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Sequent calculus for logic of correlated knowledge

Haroldas Giedra, Jūratė Sakalauskaitė

Institute of Mathematics and Informatics, Vilnius University
Akademijos 4, LT-08663 Vilnius
E-mail: haroldas.giedra@mii.vu.lt; jurate.sakalauskaite@mii.vu.lt

Abstract. Sound and complete sequent calculi for general epistemic logic and logic of correlated knowledge are presented in this paper.
Keywords: General epistemic logic, logic of correlated knowledge, sequent calculus.

Introduction

Logic of correlated knowledge is obtained from epistemic logic by adding observational capabilities to agents. Traditionally, agents can perform logical inference, positive and negative introspection and their knowledge is truthful. Applications of epistemic logic cover fields such as distributed systems, knowledge base merging, robotics or network security in computer science and artificial intelligence. By adding observational capabilities to agents, in addition we can apply it for systems, where observations or measurements can be performed and results obtained.

Alexandru Baltag and Sonja Smets introduced general epistemic logic (GEL), logic of correlated knowledge (LCK) and Hilbert style proof systems for them in [1]. In this paper sequent calculi are presented for GEL and LCK to make possibilities for mechanizing proofs. We are using the ideas of semantic internalization of Sara Negri suggested in [3] and [2].

We'll start by defining syntax and semantics of general epistemic logic in Sections 1.1 and 1.2, then introduce Hilbert style proof system for GEL in Section 1.3 and present sound and complete sequent calculus for general epistemic logic in Section 1.4. In Section 2, we'll do the same for the logic of correlated knowledge.

1 General epistemic logic

1.1 Syntax

General epistemic logic is generalized epistemic logic, where knowledge operator K_I is used for both individual agents and groups of agents. In traditional epistemic logic we have formulas like $K_{i_1}A$, where i_1 represents agent. In GEL the corresponding formula is $K_{\{i_1\}}A$. When $I = \{i_1, i_2, i_3\}$, we write $K_{\{i_1,i_2,i_3\}}$ or $K_I A$. Syntax of GEL is defined as follows:

Definition 1 [Syntax of GEL]. The language of the logic GEL has the following syntax:

$$A := p \mid \neg A \mid A \wedge A \mid K_I A,$$

where p is any atomic proposition and $I \subseteq N$, $N = \{i_1, i_2, i_3, \ldots, i_n\}$.

1.2 Semantics

Imagine the system composed of components or locations. Agents can be associated to locations, where they perform observations. If observations are the same in different states, we have equivalence relation between them. Formally, it is written as:

Definition 2 [Observational equivalence]. If s and s' are two possible states and group of agents I can make exactly the same observations in these two states, then the states are *observationally equivalent* for I. We write this as $s \overset{I}{\sim} s'$.

Semantics of general epistemic logic is multi-modal Kripke frames, where relations between states signify equivalence of observations of agent groups.

Definition 3 [General epistemic frame]. For a set of states S, a family of binary relations $\{\overset{I}{\sim}\}_{I \subseteq N} \subseteq S \times S$, *general epistemic frame* is multi-modal Kripke frame $(S, \{\overset{I}{\sim}\}_{I \subseteq N})$ meeting the following conditions:

- For each $I \subseteq N$, $\overset{I}{\sim}$ is labeled equivalence relation;
- Information is monotonic: if $I \subseteq J$ then $\overset{J}{\sim} \subseteq \overset{I}{\sim}$;
- Observability principle: if $s \overset{N}{\sim} s'$ then $s = s'$;
- Vacuous information: $s \overset{\emptyset}{\sim} s'$ for all $s, s' \in S$.

Satisfaction relation $\models$ between states of S and formulas of GEL is defined inductively in a standard way. In particular, for formula $K_I A$ we have:

$$s \models K_I A \quad \text{iff } t \models A \quad \text{for all states } t \overset{I}{\sim} s.$$

$K_I A$ means that the group of agents I carries the information that A is the case.

1.3 Hilbert style calculus HS-GEL

Hilbert style proof system of general epistemic logic has the following axioms and rules:

- Any axiomatization for propositional logic.
- Axioms for knowledge:

  K1. $K_I(A \to B) \to (K_I A \to K_I B)$,

  K2. $K_I A \to A$ (*Truthfulness*),

  K3. $K_I A \to K_I K_I A$ (*Positive introspection*),

  K4. $\neg K_I A \to K_I \neg K_I A$ (*Negative introspection*),

  K5. $K_I A \to K_J A$, when $I \subseteq J$ (*Monotonicity of group knowledge*),

  K6. $A \to K_N A$ (*Observability*).

- Rules:

$$\frac{A, A \to B}{B}\ \text{(R1)}, \qquad \frac{A}{K_I A}\ \text{(R2)},$$

where $I, J \subseteq N$ and "$\to$" stands for implication.

Alexandru Baltag and Sonja Smets have proved soundness and completeness of this calculus with respect to general epistemic frames in [1].

1.4 Gentzen style calculus GS-GEL

Using the ideas of semantic internalization of Sara Negri [3], we construct sequent calculus for general epistemic logic.

- Axiom: $s : p, \Gamma \Rightarrow \Delta, s : p$.
- Logical rules:

$$\frac{\Gamma \Rightarrow \Delta, s : A}{s : \neg A, \Gamma \Rightarrow \Delta}\ (L\neg), \qquad \frac{s : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, s : \neg A}\ (R\neg),$$

$$\frac{s : A, s : B, \Gamma \Rightarrow \Delta}{s : A \wedge B, \Gamma \Rightarrow \Delta}\ (L\wedge), \qquad \frac{\Gamma \Rightarrow \Delta, s : A \quad \Gamma \Rightarrow \Delta, s : B}{\Gamma \Rightarrow \Delta, s : A \wedge B}\ (R\wedge).$$

- Knowledge rules:

$$\frac{t : A, s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s : K_I A, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}\ (LK_I), \qquad \frac{s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, s : K_I A}\ (RK_I).$$

Rule (RK_I) requires, that $I \neq N$, $I \neq \emptyset$ and t is not in the conclusion.

$$\frac{s \overset{N}{\sim} s, \Gamma \Rightarrow \Delta, s : A}{\Gamma \Rightarrow \Delta, s : K_N A}\ (R\,K_N), \qquad \frac{s \overset{\emptyset}{\sim} t, \Gamma \Rightarrow \Delta, t : A}{\Gamma \Rightarrow \Delta, s : K_\emptyset A}\ (R\,K_\emptyset).$$

- Rules for accessibility relations of reflexivity, emptiness, transitivity, euclideanness and monotonicity:

$$\frac{s \overset{I}{\sim} s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathit{Ref}), \qquad \frac{s \overset{\emptyset}{\sim} t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathit{Emp}),$$

$$\frac{s \overset{I}{\sim} t, s \overset{I}{\sim} s', s' \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{I}{\sim} s', s' \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}\ (\mathit{Trans}), \qquad \frac{s' \overset{I}{\sim} t, s \overset{I}{\sim} s', s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{I}{\sim} s', s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}\ (\mathit{Eucl}),$$

$$\frac{s \overset{J}{\sim} t, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{J}{\sim} t, \Gamma \Rightarrow \Delta}\ (\mathit{Mon}).$$

Rule (Mon) requires, that $I \subseteq J$.

- Structural rules are as in [2].

Γ, Δ in the sequents are finite, possibly empty, multisets of labeled formulas $s : A$ and relational atoms $s \overset{I}{\sim} t$. Labels s, t are ranging in the set of states S and $s : A$ stands for $s \models A$.

Using soundness of axioms and rules of GS-GEL and completeness of HS-GEL, the following theorem can be proved.

Theorem 1 [Soundness and completeness of GS-GEL]. *Sequent calculus GS-GEL is sound and complete with respect to general epistemic frames.*

2 Logic of correlated knowledge

2.1 Syntax

The syntax of logic of correlated knowledge (LCK) is syntax of GEL enriched by possible joint observations and results. Given sets $O_{i_1}, \dots, O_{i_n}$ of possible observations for each agent, a joint observation is a tuple of observations $o = \vec{o} = (o_i)_{i \in N} \in O_{i_1} \times \cdots \times O_{i_n}$ or $o = (o_i)_{i \in I} \in O_I$, $O_I := \times_{i \in I} O_i$, when $I \subseteq N$. Associating observations to results $r \in R$, new atomic formulas o^r are obtained. Formally we have:

Definition 4 [Syntax of LCK]. The language of the logic LCK has the following syntax:

$$A := p \mid o^r \mid \neg A \mid A \wedge A \mid K_I A,$$

where p is any atomic proposition and $I \subseteq N$, $o = (o_i)_{i \in I} \in O_I$, $r \in R$.

2.2 Semantics

States of models of LCK are functions $s : O_{i_1} \times \cdots \times O_{i_n} \to R$ or $s_I : O_I \to R$, where set of results R is in structure (R, Σ) together with abstract operation $\Sigma : \mathcal{P}(R) \to R$ of composing results. For every $e \in O_I$, local state s_I is defined as:

$$s_I\big((e_i)_{i \in I}\big) := \Sigma\big\{s(o) \colon o \in O_{i_1} \times \cdots \times O_{i_n} \text{ such that } o_i = e_i \text{ for all } i \in I\big\}.$$

Definition 5 [Correlation model]. A correlation model is a general epistemic model, where observational equivalence is identity of the corresponding local states:

$$s \overset{I}{\sim} t \quad \text{iff } s_I = t_I.$$

The group knowledge K_I in a correlation model is called correlated knowledge. Semantics of o^r is defined as follows: $s \models o^r$ iff $s_I(o) = r$, where $o \in O_I$.

2.3 Hilbert style calculus HS-LCK

Fix a finite set $N = \{i_1, \dots, i_n\}$ of agents, a finite result structure (R, Σ) and a tuple of finite observation sets $\vec{O} = (O_{i_1}, \dots, O_{i_n})$. Hilbert style proof system of logic of correlated knowledge over $(R, \Sigma, \vec{O})$ contains:

- The calculus HS-GEL defined above.

- Axioms for observations:

  O1. Observations always yield results. For every $I \subseteq N$:
  $$\bigwedge_{o \in O_I} \bigvee_{r \in R} o^r.$$

  O2. Observations have unique results. For $r \neq p$, $o \in O_I$:
  $$o^r \to \neg o^p.$$

  O3. Groups know the results of their joint observations. For $o \in O_I$, $r \in R$:
  $$o^r \to K_I o^r.$$

  O4. Group knowledge is correlated knowledge (is based on joint observations). For every tuple $r = (r_o)_{o \in O_I}$ of results, one for each possible joint observation $o = (o_i)_{i \in I} \in O_I$ by group I, and setting $O_I^r := \bigwedge_{o \in O_I} o^{r_o}$, we have:
  $$\big(O_I^r \wedge K_I A\big) \to K_\emptyset\big(O_I^r \to A\big).$$

  O5. Result composition axiom. For every tuple $e \in O_I$ put $\bar{e} := \{o = (o_i)_{i \in N} \in O_{i_1} \times \cdots \times O_{i_n} \colon o_i = e_i \text{ for all } i \in I\}$, $\bar{e}^r := \bigwedge_{o \in \bar{e}} o^{r_o}$. For every tuple $r = (r_o)_{o \in \bar{e}}$, one for each global observation $o \in \bar{e}$, we have:
  $$\bar{e}^r \to e^{\Sigma\{r_o : o \in \bar{e}\}}.$$

2.4 Gentzen style calculus GS-LCK

Sequent calculus for logic of correlated knowledge contains:

- Axioms:

  $s : p, \Gamma \Rightarrow s : p, \Delta$,

  $s : o^r, \Gamma \Rightarrow s : o^r, \Delta$,

  $s : o^{r_1}, s : o^{r_2}, \Gamma \Rightarrow \Delta$, where $r_1 \neq r_2$.

- Rules of GS-GEL.

- Observability rules:

$$\left\{ \frac{\{s : o^r, \Gamma \Rightarrow \Delta\}_{r \in R}}{\Gamma \Rightarrow \Delta}\ (OYR) \right\}_{o \in O_I},$$

$$\frac{s : e^{\Sigma\{r_o : o \in \bar{e}\}}, \{s : o^{r_o}\}_{o \in \bar{e}}, \Gamma \Rightarrow \Delta}{\{s : o^{r_o}, \}_{o \in \bar{e}}, \Gamma \Rightarrow \Delta}\ (CR),$$

$$\frac{s \overset{I}{\sim} t, s : O_I^r, t : O_I^r, \Gamma \Rightarrow \Delta}{s : O_I^r, t : O_I^r, \Gamma \Rightarrow \Delta}\ (OE_1), \qquad \frac{s : O_I^r, t : O_I^r, s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}{s \overset{I}{\sim} t, \Gamma \Rightarrow \Delta}\ (OE_2).$$

Definitions of e, $\bar{e}$ and r_o in rule (CR) are the same as in axiom O5 of HS-LCK.

Theorem 2 [Soundness and completeness of GS-LCK]. *Sequent calculus GS-LCK is sound and complete with respect to correlation models over $(R, \Sigma, \vec{O})$.*

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REZIUMĖ

Sekvencinis skaičiavimas koreliatyvių žinių logikai
H. Giedra, J. Sakalauskaitė

Darbe pristatomi pagrįsti ir pilni sekvenciniai skaičiavimai bendrąjai žinių logikai ir koreliatyvių žinių logikai.

Raktiniai žodžiai: bendroji žinių logika, koreliatyvių žinių logika, sekvencinis skaičiavimas.