

A specialization of definitions in common knowledge logic

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Abstract. It is known that one of main aims of specializations of derivations in non-classical logics is the various tools which allow us to simplify the searching of termination of derivations. The traditional techniques used to ensure termination of derivations in various non-classical logics are based on loop-checking. In this paper the reflexive common knowledge logic based on modal logic K_2 is considered. For considered logic a sequent calculus with specialized non-logical (loop-type) axioms is presented.

Keywords: sequent calculus, common knowledge logic, termination.

1 Introduction

The common knowledge logics are import class of non-classical logics and play a significant role in several areas of computer science, artificial intelligence, game theory, economics and etc.

Common knowledge logics are based on multi-modal logics extended with common knowledge operator. In this paper, a reflexive common knowledge logic (RCKL in short) based on multi-modal logic K_2 and reflexive common knowledge operator [3] is considered.

Common knowledge operator satisfies some induction like axioms. In derivation this induction-like tool is realized using loop-type axioms. Determination of these loop-type axioms involves creating a new “good loop” in contrast to “bad loops” and the new loop checking along with ordinary non-induction-type loop checking. Based on history method a method of determination of “good loops” for common knowledge logic is described in [1].

In this paper for reflexive common knowledge logic (based on multi-modal logic K_2), a sequent calculus with specialized non-logical (loop-type) axioms is presented. The specialization is achieved using some splitting rules.

2 Initial calculi for considered logic

The language of considered RCKL contains:

- (1) A set of propositional symbols $P_i, P_1, \dots, Q_i, Q_1, \dots$;
- (2) A finite set of agent constants i, i_1, \dots, i_k ($I, i_l \in \{1, \dots, n\}, 1 \leq l \leq k$ for simplicity we assume than $n = 2$);

- (3) Multi-agent knowledge operator K_l , where l ($l \in \{1, \dots, n\}$) is an agent constant;
- (4) Reflexive common knowledge operator C ;
- (5) Logical operators $\supset, \wedge, \vee, \neg$.

Formulas of RCKL are defined as follows: every propositional symbol is a formula; if A, B are formulas then $A \supset B, A \wedge B, A \vee B, \neg(A)$ are formulas; if l is an agent constant, A is a formula, then $K_l(A)$ is a formula; if A is a formula, then $C(A)$ is a formula. The formula $K_l(A)$ means “Agent l knows A ” ($l \in \{1, 2\}$). The expression $E(A)$ means “every agent knows A ” and is used as abbreviation of formula $K_1(A) \wedge K_2(A)$. The formula $C(A)$ means “ A is common knowledge of all agents”. The formula $C(A)$ has the same meaning as the infinite conjunction $A \wedge E(A) \wedge E(E(A)) \dots \wedge E^K(A) \dots$ where $E^K(A) = E^{K-1}(E(A))$ and $E^0(A) = A$. Formal semantics of the formulas $K_l(A)$ and $C(A)$ is defined as RCKL [3]. Hilbert-type calculus HRC is obtained from Hilbert-type calculus for propositional logic by adding the following postulates:

- (1) $(K_i(A \supset B) \wedge K_i(A)) \supset K_i(B)$;
- (2) $(C(A \supset B) \wedge C(A)) \supset C(B)$;
- (3) $E(A) = K_1(A) \wedge K_2(A)$, where $A \equiv B = (A \supset B) \wedge (B \supset A)$;
- (4) $C(A) \supset (A \wedge E(C(A)))$;
- (5) $(C(A \supset E(A)) \wedge A) \supset C(A)$;
- (6) $\frac{A}{K_i(A)}(K_i)$;
- (7) $\frac{A}{C(A)}(C), i \in \{1, 2\}$.

The axiom (5) is called induction axiom. In [3] it is shown that the calculus HRC is sound and complete.

Along with formulas we consider sequents, i.e., formula expansions $A_1, \dots, A_K \rightarrow B_1, \dots, B_m$, where A_1, \dots, A_K and B_1, \dots, B_m is a multiset of formulas.

We consider two types of initial sequent calculi for RCKL.

1. Infinitary calculus G_wRC is obtained from classical propositional calculus (with invertible logical rules) by adding the following rules:

$$\frac{\Gamma \rightarrow A}{\Pi_1, K_l \Gamma \rightarrow \Pi_2, K_l(A)}(K_l)$$

where $K_l \Gamma$ is empty or consists of formulas of the shape $K_l(B)$.

$$\frac{\Gamma \rightarrow \Delta, K_1(A) \wedge K_2(A)}{\Gamma \rightarrow \Delta, E(A)}(\rightarrow E),$$

$$\frac{\bigwedge_{l=1}^2 K_l(A), \Gamma \rightarrow \Delta}{E(A), \Gamma \rightarrow \Delta}(E \rightarrow),$$

$$\frac{A, E(C(A)), \Gamma \rightarrow \Delta}{C(A), \Gamma \rightarrow \Delta}(C \rightarrow),$$

$$\frac{\Gamma \rightarrow \Delta, A; \Gamma \rightarrow A, E(A); \dots; \Gamma \rightarrow \Delta, E^k(A) \dots}{\Gamma \rightarrow \Delta, (A)} (\rightarrow C_w),$$

where $E^k(A) = E^{k-1}(E(A))$ and $E^0(A) = A$.

Analogously as in [2] one can prove that G_wRC is sound and complete.

2. Loop-type calculus G_LRC is obtained from the calculus G_wRC by replacing the infinity rule $(\rightarrow \square_w)$ by the following rule

$$\frac{\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, E(C(A))}{\Gamma \rightarrow \Delta, C(A)} (\rightarrow C)$$

and adding loop-type axioms as follows: a sequent S is a loop-type axiom if S is above a sequent S' and on the same branch of a derivation, such that S' can be obtained from S using structural rules of weakening and contraction and there is the right premise of the rule $(\rightarrow C)$ between S and S' .

Analogously as in [1] one can prove that G_LRC is sound and complete. Therefore the calculi G_wRC and G_LRC are equivalent.

Let us introduce a canonical form of sequents. A sequent S is a primary sequent, if S is of the following shape $\Sigma_1, K\Gamma_1, C\Delta_1 \rightarrow \Sigma_2, K\Gamma_2, C\Delta_2$, where for every i ($i \in \{1, 2\}$) Σ_i is empty or consists of propositional symbols; $K\Gamma_i$ is empty or consists of formulas of the shape $K_l(A)$ ($l \in \{1, 2\}$); $C\Delta_i$ is empty or consists of formulas of the shape CA

It is easy to see that bottom-up applying logical rules each sequent can be reduced to a set of primary sequents.

The primary sequent $\Sigma_1, K\Gamma_1, C\Delta_1 \rightarrow \Sigma_2, K\Gamma_2$ is a K -primary one; the primary sequent $\Sigma_1, K\Gamma_1, C\Delta_1 \rightarrow \Sigma_2, C\Delta_2$ is a C -primary one. The primary sequent S is a non-splittable primary one if S is either a K -primary or C -primary sequent. Otherwise, the primary sequent S is a splittable primary one.

Let $G_L^S RC$ be a calculus obtained from the calculus G_LRC by following transformations:

- (1) Adding the following splitting rule

$$\frac{S_1 \text{ or } S_2}{\Sigma_1, K\Gamma_1, C\Delta_1 \rightarrow \Sigma_2, K\Gamma_2, C\Delta_2} (Sp)$$

where the conclusion of the rule (Sp) is splittable primary sequent; $\Sigma_1 \cap \Sigma_2 = \emptyset$ (i.e. the sequent $\Sigma_1 \rightarrow \Sigma_2$ is not an axiom); S_1 is K -primary sequent $\Sigma_1, K\Gamma_1, C\Delta_1 \rightarrow \Sigma_2, K\Gamma_2$; S_2 is C -primary sequent;

- (2) Replacing a loop-type axiom by specialized loop-type axiom. A specialized loop-type axiom is a loop-type axiom, which is a C -primary sequent;
- (3) Using that the rule (Sp) is admissible in the calculus $I \in \{G_wRC, G_LRC\}$ we can prove that the calculi $G_L^S RC$ and $I \in \{G_wRC, G_LRC\}$ are equivalent, therefore the calculus $G_L^S RC$ is sound and complete.

In construction of derivation it is convenient to use the following (admissible in $G_L^S RC$) rule:

$$\frac{\Gamma \rightarrow A}{\Sigma_1, E\Gamma \rightarrow E(A), \Sigma_2} (E)$$

where $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Example 1. Let $S = P, C(P \supset E(P)) \rightarrow C(P), K_1(Q)$. Let's construct a derivation of S in $G_L^S RC$. Since S is splittable primary sequent let us backward apply to S the rule (S_p) and let us try to construct a derivation of C -primary sequent. $S_1 = P_1 C(P \supset E(P)) \rightarrow C(P)$.

$$\frac{\frac{\frac{S_1^* = P_1 C(P \supset E(P)) \rightarrow C(P)}{P \rightarrow P; P_1 E(P), E(C(P \supset E(P))) \rightarrow E(C(P))} (E)}{P_1(P \supset E(P), E(C(P \supset E(P))) \rightarrow E(C(P)))} (\supset \rightarrow)}{P_1 C(P \supset E(P)) \rightarrow P; P, C(P \supset E(P)) \rightarrow E(C(P))} (C \rightarrow)}{S_1 = P_1 C(P \supset E(P)) \rightarrow C(P)} (C \rightarrow)$$

Since $S_1 = S_1^*$, S_1^* is a loop type axiom. Therefore $G_L^S RC \vdash S$.

References

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REZIUMĖ

Išvedimų specializacija bendrojo žinojimo logikai

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Straipsnyje pateikiama ciklinių aksiomų specializacija refleksyviai bendro žinojimo logikai.

Raktiniai žodžiai: sekvencinis skaičiavimas, bendrojo žinojimo logika, ciklinės aksiomos.