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Green's function for discrete problems with nonlocal boundary conditions*

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Abstract. In this article, we investigate an m-order discrete problem with additional conditions which are described by m of linearly independent linear functionals. We have presented a formula and the existence condition of Green's function, if the general solution of a homogeneous equation is known. We have obtained the relation between two Green's functions of two inhomogeneous problems. It allows us to find Green's function for the same equation but with different additional conditions. The obtained results are applied to problems with nonlocal boundary conditions.

Keywords: m-order discrete problem, Green's function, nonlocal boundary conditions.

1 Introduction

In [1], we studied the second-order discrete problem with two additional conditions. We have obtained the solution and expression of Green's function. The results of that article are analogous to the results of a linear differential equation (see [4]).

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and $1 \leqslant m < n \in \mathbb{N}$, $X = \{0, 1, \dots, n\}$, $\tilde{X} = \{0, 1, \dots, n-m\}$, $F(X) := \{u \mid u : X \to \mathbb{K}\}$ be a linear space of real (complex) functions. We use notation $u_i = u(i)$, $i = \overline{0, n}$. A basis of this linear space is $\{\delta^j : \delta^j_i = \delta^j(i)\}$.

We consider the inhomogeneous difference equation

$$\mathcal{L}u := a_i^m u_{i+m} + \dots + a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in \tilde{X}, \tag{1}$$

with m additional conditions $\langle L_1, u\rangle = 0, \dots, \langle L_m, u\rangle = 0$, where $L_1, \dots, L_m$ are linearly independent functionals. We denote $\boldsymbol{L} = (L_1, \dots, L_m)$.

In this paper Green's function and its existence condition for a discrete m-order problem with m additional conditions are presented.

2 Notation

If we have the vector-function $\boldsymbol{u} = [u^1, \dots, u^k] \in F^k(X)$ and $\boldsymbol{i} = (i_1, \dots, i_k) \in X^k$, then we consider the matrix-function $[\boldsymbol{u}] : X^k \to M_{k\times k}(\mathbb{K})$ and its functional

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determinant $D[\boldsymbol{u}]_{\boldsymbol{i}} : X^k \to \mathbb{K}$:

$$[\boldsymbol{u}]_{\boldsymbol{i}} = \left[u^1, \dots, u^k\right]_{i_1 \dots i_k} := \begin{pmatrix} u^1_{i_1} & \cdots & u^1_{i_k} \\ \cdots & \cdots & \cdots \\ u^k_{i_1} & \cdots & u^k_{i_k} \end{pmatrix},$$

$$D[\boldsymbol{u}]_{\boldsymbol{i}} = \det[\boldsymbol{u}]_{\boldsymbol{i}} = \det\left[u^1, \dots, u^k\right]_{i_1 \dots i_k} := \begin{vmatrix} u^1_{i_1} & \cdots & u^1_{i_k} \\ \cdots & \cdots & \cdots \\ u^k_{i_1} & \cdots & u^k_{i_k} \end{vmatrix}.$$

The Wronskian determinant $W[\boldsymbol{u}]_i$ and a similar determinant $\widetilde{W}[\boldsymbol{u}]_{ij}$ in the theory of difference equations are denoted as follows

$$W[\boldsymbol{u}]_i := \begin{vmatrix} u^1_{i-k+1} & \cdots & u^k_{i-k+1} \\ \cdots & \cdots & \cdots \\ u^1_{i-1} & \cdots & u^k_{i-1} \\ u^1_i & \cdots & u^k_i \end{vmatrix} = D[\boldsymbol{u}]_{i-k+1,\dots,i}, i = k-1, \dots, n,$$

$$\widetilde{W}[\boldsymbol{u}]_{ij} := \begin{vmatrix} u^1_{i-k+1} & \cdots & u^k_{i-k+1} \\ \cdots & \cdots & \cdots \\ u^1_{i-1} & \cdots & u^k_{i-1} \\ u^1_j & \cdots & u^k_j \end{vmatrix}, \quad i = k-1, \dots, n+1,\ j \in X.$$

We consider the space $F^*(X)$ of linear functionals in the space $F(X)$. The functionals δ_j, $j = \overline{0, n}$ form a dual basis for the basis $\{\delta^i\}_{i=0}^n$. Thus, $\langle \delta_i, u \rangle = u_i$. For the functionals $\boldsymbol{f} = (f_1, \dots, f_k)$ and functions $\boldsymbol{u} = [u^1, \dots, u^k]$, the determinant is

$$D(\boldsymbol{f})[\boldsymbol{u}] := \begin{vmatrix} \langle f_1, u^1 \rangle & \cdots & \langle f_k, u^1 \rangle \\ \cdots & \cdots & \cdots \\ \langle f_1, u^k \rangle & \cdots & \langle f_k, u^k \rangle \end{vmatrix}.$$

For example, $D(\boldsymbol{\delta_i})[\boldsymbol{u}] = D[\boldsymbol{u}]_{\boldsymbol{i}}$,

$$D(\boldsymbol{f})[\boldsymbol{u}, u^0]_i := \begin{vmatrix} \langle f_1, u^1 \rangle & \cdots & \langle f_k, u^1 \rangle & u^1_i \\ \cdots & \cdots & \cdots & \cdots \\ \langle f_1, u^k \rangle & \cdots & \langle f_k, u^k \rangle & u^k_i \\ \langle f_1, u^0 \rangle & \cdots & \langle f_k, u^0 \rangle & u^0_i \end{vmatrix}.$$

Now we will use these definitions for investigation of difference equation (1). Let (if $W[\boldsymbol{u}]_{i+m} \neq 0$)

$$V[\boldsymbol{u}]_{ij} := \frac{\widetilde{W}[\boldsymbol{u}]_{i+m,j}}{W[\boldsymbol{u}]_{i+m}}, \quad i = \overline{-1, n-m},\ j \in X.$$

We introduce a function $G^c \in F(X \times \tilde{X})$:

$$G^c_{ij} := H_{i-j} V_{ji} / a^m_j, \qquad H_i := \begin{cases} 1, & i > 0, \\ 0, & i \leqslant 0. \end{cases}$$

3 Green's function

Suppose that $\mathbb{K} = \mathbb{R}$ and $X_n := X = \{0, 1, \ldots, n\}$. Let $A : F(X_n) \to F(X_{n-m}) = \operatorname{Im} A$ and $B : F(X_n) \to F(X_{M-n+m})$ be linear operators, $0 \leqslant m \leqslant n$. Consider the operator equation $Au = f$, where $u \in F(X_n)$ is unknown and $f \in F(X_{n-m})$ is given, with the additional operator equation $Bu = 0$.

If the solution of the problem $Au = f$, $Bu = 0$ allows the following representation:

$$u_i = \sum_{j=0}^{n-m} G_{ij} f_j, \quad i \in X_n, \tag{2}$$

then $G \in F(X_n \times X_{n-m})$ is called *Green's function* of operator A with the additional condition $Bu = 0$. Green's function exists, if $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$ (see [2, 3]).

For finite-difference schemes (FDS), discrete functions are defined at the points $x_i \in [0, L]$ and $f_i = f(x_i)$. In this paper, we introduce meshes:

$$\bar{\omega}^h = \{0 = x_0 < x_1 < \cdots < x_n = L\}, \qquad \omega^h = \bar{\omega}^h \setminus \{x_0, x_n\}, \tag{3}$$

with the step sizes $h_i = x_i - x_{i-1}$, $1 \leqslant i \leqslant n$, $h_0 = h_{n+1} = 0$, and a semi-integer mesh

$$\omega^h_{1/2} = \{x_{i+\frac{1}{2}} \mid x_{i+\frac{1}{2}} = (x_i + x_{i+1})/2,\ 0 \leqslant i \leqslant n-1\}$$

with the step sizes $h_{i+\frac{1}{2}} = (h_i + h_{i+1})/2$, $0 \leqslant i \leqslant n$. We define the inner product $(U, V)_{\bar{\omega}^h} := \sum_{i=0}^{n} U_i V_i h_{i+\frac{1}{2}}$, where $U, V \in F(\bar{\omega}^h)$, and the following mesh operators

$$(\delta Z)_{i+\frac{1}{2}} = \frac{Z_{i+1} - Z_i}{h_{i+1}}, \quad Z \in F\big(\bar{\omega}^h\big), \qquad (\delta Z)_i = \frac{Z_{i+\frac{1}{2}} - Z_{i-\frac{1}{2}}}{h_{i+\frac{1}{2}}}, \quad Z \in F\big(\omega^h_{1/2}\big).$$

If $A : F(\bar{\omega}^h) \to F(\omega)$ and $f \in F(\omega)$, where $\omega = \bar{\omega}^h$ or $\omega = \omega^h$, then we define the Green's function $G \in F(\bar{\omega}^h \times \omega)$:

$$u_i = \sum_{j: x_j \in \omega} G_{ij} f_j, \quad i \in X_n. \tag{4}$$

Let $\{u^1, \ldots, u^m\} \in F(X)$ be a linearly independent fundamental system of homogeneous equation (1).

Lemma 1. *Green's function for problem* (1) *with the homogeneous additional conditions* $\langle L_1, u\rangle = 0, \ \ldots, \ \langle L_m, u\rangle = 0$, *where the functionals* $L_1, \ldots, L_m$ *are linearly independent, is equal to*

$$G_{ij} = \frac{D(\boldsymbol{L}, \delta_i)[\boldsymbol{u}, G^c_{\cdot,j}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad i \in X,\ j \in \tilde{X}. \tag{5}$$

For the theoretical investigation of problems with NBCs, the next result about the relations between Green's functions G^u_{ij} and G^v_{ij} of two nonhomogeneous problems

$$\begin{cases} \mathcal{L}u = f, \\ \langle l_k, u\rangle = 0, \quad k = \overline{1,m}, \end{cases} \qquad \begin{cases} \mathcal{L}v = f, \\ \langle L_k, v\rangle = 0, \quad k = \overline{1,m}, \end{cases} \tag{6}$$

with the same f, is useful.

Theorem 1. *If Green's function G^u exists and the functionals $L_1,\dots,L_m$ are linearly independent, then*

$$G^v_{ij}=\frac{D(\boldsymbol{L},\delta_i)[\boldsymbol{u},G^u_{\cdot,j}]}{D(\boldsymbol{L})[\boldsymbol{u}]},\quad i\in X,\ j\in\tilde{X}. \tag{7}$$

4 Applications to problems with NBC

Let us investigate Green's function for the problem with nonlocal boundary conditions

$$\mathcal{L}u:=a_i^m u_{i+m}+\cdots+a_i^1u_{i+1}+a_i^0u_i=f_i,\quad i\in\tilde{X}, \tag{8}$$

$$\langle L_k,u\rangle:=\langle\kappa_k,u\rangle-\gamma_k\langle\varkappa_k,u\rangle=0,\quad k=\overline{1,m}. \tag{9}$$

We can write many problems with nonlocal boundary conditions (NBC) in this form, where $\langle\kappa_k,u\rangle:=\langle\kappa_k^i,u_i\rangle$, $k=\overline{1,m}$, is a classical part, and $\langle\varkappa_k,u\rangle:=\langle\varkappa_k^i,u_i\rangle$, $k=\overline{1,m}$, is a nonlocal part of boundary conditions (BC).

If $\gamma_1,\dots,\gamma_m=0$, then problem (8)–(9) becomes classical. Suppose that there exists Green's function G^{cl}_{ij} for the classical case. Then Green's function exists for problem (8)–(9), if $\vartheta:=D(\boldsymbol{L})[\boldsymbol{u}]\neq0$. For $L_k=\kappa_k-\gamma_k\varkappa_k$, $k=\overline{1,m}$, we derive that

$$\vartheta=\sum_{j_1,\dots,j_m=0}^{1}\gamma_1^{j_1}\cdots\gamma_m^{j_m}D\big(\big((1-j_1)\kappa_1-j_1\varkappa_1\big),\dots,\big((1-j_m)\kappa_m-j_m\varkappa_m\big)\big)[\boldsymbol{u}].$$

Since $\langle\kappa_l^k,G^{\mathrm{cl}}_{kj}\rangle=0$, $l=\overline{1,m}$, we can rewrite formula (7) as follows:

$$\begin{aligned}G_{ij}&=G^{\mathrm{cl}}_{ij}+\gamma_1\big\langle\varkappa_1^k,G^{\mathrm{cl}}_{kj}\big\rangle\frac{D(\delta_i,L_2,\dots,L_m)}{\vartheta}+\cdots+\gamma_m\big\langle\varkappa_m^k,G^{\mathrm{cl}}_{kj}\big\rangle\frac{D(L_1,\dots,L_{m-1},\delta_i)}{\vartheta}\\&=\frac{1}{\vartheta}\begin{vmatrix}\langle L_1,u^1\rangle&\cdots&\langle L_m,u^1\rangle&u_i^1\\\cdots&\cdots&\cdots&\cdots\\\langle L_1,u^m\rangle&\cdots&\langle L_m,u^m\rangle&u_i^m\\-\gamma_1\langle\varkappa_1^k,G^{\mathrm{cl}}_{kj}\rangle&\cdots&-\gamma_m\langle\varkappa_m^k,G^{\mathrm{cl}}_{kj}\rangle&G^{\mathrm{cl}}_{ij}\end{vmatrix}.\end{aligned} \tag{10}$$

Example 1. Let us consider the differential equation with three NBCs

$$u'''=f(x),\quad x\in(0,1), \tag{11}$$

$$u(0)=\gamma_0u(\xi_0),\qquad u'(0)=\gamma_1u(\xi_1),\qquad u(1)=\gamma_nu(\xi_n),\quad 0<\xi_0,\xi_1,\xi_n<1. \tag{12}$$

We introduce a mesh $\bar{\omega}^h$ (see (3)). Denote $u_i=u(x_i)$, $f_i=f(x_i)$ for $x_i\in\bar{\omega}^h$. Then problem (11), (12) can be approximated by the FDS

$$\big(\delta^3u\big)_{i+1/2}=\bar{f}_{i+1/2},\quad x_i\in\omega^h_{1/2}\setminus\{x_{1/2},x_{n-1/2}\}, \tag{13}$$

$$u_0=\gamma_0u_{s_0},\qquad u_1=\gamma_1u_{s_1},\qquad u_n=\gamma_nu_{s_n}. \tag{14}$$

We suppose that the points ξ_0, ξ_1, ξ_n are coincident with the grid points, i.e., $\xi_0=x_{s_0}$, $\xi_1=x_{s_1}$, $\xi_n=x_{s_n}$.

We rewrite Eq. (13) in the following form

$$a_i^3u_{i+3}+a_i^2u_{i+2}+a_i^1u_{i+1}+a_i^0u_i=f_i,\quad i\in\tilde{X}, \tag{15}$$

where

$$a_i^3 = \frac{1}{h_{i+2}h_{i+5/2}h_{i+3}}, \qquad a_i^2 = -\frac{2h_{i+3/2}+h_{i+3}}{h_{i+3/2}h_{i+2}^2h_{i+3}}, \qquad a_i^1 = \frac{h_{i+1}+2h_{i+5/2}}{h_{i+1}h_{i+2}^2h_{i+5/2}},$$
$$a_i^0 = -\frac{1}{h_{i+2}h_{i+1}h_{i+3/2}}, \qquad f_i = \bar{f}_{i+1/2}, \quad i \in \tilde{X}.$$

We can take the following fundamental system: $u_i^1 = 1$, $u_i^2 = x_i$, $u_i^3 = x_i^2$. Then

$$\widetilde{W}[\boldsymbol{u}]_{j+3,i} = \begin{vmatrix} 1 & 1 & 1 \\ x_{j+1} & x_{j+2} & x_i \\ x_{j+1}^2 & x_{j+2}^2 & x_i^2 \end{vmatrix} = h_{j+2}(x_i - x_{j+2})(x_i - x_{j+1}),$$
$$W_{j+3} = h_{j+3}h_{j+2}(x_{j+3} - x_{j+1}),$$
$$V_{ji} = \frac{(x_i - x_{j+2})(x_i - x_{j+1})}{(x_{j+3} - x_{j+2})(x_{j+3} - x_{j+1})},$$
$$V_{n-2,i} = V_{n-1,i} = V_{ni} = 0, \quad i \in X,\ j = -1, 0, 1, \ldots, n-3.$$

As a result, we obtain

$$G_{ij}^c = \frac{H_{i-j}V_{ji}}{a_j^3} = H_{i-j}\frac{(x_i - x_{j+2})(x_i - x_{j+1})}{2}h_{j+2}.$$

For a problem with the BCs $u_0 = u_1 = u_n = 0$ we have $D(L)[\boldsymbol{u}] = x_1(1 - x_1)$, $D(L, \delta_i)[\boldsymbol{u}, G_{\cdot,j}^c] = x_1(1 - x_1)G_{ij}^c - x_1x_i(x_i - x_1)G_{nj}^c$, and we express Green's function G^{cl} of the Dirichlet problem via Green's function G^c of the initial problem

$$G_{ij}^{\mathrm{cl}} = G_{ij}^c - \frac{x_i(x_i - x_1)}{1 - x_1}G_{nj}^c.$$

We derive expression for "classical" Green's function:

$$G_{ij}^{\mathrm{cl}} = h_{j+2}\begin{cases} \frac{(x_i-1)(x_i(x_{j+2}+x_{j+1}-x_1)+x_{j+1}x_{j+2}(x_1-x_i-1))}{2(1-x_1)}, & 0 \leqslant j+1 \leqslant i \leqslant n, \\ -\frac{x_i(x_i-x_1)(1-x_{j+2})(1-x_{j+1})}{2(1-x_1)}, & 0 \leqslant i \leqslant j+1 \leqslant n-2, \end{cases}$$

Now we consider the conditions (14), where $s_0 \neq 0$, $s_1 \neq 1$, $s_n \neq n$.

For a "nonlocal" problem with the boundary conditions (14), we have

$$\begin{aligned} \vartheta :=& D(L)[\boldsymbol{u}] \\ =& \begin{vmatrix} \langle L_1, 1\rangle & \langle L_2, 1\rangle & \langle L_3, 1\rangle \\ \langle L_1, x\rangle & \langle L_2, x\rangle & \langle L_3, x\rangle \\ \langle L_1, x^2\rangle & \langle L_2, x^2\rangle & \langle L_3, x^2\rangle \end{vmatrix} = \begin{vmatrix} 1-\gamma_0 & 1-\gamma_1 & 1-\gamma_n \\ -\gamma_0\xi_0 & x_1-\gamma_1\xi_1 & 1-\gamma_n\xi_n \\ -\gamma_0\xi_0^2 & x_1^2-\gamma_1\xi_1^2 & 1-\gamma_n\xi_n^2 \end{vmatrix} \\ =& x_1(1-x_1) - \gamma_0(1-\xi_0)(x_1-\xi_0)(1-x_1) - \gamma_1\xi_1(1-\xi_1) + \gamma_n\xi_n x_1(x_1-\xi_n) \\ &+ \gamma_0\gamma_1(1-\xi_0)(1-\xi_1)(\xi_1-\xi_0) - \gamma_0\gamma_n(x_1-\xi_0)(x_1-\xi_n)(\xi_n-\xi_0) \\ &+ \gamma_1\gamma_n\xi_1\xi_n(\xi_n-\xi_1) - \gamma_0\gamma_1\gamma_n(\xi_1-\xi_0)(\xi_n-\xi_0)(\xi_n-\xi_1). \end{aligned}$$

It follows from Eq. (10) that

$$\begin{aligned} G_{ij} = & G^{\mathrm{cl}}_{ij} + \frac{\gamma_0}{\vartheta}\bigl(-(x_i - x_1)(1 - x_i)(1 - x_1) + \gamma_1(1 - x_i)(x_i - \xi_1)(1 - \xi_1) \\ & + \gamma_n(x_i - x_1)(x_i - \xi_n)(x_1 - \xi_n) + \gamma_1\gamma_n(x_i - \xi_1)(x_i - \xi_n)(\xi_n - \xi_1)\bigr) G^{\mathrm{cl}}_{s_0 j} \\ & + \frac{\gamma_1}{\vartheta}\bigl(x_i(1 - x_i) - \gamma_0(1 - x_i)(x_i - \xi_0)(1 - \xi_0) + \gamma_n x_i \xi_n (x_i - \xi_n) \\ & - \gamma_0\gamma_n(x_i - \xi_0)(x_i - \xi_n)(\xi_n - \xi_0)\bigr) G^{\mathrm{cl}}_{s_1 j} \\ & + \frac{\gamma_n}{\vartheta}\bigl(x_1(1 - x_1) - \gamma_0(x_i - x_1)(x_i - \xi_0)(x_1 - \xi_0) - \gamma_1 x_i \xi_1 (x_i - \xi_1) \\ & + \gamma_0\gamma_1(x_i - \xi_0)(x_i - \xi_1)(\xi_1 - \xi_0)\bigr) G^{\mathrm{cl}}_{s_n j} \end{aligned}$$

if $\vartheta \neq 0$. Green's function does not exist for $\vartheta = 0$.

Green's function for problems with additional conditions is related with Green's function of a similar problem and this relation is expressed by formulae (7). Green's function exists, if $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$. If we know Green's function for the problem with additional conditions and the fundamental basis of a homogeneous difference equation, then we can obtain Green's function for a problem with the same equation, but with different additional conditions.

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REZIUMĖ

Diskretaus uždavinio su nelokaliosiomis kraštinėmis sąlygomis Gryno funkcija

S. Roman and A. Štikonas

Straipsnyje tiriamas m-tosios eilės diskretusis uždavinys su papildomomis sąlygomis, kurios yra aprašytos tiesiškai nepriklausomais tiesiniais funkcionalais. Pateikiama Gryno funkcijos išraiška ir jos egzistavimo sąlyga, jei žinoma homogeninės lygties fundamentalioji sistema. Gautas dviejų Gryno funkcijų sąryšis uždaviniams su ta pačia lygtimi, bet su skirtingomis papildomomis sąlygomis. Rezultatai taikomi konstruojant Gryno funkcijas uždaviniams su nelokaliosiomis kraštinėmis sąlygomis.

Raktiniai žodžiai: m-tos eilės diskretusis uždavinys, Gryno funkcija, nelokaliosios kraštinės sąlygos.