H(K_r)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\mathcal{L}(s + ikh) - f(s)| < \varepsilon,$$
$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

*Moreover, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\mathcal{L}(s + ikh) - f(s)| < \varepsilon,$$
$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon\Big\} > 0$$

*exists for all but at most countably many $\varepsilon > 0$.*

For positive $h, h_1, \ldots, h_r$, define one more set

$$L(\mathbb{P}; \alpha_1, \ldots, \alpha_r; h, h_1, \ldots, h_r; \pi) = \{(h \log p : p \in \mathbb{P}), (h_j \log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \ldots, r), 2\pi\}.$$

Then we have the following generalization of Theorem 6

THEOREM 7. *Suppose that the set* $L(\mathbb{P};\alpha_1,\dots,\alpha_r;h,h_1,\dots,h_r;\pi)$ *is linearly independent over* $\mathbb{Q}$ *and* $\mathcal{L}\in\widetilde{\mathcal{S}}$. *Let* $K\in\mathcal{K}_{\mathcal{L}}$, $K_1,\dots,K_r\in\mathcal{K}$, *and* $f(s)\in H_{0\mathcal{L}}(K)$, $f_1(s)\in H(K_1),\dots,f_r(s)\in H(K_r)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}|\mathcal{L}(s+ikh)-f(s)|<\varepsilon,$$
$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+ikh_j,\alpha_j;\mathfrak{a}_j)-f_j(s)|<\varepsilon\Big\}>0.$$

*Moreover, the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}|\mathcal{L}(s+ikh)-f(s)|<\varepsilon,$$
$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+ikh_j,\alpha_j;\mathfrak{a}_j)-f_j(s)|<\varepsilon\Big\}>0$$

*exists for all but at most countably many* $\varepsilon>0$.

The latter theorem can be generalized in the following manner. Suppose that $\mathfrak{a}_{jl}=\{a_{mjl}: m\in\mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $q_{jl}\in\mathbb{N}$, $j=1,\dots,r$, $l=1,\dots,l_j$. For $j=1,\dots,r$, let $q_j$ be the least common multiple of the periods $q_{j1},\dots,q_{jl_j}$, and

$$A_j=\begin{pmatrix} a_{0j1} & a_{0j2} & \dots & a_{0jl_j}\\ a_{1j1} & a_{1j2} & \dots & a_{1jl_j}\\ \dots & \dots & \dots & \dots\\ a_{q_j-1,j1} & a_{q_j-1,j2} & \dots & a_{q_j-1,jl_j}\end{pmatrix}.$$

THEOREM 8. *Suppose that* $\mathcal{L}\in\widetilde{\mathcal{S}}$, *the set* $L(\mathbb{P};\alpha_1,\dots,\alpha_r;h,h_1,\dots,h_r;\pi)$ *is linearly independent over* $\mathbb{Q}$ *and that* $\operatorname{rank}(A_j)=l_j$, $j=1,\dots,r$. *Let* $K\in\mathcal{K}_{\mathcal{L}}$ *and* $f(s)\in H_{0\mathcal{L}}(K)$, *and for* $j=1,\dots,r$, $l=1,\dots,l_j$, *let* $K_{jl}\in\mathcal{K}$, $f_{jl}(s)\in H(K_{jl})$. *Then, for every* $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}|\mathcal{L}(s+ikh)-f(s)|<\varepsilon,$$
$$\sup_{1\leqslant j\leqslant r}\sup_{1\leqslant l\leqslant l_j}\sup_{s\in K_{jl}}|\zeta(s+ikh_j,\alpha_j;\mathfrak{a}_{jl})-f_{jl}(s)|<\varepsilon\Big\}>0.$$

*Moreover, the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}|\mathcal{L}(s+ikh)-f(s)|<\varepsilon,$$
$$\sup_{1\leqslant j\leqslant r}\sup_{1\leqslant l\leqslant l_j}\sup_{s\in K_{jl}}|\zeta(s+ikh_j,\alpha_j;\mathfrak{a}_{jl})-f_{jl}(s)|<\varepsilon\Big\}>0$$

*exists for all but at most countably many* $\varepsilon>0$.

We see that Theorem 6 is a partial case of Theorem 7 with $h_1=\dots=h_r=h$, and Theorem 7 is a partial case of Theorem 8 with $l_1=\dots=l_r=1$. Therefore, it suffices to prove Theorem 8.

The next section is of probabilistic character. It is devoted to limit theorems on weakly convergent certain probability measures connected to the functions $\mathcal{L}(s)$ and $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$.

## 2. Probabilistic results

Let $G$ be a region on the complex plane, and $H(G)$ be the space of analytic functions on $G$ endowed with the topology of uniform convergence on compacta. We preserve the notation of [17]. Thus, let

$$u = \sum_{j=1}^{r} l_j, \quad v = u + 1,$$

and

$$H^v = H^v(D_{\mathcal{L}}, D) = H(D_{\mathcal{L}}) \times H^u(D),$$

where $H^u(D) = \underbrace{H(D) \times \cdots \times H(D)}_{u}$. Denote by $\mathcal{B}(\mathbb{X})$ the Borel $\sigma$-field of the space $\mathbb{X}$, and use the notation

$$Z\left(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}\right) = \left(\mathcal{L}(\hat{s}), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\right),$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$. For $A \in \mathcal{B}(H^v)$ and $N \in \mathbb{N}_0$, define

$$P_N(A) = \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : Z\left(\hat{s} + ikh, \underline{s} + ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}\right) \in A\right\},$$

where $s + ik\underline{h} = (s + ikh_1, \dots, s + ikh_r)$. In this section, we will consider the weak convergence of $P_N$ as $N \to \infty$. For the definition of the limit measure, we need a certain $H^v$-valued random element. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\hat{\Omega} = \prod_{p \in \mathbb{P}} \gamma_p, \quad \Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The classical Tikhonov theorem implies that the infinite-dimensional tori $\hat{\Omega}$ and $\Omega$ with the product topology and pointwise multiplication are compact topological Abelian groups. Hence,

$$\underline{\Omega} = \hat{\Omega} \times \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$, is again a compact topological Abelian group. Therefore, on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$, the probabilistic Haar measure $m_H$ can be defined, and we obtain the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$. Denote by $\hat{\omega}(p)$ the $p$th component of $\hat{\omega} \in \hat{\Omega}$, $p \in \mathbb{P}$, and by $\omega_j(m)$ the $m$th component of $\omega_j \in \Omega_j$, $m \in \mathbb{N}_0$, $j = 1, \dots, r$. Moreover, let $\omega = (\hat{\omega}, \omega_1, \dots, \omega_r)$ be elements of $\underline{\Omega}$. Now, on the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$, define the $H^v$-valued random element $Z(\hat{s}, s, \omega, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})$ by the formula

$$\begin{aligned} \underline{Z}(\omega) = Z(\hat{s}, s, \omega, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) = \big(&\mathcal{L}(\hat{s}, \hat{\omega}), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\ &\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big), \end{aligned}$$

where

$$\mathcal{L}(\hat{s}, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{a(m)\hat{\omega}(m)}{m^{\hat{s}}}, \quad \hat{s} \in D_{\mathcal{L}},$$

with

$$\hat{\omega}(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \hat{\omega}^l(p), \quad m \in \mathbb{N},$$

and, for $s \in D$,

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r, \ l = 1, \dots, l_j.$$

We observe that, for almost all $\hat{\omega} \in \hat{\Omega}$, the equality

$$\mathcal{L}(\hat{s}, \hat{\omega}) = \exp\left\{ \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{b(p^k)\hat{\omega}^k(p)}{p^{k\hat{s}}} \right\}$$

holds [21] with certain coefficients $b(p^k)$.

Denote by $P_Z$ the distribution of the random element $Z(\hat{s}, s, \omega, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})$, i.e., $P_Z$ is a probability measure on $(H^v, \mathcal{B}(H^v))$ defined by

$$P_Z(A) = m_H\left\{\omega \in \underline{\Omega} : Z(\hat{s}, s, \omega, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\right\}, \quad A \in \mathcal{B}(H^v).$$

Now, we are able to state a limit theorem for $P_N$.

THEOREM 9. *Suppose that $\mathcal{L} \in \widetilde{\mathcal{S}}$, the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$ and that $\operatorname{rank}(A_j) = l_j$, $j = 1, \dots, r$. Then $P_N$ converges weakly to $P_Z$ as $N \to \infty$. Moreover, the support of the measure $P_Z$ is the set $S_{\mathcal{L}} \times H^u(D)$, where*

$$S_{\mathcal{L}} = \left\{g \in H(D_{\mathcal{L}}) : g(s) \neq 0 \,\text{or}\, g(s) \equiv 0\right\}.$$

We divide the proof of Theorem 9 into Lemmas. The first of them deals with weak convergence on the group $\underline{\Omega}$.

LEMMA 1. *Suppose that the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$Q_N(A) \overset{def}{=} \frac{1}{N+1}\Big\{0 \leqslant k \leqslant N : \Big(\Big(p^{-ikh} : p \in \mathbb{P}\Big), \Big((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0\Big), \dots, \Big((m+\alpha_r)^{-ikh_r} m \in \mathbb{N}_0\Big)\Big) \in A\Big\}, \quad A \in \mathcal{B}(\underline{\Omega}),$$

*converges weakly to the Haar measure $m_H$ as $N \to \infty$.*

PROOF. We apply the Fourier transform method. Denote by $g_N(\underline{k}, \underline{l}_1, \dots, \underline{l}_r)$, where $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$, $\underline{l}_1 = (l_{1m} : l_{1m} \in \mathbb{Z},\ m \in \mathbb{N}_0), \dots, \underline{l}_r = (l_{rm} : l_{rm} \in \mathbb{Z},\ m \in \mathbb{N}_0)$, the Fourier transform of $Q_N$. Since the characters of the group $\underline{\Omega}$ are of the form [13], [17]

$$\prod_{p \in \mathbb{P}}{}' \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}{}' \omega_j^{l_{jm}}(m),$$

where the sign "$\,'\,$" shows that only a finite number of integers $k_p$ and $l_{jm}$ are distinct from zero, we have that

$$g_N(\underline{k}, \underline{l}_1, \dots, \underline{l}_r) = \int_{\underline{\Omega}} \left( \prod_{p \in \mathbb{P}}{}' \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}{}' \omega_j^{l_{jm}}(m) \right) \mathrm{d}Q_N.$$

Therefore, by the definition of $Q_N$,

$$g_N(\underline{k}, \underline{l}_1, \dots, \underline{l}_r) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{p \in \mathbb{P}}{}' p^{-ikk_p h} \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}{}' (m+\alpha_j)^{-ikh_j l_{jm}}$$

$$= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ -ik \left( \sum_{p \in \mathbb{P}}{}' hk_p \log p + \sum_{j=1}^{r} \sum_{m \in \mathbb{N}_0}{}' h_j l_{jm} \log(m+\alpha_j) \right) \right\}. \qquad (2)$$

Obviously,

$$g_N(\underline{0}, \underline{0}, \dots, \underline{0}) = 1. \tag{3}$$

Since the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$, we have that

$$\exp\left\{-ik\left(\sideset{}{'}\sum_{p\in\mathbb{P}} hk_p \log p + \sum_{j=1}^{r} \sideset{}{'}\sum_{m\in\mathbb{N}_0} h_j l_{jm} \log(m+\alpha_j)\right)\right\} \neq 1 \tag{4}$$

for $(\underline{k}, \underline{l}_1, \dots, \underline{l}_r) \neq (\underline{0}, \underline{0}, \dots, \underline{0})$. Actually, if inequality (4) is not true, then

$$A \stackrel{def}{=} \sideset{}{'}\sum_{p\in\mathbb{P}} hk_p \log p + \sum_{j=1}^{r} \sideset{}{'}\sum_{m\in\mathbb{N}_0} h_j l_{jm} \log(m+\alpha_j) = 2\pi a$$

with a certain $a \in \mathbb{Z}$, and this contradicts the linear independence of the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$. Thus, inequality (4) is true, and, in view of (2), we find that, for

$$(\underline{k}, \underline{l}_1, \dots, \underline{l}_r) \neq (\underline{0}, \underline{0}, \dots, \underline{0}),$$

$$g_N(\underline{k}, \underline{l}_1, \dots, \underline{l}_r) = \frac{1 - \exp\{-i(N+1)A\}}{(N+1)(1-\exp\{-iA\})}.$$

This and (3) show that

$$\lim_{N\to\infty} g_N(\underline{k}, \underline{l}_1, \dots, \underline{l}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}, \underline{l}_1, \dots, \underline{l}_r) = (\underline{0}, \underline{0}, \dots, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}, \underline{l}_1, \dots, \underline{l}_r) \neq (\underline{0}, \underline{0}, \dots, \underline{0}), \end{cases} \tag{5}$$

and the lemma is proved because the right-hand side of (5) is the Fourier transform of the Haar measure $m_H$. $\square$

The next lemma considers probability measures on the space $(H^v, \mathcal{B}(H^v))$ defined by collections consisting of absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}, \quad m, n \in \mathbb{N},$$

$$v_n(m, \alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^\theta\right\}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N},\ j = 1, \dots, r.$$

Define the functions

$$\mathcal{L}_n(s) = \sum_{m=1}^{\infty} \frac{a(m) v_n(m)}{m^s}$$

and

$$\zeta_n(s, \alpha_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl} v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,\ l = 1, \dots, l_j.$$

Then it is known that the series for $\mathcal{L}_n(s)$ is absolutely convergent for $\sigma > \max\left(\frac{1}{2}, 1 - \frac{1}{d_{\mathcal{L}}}\right) \stackrel{def}{=} \sigma_{\mathcal{L}}$ [21], and the series for $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ are absolutely convergent for $\sigma > \frac{1}{2}$ [12]. Additionally, we define the series

$$\mathcal{L}_n(s, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{a(m) \hat{\omega}(m) v_n(m)}{m^s}$$

and

$$\zeta_n(s, \omega_j, \alpha_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl} \omega_j(m) v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,\ l = 1, \dots, l_j,$$

which, obviously, are also absolutely convergent in the above regions.

Let, for brevity,

$$Z_n(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) = (\mathcal{L}_n(\hat{s}), \zeta_n(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{rl_r})),$$

$$Z_n(\hat{s}, s, \omega, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) = (\mathcal{L}_n(\hat{s}, \hat{\omega}), \zeta_n(s, \omega_1, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \omega_1, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta_n(s, \omega_r, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \omega_r, \alpha_r; \mathfrak{a}_{rl_r})),$$

and

$$P_{N,n}(A) = \frac{1}{N+1} \# \{0 \leqslant k \leqslant N : Z_n(\hat{s} + ikh, s + ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\}, \quad A \in \mathcal{B}(H^v).$$

LEMMA 2. *Suppose that $\mathcal{L} \in \widetilde{S}$ and the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n}$ converges weakly to the measure $\hat{P}_n$ on $(H^v, \mathcal{B}(H^v))$ as $N \to \infty$, where $\hat{P}_n = m_H u_n^{-1}$, and the function $u_n : \underline{\Omega} \to H^v$ is given by the formula*

$$u_n(\omega) = Z_n(\hat{s}, s, \omega, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}).$$

PROOF. We have that

$$u_n\left(\left(p^{-ikh} : p \in \mathbb{P}\right), \left((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0\right), \dots, \left((m+\alpha_r)^{-ikh_r} m \in \mathbb{N}_0\right)\right) = Z_n(\hat{s} + ikh, s + ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})$$

Therefore,

$$P_{N,n} = Q_N u_n^{-1}, \tag{6}$$

where $Q_N$ is from Theorem 9, and the equality is understand as $P_{N,n}(A) = Q_N(u_n^{-1}A)$, $A \in \mathcal{B}(H^v)$. Moreover, the absolute convergence of the series for $\mathcal{L}_n(s, \hat{\omega})$ and $\zeta_n(s, \omega_j, \alpha_j; \mathfrak{a}_{jl})$ implies the continuity of the function $u_n$. Therefore, the lemma is a consequence of (6), Lemma 1 and Theorem 5.1 of [2]. □

Now, we will approximate $Z$ by $Z_n$ in the mean. For this, we need the metric in $H^v$. Let $G$ be a region in $\mathbb{C}$. Then it is known [5] that there exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\} \subset G$ such that

$$G = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some $l$. Taking

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(G),$$

gives a metric in $H(D)$ inducing its topology of uniform convergence on compacta. Define by $\rho_{\mathcal{L}}$ the above metric in $H(D_{\mathcal{L}})$, and by $\rho$ the metric in $H(D)$. Let

$$\underline{g} = (g, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}), \ \underline{f} = (f, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r}) \in H^v.$$

Then

$$\rho_v(\underline{g}, \underline{f}) = \max\left(\rho_{\mathcal{L}}(g, f), \max_{1 \leqslant j \leqslant r} \max_{1 \leqslant l \leqslant l_j} \rho(g_{jl}, f_{jl})\right)$$

is a desired metric in $H^v$ inducing its product topology.

LEMMA 3. *Let $\mathcal{L} \in \widetilde{\mathcal{S}}$. Then the equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho_v \left( Z(\hat{s}+ikh, s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}), Z_n(\hat{s}+ikh, s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \right) = 0$$

*holds.*

PROOF. By the definition of the metric $\rho_v$, it suffices to prove that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho_{\mathcal{L}}(\mathcal{L}(s+ikh), \mathcal{L}_n(s+ikh)) = 0,$$

and, for $j = 1, \dots, r$, $l = 1, \dots, l_j$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(\zeta(s+ikh_j, \alpha_j; \mathfrak{a}_{jl}), \zeta_n(s+ikh_j, \alpha_j; \mathfrak{a}_{jl})) = 0.$$

However, the first equality was obtained in [21], while the second equality follows from [10]. □

Now, we will consider the limit measure $\hat{P}_n$ of Lemma 2, and will prove that the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H^v$ such that

$$\hat{P}_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

LEMMA 4. *Suppose that $\mathcal{L} \in \widetilde{\mathcal{S}}$ and the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$. Then the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight.*

PROOF. On a certain probability space with the measure $\mu$, define the random variable $\theta_N$ by

$$\mu\{\theta_N = k\} = \frac{1}{N+1}, \quad k = 0, 1, \dots, N.$$

Define the $H^v$-valued random element $X_{N,n} = X_{N,n}(\hat{s}, s) = (X_{N,n}(\hat{s}), X_{N,n,1,1}(s), \dots, X_{N,n,1,l_1}(s), \dots, X_{N,n,r,1}(s), \dots, X_{N,n,r,l_r}(s)) = Z_n(\hat{s}+i\theta_N h, s+i\theta_N \underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})$. Moreover, let

$$\hat{X}_n = \hat{X}_n(\hat{s}, s) = (X_n(\hat{s}), X_{n,1,1}(s), \dots, X_{n,1,l_1}(s), \dots, X_{n,r,1}(s), \dots, X_{n,r,l_r}(s))$$

be $H^v$-valued random element with the distribution $\hat{P}_n$, where $\hat{P}_n$ is the limit measure in Lemma 2. Then the assertion of Lemma 2 can be written as

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \hat{X}_n, \tag{7}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Since the series for $\mathcal{L}_n(s)$ is absolutely convergent for $\sigma > \sigma_{\mathcal{L}}$, we have that, for $\frac{1}{2} < \sigma < \sigma_{\mathcal{L}}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\mathcal{L}_n(\sigma+it)|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{|a(m)|^2 v_n^2(m)}{m^{2\sigma}} \leqslant \sum_{m=1}^{\infty} \frac{|a(m)|^2}{m^{2\sigma}} \leqslant C_\sigma < \infty$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\mathcal{L}'_n(\sigma+it)|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{|a(m)|^2 v_n^2(m) \log^2 m}{m^{2\sigma}} \leqslant C_{\sigma,1} < \infty.$$

These estimates and an application of the Gallagher lemma [20, Lemma 1.4], which connects discrete and continuous mean squares of some functions, lead, for $\frac{1}{2} < \sigma < \sigma_{\mathcal{L}}$ and all $n \in \mathbb{N}$, to

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} |\mathcal{L}_n(\sigma + ikh)|^2 \leqslant C_{\sigma_{\mathcal{L}}} < \infty. \tag{8}$$

Let $\hat{K}_m$ be a compact set from the definition of the metric $\rho_{\mathcal{L}}$. Then (8) and the Cauchy integral formula imply, for all $n \in \mathbb{N}$,

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{\hat{s}\in\hat{K}_m} |\mathcal{L}_n(\hat{s} + ikh)| \leqslant C_m < \infty. \tag{9}$$

Let $K_m$ be a compact set from the definition of the metric $\rho$. Then, in a similar way, we obtain that, for all $j = 1, \dots, r$, $l = 1, \dots, l_j$,

$$\limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K_m} |\zeta_n(s + ikh_j, \alpha_j; \mathfrak{a}_{jl})| \leqslant C_{j,l,m} < \infty \tag{10}$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be an arbitrary number, and, for $m \in \mathbb{N}$,

$$M_m = M_m(\varepsilon) = C_m 2^{m+1}\varepsilon^{-1}, \quad M_{j,l,m} = M_{j,l,m}(\varepsilon) = C_{j,l,m} 2^{n+m+1}\varepsilon^{-1}.$$

Now, using (9) and (10), we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{N\to\infty} \mu\left\{\left(\sup_{\hat{s}\in\hat{K}_m} |X_{N,n}(\hat{s})| > M_m\right) \text{ or } \left(\exists j,l : \sup_{s\in K_m} |X_{N,n,j,l}(s)| > M_{j,l,m}\right)\right\} \\
&\quad\leqslant \limsup_{N\to\infty} \mu\left\{\sup_{\hat{s}\in\hat{K}_m} |X_{N,n}(\hat{s})| > M_m\right\} + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{N\to\infty} \mu\left\{\sup_{s\in K_m} |X_{N,n,j,l}(s)| > M_{j,l,m}\right\} \\
&\quad= \limsup_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{\hat{s}\in\hat{K}_m} |\mathcal{L}_n(\hat{s} + ikh)| > M_m\right\} \\
&\qquad+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{s\in K_m} |\zeta_n(s + ikh_j, \alpha_j; \mathfrak{a}_{jl})| > M_{j,l,m}\right\} \\
&\quad\leqslant \limsup_{N\to\infty} \frac{1}{M_m(N+1)} \sum_{k=0}^{N} \sup_{\hat{s}\in\hat{K}_m} |\mathcal{L}_n(\hat{s} + ikh)| \\
&\qquad+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{N\to\infty} \frac{1}{M_{j,l,m}(N+1)} \sum_{l=0}^{N} \sup_{s\in K_m} |\zeta_n(s + ikh_j, \alpha_j; \mathfrak{a}_{jl})| \leqslant \frac{\varepsilon}{2^m}
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, in virtue of (7),

$$\mu\left\{\left(\sup_{\hat{s}\in\hat{K}_m} |X_n(\hat{s})| > M_m\right) \text{ or } \left(\exists j,l : \sup_{s\in K_m} |X_{n,j,l}| > M_{j,l,m}\right)\right\} \leqslant \frac{\varepsilon}{2^m} \tag{11}$$

for all $n \in \mathbb{N}$. Define the set

$$\begin{aligned}
K^v(\varepsilon) = \biggl\{&(g, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^v : \sup_{\hat{s}\in\hat{K}_m} |g(\hat{s})| \leqslant M_m, \sup_{s\in K_m} |g_{11}(s)| \leqslant M_{1,1,m}, \dots, \\
&\sup_{s\in K_m} |g_{1l_1}(s)| \leqslant M_{1,l_1,m}, \dots, \sup_{s\in K_m} |g_{r1}(s)| \leqslant M_{r,1,m}, \dots, \sup_{s\in K_m} |g_{rl_r}(s)| \leqslant M_{r,l_r,m}, m \in \mathbb{N}\biggr\}.
\end{aligned}$$

Then the set $K^v(\varepsilon)$ is compact in $H^v$, and, in virtue of (11),

$$\mu\left\{\hat{X}_n \in K^v(\varepsilon)\right\} \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$, or equivalently,

$$\hat{P}_n(K^v(\varepsilon)) \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$. This shows that the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight. □

Proof. [Proof of Theorem 9] Since, by Lemma 4, the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, in virtue of the Prokhorov theorem [2, Theorem 6.1], it is relatively compact. Therefore, every sequence of $\{\hat{P}_n\}$ contains a subsequence $\{\hat{P}_{n_k}\}$ such that $\hat{P}_{n_k}$ converges to a certain probability measure $P$ on $(H^v, \mathcal{B}(H^v))$ as $k \to \infty$. Hence,

$$\hat{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{12}$$

Now, define the $H^v$-valued random element $X_N$ by the formula

$$X_N = X_N(\hat{s}, s) = Z(\hat{s} + i\theta_N h, s + i\theta_N \underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}).$$

Then an application of Lemma 3 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\{\rho_v(X_N, X_{N,n}) \geqslant \varepsilon\} \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\{0 \leqslant k \leqslant N : \\
&\quad \rho_v(Z(\hat{s}+ikh, s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}), Z_n(\hat{s}+ikh, s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})) \geqslant \varepsilon\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \rho_v((\hat{s}+ikh, s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}), Z_n(\hat{s}+ikh, s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})) = 0.
\end{aligned}$$

The latter equality, relations (7), (11) and Theorem 4.2 of [2] imply that

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{13}$$

or, in other words, $P_N$ converges weakly to $P$ as $N \to \infty$. Moreover, (13) shows that the measure $P$ is independent of the choice of the sequence $\{X_{n_k}\}$. Therefore,

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

This means that $P_N$, as $N \to \infty$, converges weakly to the limit measure $P$ of $\hat{P}_n$ as $n \to \infty$.

Denote by

$$X = (X_0, X_1, \dots, X_r), \quad X_j = (X_{j1}, \dots, X_{jl_j}), \quad j = 1, \dots, r,$$

the $H^v$-valued random element with distribution $P$. Moreover, let $\hat{P}_{n,0}, \hat{P}_{n,1}, \dots, \hat{P}_{n,r}$ be the marginal measures of $\hat{P}_n$. Then it is known [21] that $\hat{P}_{n,0}$ converges weakly to the distribution of the $H(D)$-valued random element

$$\mathcal{L}(\hat{s}, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{a(m)\hat{\omega}(m)}{m^{\hat{s}}}, \quad \hat{s} \in D_{\mathcal{L}},$$

as $n \to \infty$. The linear independence over $\mathbb{Q}$ of the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_r; h, h_1, \dots, h_r; \pi)$ implies that for the sets

$$L(\alpha_j) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0\}, \quad j = 1, \dots, r.$$

Therefore, repeating the arguments of [12], we obtain that $\hat{P}_{n,j}$ converges weakly to the distribution of the $H^{l_j}$-valued random element

$$\zeta_j = \zeta_j(\omega) = (\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl_1}), \dots, (\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl_j})$$

as $n \to \infty$, $j = 1, \dots, r$. This and the definition of the random element $X$ show that

$$X_0 \overset{\mathcal{D}}{=} \mathcal{L}(s, \hat{\omega}) \quad \text{and} \quad X_j \overset{\mathcal{D}}{=} \zeta_j, \quad j = 1, \dots, r.$$

Therefore, $P$ is the distribution of the $H^v$-valued random element

$$(\mathcal{L}(\hat{s}, \hat{\omega}), \zeta_1, \dots, \zeta_r),$$

in other words, $P_N$ converges weakly to the distribution $P_Z$ of the random element $Z$.

It remains to find the support of $P_Z$.

It is known [21] that the support of the random element $\mathcal{L}(\hat{s}, \hat{\omega})$ is the set $S_{\mathcal{L}}$. Denote by $\hat{m}_H$ the Haar measure on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$, and by $m_{H,j}$ the Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Then we have that $m_H$ is the product of the measures $\hat{m}_H$ and $m_{H,1}, \dots, m_{H,r}$. This means that, for

$$\begin{aligned} A = A_0 \times A_1 \times \dots \times A_r,\ A_0 \in \mathcal{B}(H(D_{\mathcal{L}})),\ A_j \in \mathcal{B}(H^{l_j}(D)),\ j = 1, \dots, r, \\ m_H(A) = \hat{m}_H(A_0) \cdot m_{H,1}(A_1) \cdots m_{H,r}(A_r). \end{aligned} \tag{14}$$

The spaces $H(D_{\mathcal{L}})$ and $H(D)$ are separable, therefore [2]

$$\mathcal{B}(H^v) = \mathcal{B}(H(D_{\mathcal{L}})) \times \mathcal{B}(H^{l_1}(D)) \times \dots \times \mathcal{B}(H^{l_r}(D)).$$

Hence, it suffices to consider the measure $m_H$ on sets of the type (14). Since the sets $L(\alpha_j)$ are linearly independent over $\mathbb{Q}$ and $\operatorname{rank}(A_j) = l_j$, $j = 1, \dots, r$, we have that the support of $\zeta_j$ is the set $H^{l_j}(D)$, $j = 1, \dots, r$ [13]. Therefore, using the equality (14), we obtain that

$$\begin{aligned} m_H\{\omega \in \underline{\Omega} : Z(\omega) \in A\} = \hat{m}_H\{\hat{\omega} \in \hat{\Omega} : \mathcal{L}(\hat{s}, \hat{\omega}) \in A_0\} \cdot m_{H,1}\{\omega_1 \in \Omega_1 : \zeta_1(\omega_1) \in A_1\} \cdots \\ m_{H,r}\{\omega_r \in \Omega_r : \zeta_r(\omega_r) \in A_r\}. \end{aligned}$$

This, the minimality of the support and the supports of the random elements $\mathcal{L}(\hat{s}, \hat{\omega}), \zeta_1(\omega_1), \dots, \zeta_r(\omega_r)$ imply that the support of the measure $P_Z$ is the set $S_{\mathcal{L}} \times H^u(D)$. The theorem is proved. □

## 3. Proof of universality

First we recall the Mergelyan theorem on the approximation of analytic functions by polynomials [18].

LEMMA 5. *Let $K \subset \mathbb{C}$ be a compact set with connected complements, and $f(s)$ be a continuous function on $K$ and analytic in the interior of $K$. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

PROOF. [Proof of Theorem 8] In view of Lemma 5, there exist polynomials $p(s)$ and $p_{jl}(s)$ such that

$$\sup_{s \in K_{\mathcal{L}}} \left| f(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \tag{15}$$

and

$$\sup_{s \in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2}, \quad j = 1, \dots, r,\ l = 1, \dots, l_j. \tag{16}$$

Define the set

$$G_\varepsilon = \Bigg\{ (g, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^v : \sup_{s\in K_{\mathcal{L}}} \left| f(s) - e^{p(s)} \right| < \frac{\varepsilon}{2}, \\ \sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2} \Bigg\}.$$

Then, by the second part of Theorem 9, the set $G_\varepsilon$ is an open neighborhood of the element $\big(e^{p(s)}, p_{11}, \dots p_{1l_1}, \dots, p_{r1}, \dots, p_{rl_r}\big)$ of the support of the measure $P_Z$. Hence,

$$P_Z(G_\varepsilon) > 0. \tag{17}$$

Moreover, by Theorem 9 and the equivalent of weak convergence of probability measures in terms of open sets ([2, Theorem 2.1]), we have that

$$\liminf_{N\to\infty} P_N(G_\varepsilon) \geqslant P_Z(G_\varepsilon).$$

This, the definitions of $P_N$ and $G_\varepsilon$, and (15) – (17) prove the first assertion of the theorem.

To prove the second assertion of the theorem, define the set

$$\hat{G}_\varepsilon = \Bigg\{ (g, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^v : \sup_{s\in K_{\mathcal{L}}} |g(s) - f(s)| < \varepsilon, \\ \sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} |g_{jl}(s) - f_{jl}(s)| < \varepsilon \Bigg\}.$$

Then the boundaries $\partial\hat{G}_{\varepsilon_1}$ and $\partial\hat{G}_{\varepsilon_2}$ do not intersect for different positive $\varepsilon_1$ and $\varepsilon_2$. Hence, the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_Z$ ($P_Z(\partial\hat{G}_\varepsilon) = 0$) for all but at most countably many $\varepsilon > 0$. Using of Theorem 9 and the equivalent of weak convergence of probability measures in terms of continuity sets ([2, Theorem 2.1]) yields the equality

$$\lim_{N\to\infty} P_N(\hat{G}_\varepsilon) = P_Z(\hat{G}_\varepsilon) \tag{18}$$

for all but at most countably many $\varepsilon > 0$. Inequalities (15) and (16) imply that $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, in virtue of (17), we have that $P_Z(\hat{G}_\varepsilon) > 0$. This, the definitions of $P_N$ and $\hat{G}_\varepsilon$, and (18) prove the second assertion of the theorem. □

## 4. Conclusions

In the paper, the joint discrete universality of the $L$-functions from the modified Selberg class and periodic Hurwitz zeta-functions is obtained. This means that wide collections of analytic functions $(f, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r})$ can be approximated by discrete shifts

$$(\mathcal{L}(\hat{s} + ikh), \zeta(s + ikh_1, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s + ikh_1, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s + ikh_r, \alpha_r; \mathfrak{a}_{r1}), \dots, \\ \zeta(s + ikh_r, \alpha_r; \mathfrak{a}_{rl_r})).$$

For this, the linear independence over $\mathbb{Q}$ for the set

$$\{(h \log p : p \in \mathbb{P}), (h_j \log(m + \alpha_j) : m \in \mathbb{N}_0, \ j = 1, \dots, r), 2\pi\},$$

where $\alpha_1, \dots, \alpha_r$ are parameters of periodic Hurwitz zeta-functions, and $h; h_1, \dots, h_r$ are positive numbers, is applied.

We note that theorems of the paper can be extended for collections having several $L$-functions from the Selberg class. For this, the linear independence over $\mathbb{Q}$ for the set

$$\Big\{(\hat{h}_k \log p : p \in \mathbb{P},\ k = 1, \dots, m), (h_j \log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r), 2\pi\Big\}$$

would be used.

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