Lietuvos matematikos rinkinys. LMD darbai ISSN 0132-2818
52 tomas, 2011, 257–261 www.mii.lt/LMR/

Partial cut elimination for combinations of propositional multi-modal logics with past time

Jūratė Sakalauskaitė

Institute of Mathematics and Informatics, Vilnius University
Akademijos 4, LT-08663 Vilnius
E-mail: jurate.sakalauskaite@vu.mii.lt

Abstract. We consider combinations of nine propositional multi-modal logics with propositional discrete linear time temporal logic with past time. For these combinations, we present sound and complete Gentzen-type sequent calculi with a restricted cut rule.

Keywords: modal logic, temporal logic, sequent calculus, cut.

1 Introduction

Recently, there has been a significant interest in multi-modal logics combining operators of knowledge and time [1, 2, 3, 5, 7, 8, 9, 12]. With only a few exceptions, this literature deals with *future time* temporal operators. As indicated in [3] the logic of future time axioms is too weak to fully express the unique initial states, synchrony and some other properties of computer systems. Another reason to consider knowledge in combination with past time operators is that *knowledge-based* programs behave better with past-time operators than with future-time ones.

In [9] we consider logic of knowledge and past time introduced in [3]. In this paper we consider combinations of nine familiar propositional multi-modal logics with propositional discrete linear time temporal logic with past-time operators. Such a combination is denoted by LT. As far as we know such combinations of logics (except logic of knowledge and past time) has not been considered earlier.

For each formula φ of LT we introduce sequent calculus $\mathcal{G}_{LT}(\varphi)$ with a restricted cut rule. We call a cut rule *restricted* if cut formulas used in the rule are taken from some finite set (say, $\Pi_{\boldsymbol{L}}(\varphi)$). We denote such a rule ($\Pi_{\boldsymbol{L}}(\varphi)$-cut).

We prove soundness and weak completeness (with respect to the certain class of models) for the presented calculi. Decidability of provability in $\mathcal{G}_{LT}(\varphi)$ is the consequence of restricted cut rule.

2 Syntax and semantics

Let L be a logic from the set of nine well-known propositional multi-modal logics $\{\boldsymbol{K}_m, \boldsymbol{T}_m(\boldsymbol{KT}_m)$, $\boldsymbol{KD}_m$, $\boldsymbol{K4}_m, \boldsymbol{KD4}_m$, $\boldsymbol{S4}_m(\boldsymbol{KT4}_m)$, $\boldsymbol{K45}_m$, $\boldsymbol{KD45}_m$, $\boldsymbol{S5}_m$ $(\boldsymbol{KT5}_m)\}$ (see e.g. [4]). Let LTL^- be the propositional discrete linear time temporal logic with past operators (see e.g. [6, 9]). As above we denote the combination of logic L and LTL^- by LT.

The combinations of logics we are considering are all propositional and share a common syntax. Let P be a nonempty set of primitive propositions. Let $\{1, \ldots, m\}$ be a set of agents, $m \geqslant 1$. The language $\mathcal{L}$ is given by the abstract syntax

$$\phi = p|\mathit{false}|\neg\phi|\phi \vee \phi'|\bigcirc\phi|\bigcirc_W\phi|\phi U\phi'|\phi S\phi'|\Box_i\phi,$$

where $p \in P$, and $1 \leqslant i \leqslant m$ is the index of an agent. The operators are, respectively *not, or, next* (*tomorrow*), *weak yesterday, until, since*, and *necessity for i* with their usual meanings. We use the familiar propositional abbreviations (*true*, $\wedge, \supset$).

We recall the definition of models for LT (introduced in [12]). Let S be a nonempty set of states. A *timeline* r is an infinitely long to the future, bounded to the past, linear, discrete sequence of states, indexed by natural numbers. Let *Tlines* be the set of all timelines. A *point* is a pair (r, n), where r is a timeline and $n \in N$ is a temporal index into r. Let the set of all points (over S) be *Points*.

A *model*, M, for $\mathcal{L}$, is a structure $M = (TL, R_1, \ldots, R_m, \pi)$, where: $TL \subseteq \mathit{Tlines}$ is a set of timelines; R_i, for all $i \in \{1, \ldots, m\}$, is an agent accessibility relation over *Points*, i.e. $R_i \subseteq \mathit{Points} \times \mathit{Points}$; $\pi : \mathit{Points} \times P \to \{T, F\}$ is a valuation.

As usually, we inductively define the semantics of the language via satisfaction relation "$\models$". We present semantics only for modal and past temporal operators. (Semantics for future operators see e.g. in [2, 12].)

- $(M, (r, n)) \models \Box_i\phi$ iff $\forall r' \in TL, \forall n' \in N$, if $((r, n), (r', n')) \in R_i$ then $(M, (r', n')) \models \phi$;
- $(M, (r, n)) \models \bigcirc_W\phi$ iff $n = 0$ or $(M, (r, n-1)) \models \phi$;
- $(M, (r, n)) \models \phi S\psi$ iff $\exists n' \in N$ such that $n' \leqslant n$ and $(M, (r, n')) \models \psi$ and, $\forall k \in N$ if $n' < k \leqslant n$ then $(M, (r, k)) \models \phi$.

We say that formula ϕ is *valid* in a model M iff $(M, (r, n)) \models \phi$ for every point $(r, n) \in M$. Let $\mathcal{C}$ be a class of models. We say that ϕ is $\mathcal{C}$-*valid* iff ϕ is valid in every model from $\mathcal{C}$.

We get the class of models (denoted by $\mathcal{C}_{\boldsymbol{L}}$) for LT by imposing the familiar corresponding conditions on the accessibility relations $R_1, \ldots, R_m$ (see e.g. [4]). For example, $\mathcal{C}_{\boldsymbol{K}_m}$ is the class of models with no conditions on each accessibility relation; $\mathcal{C}_{\boldsymbol{KD45}_m}$ is the class of models such that each accessibility relation is serial, transitive and Euclidean.

3 Sequent calculi for combinations of logics

3.1 Preliminaries

Small greek letters stand for arbitrary formulas. The capital greek letters $\Gamma, \Delta, \Sigma, \ldots$ stand for finite sets (possibly, empty) of formulas of the language $\mathcal{L}$. Let Γ be a set of formulas $\{\phi_1, \ldots, \phi_n\}$. We use the following convenient abbreviations: $\Box_i\Gamma = \{\Box_i\phi_1, \ldots, \Box_i\phi_n\}$; $\vee\Gamma = \phi_1 \vee \ldots \vee \phi_n$; $\wedge\Gamma = \phi_1 \wedge \ldots \wedge \phi_n$. $\Gamma \Rightarrow \Delta$ is called a *sequent*. The semantical meaning of sequent is $\wedge(\Gamma) \supset \vee(\Delta)$. For any sets Γ, Δ and formulas ϕ, ψ, the set $\Gamma \cup \Delta \cup \{\phi\} \cup \{\psi\}$ is denoted by $\phi, \psi, \Gamma, \Delta$.

3.2 Construction of closure sets $FL_{\boldsymbol{L}}(\varphi)$ and $\Pi_{\boldsymbol{L}}(\varphi)$

In the introduction we have presented the notion of ($\Pi_{\boldsymbol{L}}(\varphi)-$ cut) rule. At we end of this subsection we construct the finite set of formulas $\Pi_{\boldsymbol{L}}(\varphi)$.

At first we define the *Fisher–Ladner closure* $FL(\varphi)$. $FL(\varphi)$ is obtained by addition of the following clause to the respective definition in [10]: if $\Box_i\psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$.

Now we define the closure set $FL'_{\boldsymbol{L}}(\varphi)$. The set $FL'_{\boldsymbol{K}_m}(\varphi) = FL(\varphi) \cup \{\bigcirc\neg\psi | \bigcirc\psi \in FL(\varphi)\} \cup \{\bigcirc_W\neg\psi | \bigcirc_W\psi \in FL(\varphi)\}$. $FL'_{\boldsymbol{KD}_m}(\varphi)$, $FL'_{\boldsymbol{T}_m}(\varphi)$, and $FL'_{\boldsymbol{S}5_m}(\varphi)$ are defined to be $FL'_{\boldsymbol{K}_m}(\varphi)$. The set $FL_{\boldsymbol{K}_m}(\varphi)$ is defined to be $FL'_{\boldsymbol{K}_m}(\varphi) \cup \{\neg\psi | \psi \in FL'_{\boldsymbol{K}_m}(\varphi)\}$. $FL'_{\boldsymbol{K4}_m}(\varphi)$, $FL'_{\boldsymbol{KD4}_m}(\varphi)$ and $FL'_{\boldsymbol{S4}_m}(\varphi)$ are defined to be $FL'_{\boldsymbol{K}m}(\varphi) \cup \{\Box_i\Box_i\psi | \Box_i\psi \in FL(\varphi), 1 \leqslant i \leqslant m\}$. The set $FL'_{\boldsymbol{KD45}_m}(\varphi)$ is defined to be $FL'_{\boldsymbol{K4}_m}(\varphi) \cup \{\Box_i\neg\Box_i\psi | \neg\Box_i\psi \in FL_{\boldsymbol{K}_m}(\varphi), 1 \leqslant i \leqslant m\}$. The set $FL'_{\boldsymbol{K45}_m}(\varphi)$ is defined to be $FL'_{\boldsymbol{KD45}_m}(\varphi) \cup \{\Box_i(\Box_i false \supset false), \Box_i false | 1 \leqslant i \leqslant m\}$.

For each logic LT the set $FL_{\boldsymbol{L}}(\varphi)$ is defined to be $FL'_{\boldsymbol{L}}(\varphi) \cup \{\neg\psi | \psi \in FL'_{\boldsymbol{L}}(\varphi)\}$.

The closure set (set of cut formulas) $\Pi_{\boldsymbol{L}}(\varphi)$ is to be defined as the following extension of the closure set $FL_{\boldsymbol{L}}(\varphi)$, i.e. $\Pi_{\boldsymbol{L}}(\varphi)$ is the set $\{(\wedge M_1) \vee \cdots \vee (\wedge M_k), \bigcirc((\wedge M_1) \vee \cdots \vee (\wedge M_k)), \bigcirc_W((\wedge M_1) \vee \cdots \vee (\wedge M_k)) | M_1, \ldots, M_k \subseteq FL_{\boldsymbol{L}}(\varphi), k \geqslant 1\}$.

Remark 1. We get the set $\Pi_{\boldsymbol{L}}(\varphi)$ by looking through the proofs of statements which are used to prove the truth theorem and Lemma 2 (see below) (similar as in [9]). These statements are omitted here due to the lack of space.

3.3 Gentzen-type inference rules for operators of necessity

Now we list the well-known inference rules for operators of necessity $\Box_i$ for each multi-modal logic L we are considering (see e.g. [11]). Let $i = 1, \ldots, m$.

For $\boldsymbol{K}_m$: $\frac{\Gamma \Rightarrow \phi}{\Sigma, \Box_i\Gamma \Rightarrow \Box_i\phi, \Pi}(\Box_i)_K$;

For $\boldsymbol{KD}_m$: $(\Box_i)_K$ and $\frac{\Gamma, \phi \Rightarrow}{\Sigma, \Box_i\Gamma, \Box_i\phi \Rightarrow, \Pi}(\Box_i)_D$;

For $\boldsymbol{K4}_m$: $\frac{\Gamma, \Box_i\Gamma \Rightarrow \phi}{\Sigma, \Box_i\Gamma \Rightarrow \Box_i\phi, \Pi}(\Box_i)_{K4}$;

For $\boldsymbol{T}_m$: $(\Box_i)_K$ and $\frac{\phi, \Gamma \Rightarrow \Delta}{\Box_i\phi, \Gamma \Rightarrow \Delta}(\Box_i)_T$;

For $\boldsymbol{KD4}_m$:$(\Box_i)_D$ and $\frac{\Gamma' \Rightarrow \phi}{\Sigma, \Box_i\Gamma \Rightarrow \Box_i\phi, \Pi}(\Box_i)_{KD4}$, where Γ' is obtained from Γ by prefixing zero or more formulas in Γ by $\Box_i$;

For $\boldsymbol{S4}_m$: $(\Box_i)_T$ and $\frac{\Box_i\Gamma \Rightarrow \phi}{\Sigma, \Box_i\Gamma \Rightarrow \Box_i\phi, \Pi}(\Box_i)_{S4}$;

For $\boldsymbol{KD45}_m$: $\frac{\Gamma, \Box_i\Gamma_1 \Rightarrow \Box_i\Delta, \Theta}{\Sigma, \Box_i\Gamma, \Box_i\Gamma_1 \Rightarrow \Box_i\Delta, \Box_i\Theta, \Pi}(\Box_i)_{K45}$, where $\Theta = \emptyset$ or $\Theta = \{\phi\}$ and if $\Theta = \emptyset$ then $\Gamma \cup \Gamma_1 \cup \Delta \neq \emptyset$;

For $\boldsymbol{K45}_m$: $(\Box_i)_{K45}$ with the additional requirement that the set $\Delta \cup \Theta$ is nonempty;

For $\boldsymbol{S5}_m$: $(\Box_i)_T$ and $\frac{\Box_i\Gamma \Rightarrow \Box_i\Delta, \phi}{\Sigma, \Box_i\Gamma \Rightarrow \Box_i\Delta, \Box_i\phi, \Pi}(\Box_i)_{S5}$.

3.4 Gentzen-type sequent calculi for LT, soundness

Let $GLTL^-$ be a Gentzen type calculus for temporal logic LTL^- introduced in Section 3.3 of [10]. Gentzen-type sequent calculi $\mathcal{G}_{LT}(\varphi)$ for the logic LT and a formula φ is defined as follows: $GLTL^-$ + inference rules for operators of necessity for logic L + ($\Pi_{\boldsymbol{L}}(\varphi)$-cut) rule.

The notion of a proof of a given sequent in the introduced calculi is defined in the usual way. By the induction on the height of the proof of a sequent $\Gamma \Rightarrow \Delta$ we can verify the following:

Theorem 1 [Soundness of $\mathcal{G}_{LT}(\varphi)$]. *If a sequent $\Gamma \Rightarrow \Delta$ is provable in $\mathcal{G}_{LT}(\varphi)$ then $\wedge(\Gamma) \supset \vee(\Delta)$ is $\mathcal{C}_{\boldsymbol{L}}$-valid.*

3.5 Completeness with the restricted cut rule

Theorem 2 [Completeness of $\mathcal{G}_{LT}(\varphi)$]. *If a formula φ (of the language $\mathcal{L}$) is $\mathcal{C}_{\boldsymbol{L}}$-valid then $\Rightarrow \varphi$ is provable in $\mathcal{G}_{LT}(\varphi)$.*

We give a sketch of a proof of the completeness theorem. We say that a finite set of formulas Γ is $\Pi_{\boldsymbol{L}}(\varphi)$-*consistent* if the sequent $\Gamma \Rightarrow$ is not provable in the calculus $\mathcal{G}_{LT}(\varphi)$. We call a maximal $\Pi_{\boldsymbol{L}}(\varphi)$-consistent subset of the set $FL_{\boldsymbol{L}}(\varphi)$ an *atom*(of $FL_{\boldsymbol{L}}(\varphi)$). We prove completeness theorem by contraposition (similar as in [9] and [10]):

Lemma 1. *If $\neg\varphi$ is $\Pi_{\boldsymbol{L}}(\varphi)$-consistent formula then there exists a model $M \in C_{\boldsymbol{L}}$ and a point (r,n) such that $(M,(r,n)) \models \neg\varphi$.*

In order to prove Lemma 1 we construct a *canonical model*, denoted by $M_L^c(\varphi)$ which belongs to the class of models $\mathcal{C}_{\boldsymbol{L}}$. At first, we define a *pre-model* $M_L(\varphi) = (W_L(\varphi), <, R_1, \ldots, R_m)$, where the set of states $W_L(\phi)$ is defined to be the set of all atoms of $FL_{\boldsymbol{L}}(\varphi)$; $s < t$ iff $\{\psi | \bigcirc\psi \in s\} \subseteq t$ and $\{\psi | \bigcirc_W \psi \in t\} \subseteq s$; for all considered logics except the logic $\boldsymbol{S5}_m$ $(s,t) \in R_i$ iff $\{\psi | \Box_i \psi \in s\} \subseteq t\}$, $i = 1, \ldots, m$; for the logic $\boldsymbol{S5}_m$ $(s,t) \in R_i$ iff $\{\Box_i\phi | \Box_i\phi \in s\} = \{\Box_i\phi | \Box_i\phi \in t\}$, $i = 1, \ldots, m$.

We define the notion of *acceptable* sequence of atoms $s_0, s_1, \ldots$ as in [10].

Now we construct a canonical model $M_L^c(\varphi)$. Let $TL_L(\varphi)$ be the set of timelines $r : N \to W_L(\varphi)$ such that: $\bigcirc_W false \in r(0)$; for all n, $r(n) < r(n+1)$; $r(0), r(1), \ldots$ is an acceptable sequence of atoms. We define $((r,n),(r',n')) \in \tilde{R}_i$ if $(r(n), r'(n')) \in R_i$, $1 \leqslant i \leqslant m$. $M_L^c(\varphi)$ is defined to be a tuple $\langle TL_L(\varphi), \tilde{R}_1, \ldots \tilde{R}_m, \pi\rangle$, where $\pi((r,n),p) = T$ iff $p \in r(n)$, $p \in P$.

By induction on the complexity of formula ψ we prove the following

Theorem 3 [Truth theorem]. $(M_L^c(\varphi),(r,n)) \models \psi$ *iff* $\psi \in r(n)$. *(Here (r,n) is some point from $M_L^c(\varphi)$ and $r(n)$ is the n-th state of timeline r.)*

Using definition of canonical model one can prove the following

Lemma 2. *If $\psi \in FL_{\boldsymbol{L}}(\varphi)$ and ψ is a $\Pi_{\boldsymbol{L}}(\varphi)$-consistent formula then there exists a point $(r,n) \in M_L^c(\varphi)$ such that $\psi \in r(n)$.*

Lemma 1 follows by Lemma 2 and Theorem 3. Theorem 2 is proved.

References

[1] C. Dixon, M. Fisher and A. Bolotov. Clausal resolution in a logic of rational agency. *Art. Int.*, **139**:47–89, 2002.

[2] C. Dixon, M. Fisher and M. Wooldridge. Resolution for temporal logics of knowledge. *J. Logic Computat.*, **8**(3):345–372, 1998.

[3] T. French, R. van Meyden and M. Reynolds. Axioms for logics of knowledge and past time: synchrony and unique initial states. *Adv. Modal Logic*, **5**:53–72, 2005.

[4] R. Gore. Tableau methods for modal and temporal logic. In M. D'Agostino, D.Gabbay, R. Haehnle and J. Possega(Eds.), *Handbook of Tableau Methods*, pp. 297–396. Kluwer Academic Publishers, 1999.

[5] J.Y. Halpern, R. van der Meyden and M.Y. Vardi. Complete axiomatizations for reasoning about knowledge and time. *SIAM J. Comput.*, **33**:674–703, 2004.

[6] O. Lichtenstein and A. Pnueli. Propositional temporal logics: Decidability and completeness. *Logic J. IGPL*, **8**(1):55–85, 2000.

[7] A. Maruyama, S. Tojo and H. Ono. Decidability of temporal epistemic logics for multi-agent models. In *Proc. ICLP'01 Workshop Computational Logic in Multi-agent Systems*, pp. 31–40, 2001.

[8] R. van der Meyden and K. Wong. Complete axiomatizations for reasoning about knowledge and branching time. *Stud. Logica*, **75**(1):93–123, 2003.

[9] J. Sakalauskaitė. A sequent calculus for logic knowledge and past time: completeness and decidability. *Lith. Math. J.*, **46**(3):427–437, 2006.

[10] J. Sakalauskaitė. Partial cut elimination for propositional discrete linear time temporal logic. *Lith. Math. J., LMD works*, **51**:347–351, 2010.

[11] H. Wansing. Sequent systems for modal logics. In *Handbook of Philosophical Logic*, Handbook of Philosophical Logic, vol. 8, pp. 61–145. Kluwer Academic Publishers, 2002.

[12] M. Wooldridge, C. Dixon and M. Fisher. A tableau-based proof method for temporal logic of knowledge and belief. *J. Appl. Non-classical Logics*, **8**(3):225–258, 1998.

REZIUMĖ

Dalinis pjūuvio eliminavimas daugiamodalumų logikų junginiams su praeities laiku

J. Sakalauskaitė

Pateikiami sekvenciniai skaičiavimai daugiamodalumų logikų ir praeities laiko logikos junginiams. Įrodomas šių skaičiavimų pilnumas su apribota pjūvio taisykle. Kaip išvada plaukia šių junginių išsprendžiamumas.

Raktiniai žodžiai: modalumų logika, laiko logika, sekvencinis skaičiavimas, pjūvis.