

# On the periodic zeta-function with rational parameter

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**Abstract.** We obtain an asymptotic formula with estimated error term for the fourth power moment of the periodic zeta-function with rational parameter.

**Keywords:** approximate functional equation, periodic zeta-function, Riemann zeta-function.

Let  $s = \sigma + it$  and  $\lambda \in \mathbb{R}$ . The periodic zeta-function  $\zeta_\lambda(s)$ , with parameter  $\lambda$  is defined, for  $\sigma > 1$ , by the Dirichlet series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and, in the case  $0 < \lambda < 1$ , is analytically continued to entire function. In [3], we considered the case of irrational parameter  $\lambda$  and obtained the following asymptotic formula for the fourth power moment of  $\zeta_\lambda(s)$ . Let  $\frac{1}{2} < \sigma < 1$  and  $T \rightarrow \infty$ . Then for every  $\varepsilon > 0$ ,

$$\int_1^T |\zeta_\lambda(\sigma + it)|^4 dt = T \left( \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \right) + O(T^{\frac{3}{2} - \sigma + \varepsilon}).$$

Here  $\zeta(s)$  denotes the Riemann zeta-function. The aim of this note is to obtain a similar formula for the rational parameter  $\lambda$ . We notice that the case of rational  $\lambda$  is more complicated, and the formula obtained is valid in a more narrow region than that in the case of irrational  $\lambda$ .

**Theorem 1.** *Suppose that the number  $\lambda$  is rational,  $0 < \lambda < 1$ ,  $\frac{3}{4} < \sigma < 1$  and  $T \rightarrow \infty$ . Then, for every  $\varepsilon > 0$ ,*

$$\int_1^T |\zeta_\lambda(\sigma + it)|^4 dt = T \left( \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \right) + O(T^{\frac{7}{4} - \sigma + \varepsilon}).$$

For the proof of Theorem 1, we apply, as in the case of irrational  $\lambda$ , an approximate functional equation for  $\zeta_\lambda(s)$ . Let  $[u]$  denote the integer part of  $u$ . Define the functions

$$p(t) = \left[ \sqrt{\frac{t}{2\pi}} - 1 \right], \quad q(t) = \left[ \sqrt{\frac{t}{2\pi}} \right], \quad g(\lambda, t) = 2\sqrt{\frac{t}{2\pi}} - p(t) - q(t) - \lambda - 1,$$

$$\begin{aligned} f(\lambda, t) = & -\frac{t}{2\pi} \log \frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(1 - \lambda^2) + p(t) - q(t) \\ & + 2\sqrt{\frac{t}{2\pi}}(q(t) - p(t) + \lambda - 1) - \frac{1}{2}(q(t) + p(t)) - \lambda(1 + q(t) - p(t)) \end{aligned}$$

and

$$\psi(t) = \frac{\cos \pi \left( \frac{t^2}{2} - t - \frac{1}{8} \right)}{\cos \pi t}.$$

**Lemma 1.** *Suppose that  $0 < \lambda < 1$ ,  $0 \leq \sigma \leq 1$  and  $t \geq t_0 > 0$ . Then*

$$\begin{aligned} \zeta_\lambda(s) = & \sum_{1 \leq m \leq p(t)} \frac{e^{2\pi i \lambda m}}{m^s} + \left( \frac{t}{2\pi} \right)^{\frac{1}{2} - \sigma - it} e^{it + \frac{\pi i}{4}} \sum_{0 \leq m \leq q(t)} \frac{1}{(m + \lambda)^{1-s}} \\ & + \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} e^{\pi i f(\lambda, t) + 2\pi i \lambda} \psi(g(\lambda, t)) + O(t^{\frac{\sigma}{2} - 1}). \end{aligned}$$

The lemma is a special case of the approximate functional equation obtained in [1] for the Lerch zeta-function.

Denote

$$S_1(s) = \sum_{1 \leq m \leq p(t)} \frac{e^{2\pi i \lambda m}}{m^s}, \quad S_2(s) = \left( \frac{t}{2\pi} \right)^{\frac{1}{2} - \sigma - it} e^{it + \frac{\pi i}{4}} \sum_{0 \leq m \leq q(t)} \frac{1}{(m + \lambda)^{1-s}}.$$

Then Lemma 1 implies, for  $0 \leq \sigma \leq 1$ , the estimate

$$\zeta_\lambda(s) = S_1(s) + S_2(s) + O(t^{-\frac{1}{4}}). \quad (1)$$

Therefore, for the proof of Theorem 1, it suffices to find the fourth power moments for the sums  $S_1(s)$  and  $S_2(s)$ . We observe that the fourth moment of  $S_1(s)$  does not depend of the arithmetical nature of the parameter  $\lambda$ . Therefore, we have the following statement.

**Lemma 2.** *Suppose that  $\lambda$  is irrational,  $0 < \lambda < 1$ ,  $\frac{1}{2} < \sigma < 1$  and  $T \rightarrow \infty$ . Then, for every  $\varepsilon > 0$ ,*

$$\begin{aligned} & \int_1^T |S_1(\sigma + it)|^4 dt \\ & = T \left( \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \right) + O(T^{\frac{3}{2} - \sigma + \varepsilon}). \end{aligned}$$

Proof of the lemma is given in [3].

**Lemma 3.** *Suppose that the number  $\lambda$ ,  $0 < \lambda < 1$ , is rational,  $\frac{3}{4} < \sigma < 1$  and  $T \rightarrow \infty$ . Then, for every  $\varepsilon > 0$ ,*

$$\int_1^T |S_2(\sigma + it)|^4 dt = O(T^{\frac{5}{2}-2\sigma+\varepsilon}).$$

*Proof.* Consider the fourth moment

$$J(T) = \int_1^T \left| \sum_{0 \leq m \leq q(t)} \frac{1}{(m + \lambda)^{1-\sigma-it}} \right|^4 dt.$$

Clearly, we have that

$$J(T) = J_1(T) + O_\lambda(T), \tag{2}$$

where

$$J_1(T) = \int_1^T \left| \sum_{1 \leq m \leq q(t)} \frac{1}{(m + \lambda)^{1-\sigma-it}} \right|^4 dt.$$

It is not difficult to see that

$$\begin{aligned} J_1(T) &= \int_1^T \sum_{1 \leq m_1, n_1, m_2, n_2 \leq q(t)} \frac{1}{((m_1 + \lambda)(n_1 + \lambda)(m_2 + \lambda)(n_2 + \lambda))^{1-\sigma}} \\ &\quad \times \left( \frac{(m_1 + \lambda)(n_1 + \lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right)^{it} dt. \end{aligned}$$

Let  $T_1 = 2\pi \max(m_1^2, n_1^2, m_2^2, n_2^2)$ . Then

$$\begin{aligned} J_1(T) &= \sum_{1 \leq m_1, n_1, m_2, n_2 \leq q(t)} \frac{1}{((m_1 + \lambda)(n_1 + \lambda)(m_2 + \lambda)(n_2 + \lambda))^{1-\sigma}} \\ &\quad \times \int_{T_1}^T \left( \frac{(m_1 + \lambda)(n_1 + \lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right)^{it} dt. \end{aligned} \tag{3}$$

We investigate two cases:  $(m_1 + \lambda)(n_1 + \lambda) = (m_2 + \lambda)(n_2 + \lambda)$  and  $(m_1 + \lambda)(n_1 + \lambda) \neq (m_2 + \lambda)(n_2 + \lambda)$ . In the first case, the integral in (3) is equal to  $T - T_1$ , in the second case, we find after integration that, for that integral, the estimate

$$O\left( \left| \log \frac{(m_1 + \lambda)(n_1 + \lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right|^{-1} \right)$$

holds. The above remarks together with (3) show that

$$\begin{aligned} J_1(T) &\ll \sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)}^* \frac{T - T_1}{((m_1 + \lambda)(n_1 + \lambda))^{2-2\sigma}} \\ &\quad + \sum_{(m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)}^* \frac{\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right|^{-1}}{((m_1 + \lambda)(n_1 + \lambda)(m_2 + \lambda)(n_2 + \lambda))^{1-\sigma}}, \end{aligned} \tag{4}$$

where the star “\*” means that each  $m_1, n_1, m_2, n_2$  runs over  $[1, q(T)]$ . Denoting by  $d(m)$  the number of divisors of  $m$ , we have that

$$\begin{aligned}
T & \sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)}^* \frac{1}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} \\
& \ll T \sum_{m \leq q^2(T)} \sum_{m-\sqrt{T} \leq n \leq m+\sqrt{T}} \frac{d(m)d(n)}{m^{2-2\sigma}} \\
& \ll T^{\frac{3}{2}+\varepsilon} \sum_{m \leq q^2(T)} \frac{d(m)}{m^{2-2\sigma}} \ll T^{\frac{1}{2}+2\sigma+\varepsilon}.
\end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned}
& \sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)}^* \frac{T_1}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} \\
& \ll \sum_{m_1, n_1 \leq q(T)} \sum_{m_1 n_1 - \sqrt{T} \leq n \leq m_1 n_1 + \sqrt{T}} \frac{m_1^2 d(m_1 n_1) d(m)}{(m_1 n_1)^{2-2\sigma}} \\
& \ll T^{\frac{1}{2}+\varepsilon} \left( \sum_{m_1 \leq q(T)} m_1^{2\sigma} \sum_{n_1 \leq q(T)} \frac{1}{n_1^{2-2\sigma}} \right) \ll T^{\frac{1}{2}+2\sigma+\varepsilon}.
\end{aligned} \tag{6}$$

The inequality

$$\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right| \geq c(\lambda) \min \left( \frac{1}{(m_1+\lambda)(n_1+\lambda)}, \frac{1}{(m_2+\lambda)(n_2+\lambda)} \right),$$

with a certain  $c(\lambda) > 0$  and application of the Montgomery–Vaughan theorem [2] yield

$$\sum_{(m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)}^* \frac{\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right|^{-1}}{(m_1 n_1 m_2 n_2)^{1-\sigma}} \ll_{\lambda} T^{\varepsilon} \sum_{1 \leq m \leq q^2(T)} \frac{m}{m^{2-2\sigma}} \ll_{\lambda} T^{2\sigma+\varepsilon}.$$

This and (4)–(6) give the estimate

$$J_1(T) \ll_{\lambda} T^{\frac{1}{2}+2\sigma+\varepsilon}.$$

and, in view of (1),

$$J(T) \ll_{\lambda} T^{\frac{1}{2}+2\sigma+\varepsilon}.$$

Hence,

$$\int_1^T |S_2(\sigma + it)|^4 \ll \int_1^T t^{2-4\sigma} dJ(t) \ll_{\lambda} T^{\frac{5}{2}-2\sigma+\varepsilon}. \quad \square$$

*Proof of Theorem 1.* In view of (1), we have that

$$\begin{aligned}
 |\zeta_\lambda(s)|^4 &= (S_1(s) + S_2(s) + O(t^{-\frac{1}{4}}))^2 (\overline{S_1(s)} + \overline{S_2(s)} + O(t^{-\frac{1}{4}}))^2 \\
 &= |S_1(s)|^4 + S_1^2(s)\overline{S_2^2(s)} + 2S_1^2(s)\overline{S_1(s)S_2(s)} + \overline{S_1^2(s)S_2^2(s)} + |S_2(s)|^4 \\
 &\quad + 2\overline{S_1(s)S_2^2(s)S_2(s)} + 2S_1(s)\overline{S_1^2(s)S_2(s)} + 2S_1(s)\overline{S_2^2(s)S_2(s)} \\
 &\quad + 4|S_1(s)|^2|S_2(s)|^2 + O(|S_1(s)|^3 t^{-\frac{1}{4}}) + O(|S_1(s)|^2|S_2(s)| t^{-\frac{1}{4}}) \\
 &\quad + O(|S_1(s)|^2 t^{-\frac{1}{2}}) + O(|S_1(s)||S_2(s)|^2 t^{-\frac{1}{4}}) + O(|S_2(s)|^3 t^{-\frac{1}{4}}) \\
 &\quad + O(|S_2(s)|^2 t^{-\frac{1}{2}}) + O(|S_1(s)||S_2(s)| t^{-\frac{1}{2}}) + O|S_1(s)| t^{-\frac{3}{4}} \\
 &\quad + O(|S_2(s)| t^{-\frac{3}{4}}) + O(t^{-1}). \quad \square
 \end{aligned} \tag{7}$$

By Lemmas 2 and 3, and the Cauchy–Schwarz inequality, we find that

$$\begin{aligned}
 &\int_1^T (S_1^2(\sigma + it)\overline{S_2^2(\sigma + it)} + \overline{S_1^2(\sigma + it)S_2^2(\sigma + it)} + 4|S_1^2(\sigma + it)|^2|S_2^2(\sigma + it)|^2) dt \\
 &\ll \left( \int_1^T |S_1(\sigma + it)|^4 dt \int_1^T |S_2(\sigma + it)|^4 dt \right)^{\frac{1}{2}} \ll_\lambda T^{\frac{7}{4}-\sigma+\varepsilon}, \\
 &\int_1^T (2S_1^2(\sigma + it)\overline{S_1(\sigma + it)S_2(\sigma + it)} + 2S_1(\sigma + it)\overline{S_1^2(\sigma + it)S_2(\sigma + it)} \\
 &\quad + 2\overline{S_1(\sigma + it)S_2^2(\sigma + it)S_2(\sigma + it)} + 2S_1(\sigma + it)\overline{S_2^2(\sigma + it)S_2(\sigma + it)}) dt \\
 &\ll \left( \int_1^T |S_1(\sigma + it)|^2|S_2(\sigma + it)|^2 dt \right)^{\frac{1}{2}} \\
 &\quad \times \left( \left( \int_1^T |S_1(\sigma + it)|^4 dt \right)^{\frac{1}{2}} + \left( \int_1^T |S_2(\sigma + it)|^4 dt \right)^{\frac{1}{2}} \right) \\
 &\ll_\lambda \left( \int_1^T |S_1(\sigma + it)|^4 dt \int_1^T |S_2(\sigma + it)|^4 dt \right)^{\frac{1}{4}} \\
 &\quad \times (T^{\frac{1}{2}} + T^{\frac{5}{4}-\sigma+\varepsilon}) \ll_\lambda (T^{\frac{7}{8}-\frac{\sigma}{2}+\varepsilon}(T^{\frac{1}{2}} + T^{\frac{5}{4}-\sigma+\varepsilon})) \ll_\lambda T^{\frac{11}{8}-\frac{\sigma}{2}+\varepsilon}.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 &\int_1^T |S_1(\sigma + it)|^3 t^{-\frac{1}{4}} dt \ll_\lambda T^{\frac{3}{4}+\varepsilon}, \\
 &\int_1^T |S_2(\sigma + it)|^3 t^{-\frac{1}{4}} dt \ll_\lambda T^{\frac{15}{8}-\frac{3}{2}\sigma+\varepsilon}, \\
 &\int_1^T |S_1(\sigma + it)||S_2(\sigma + it)|^2 t^{-\frac{1}{4}} dt \ll_\lambda T^{\frac{3}{2}-\sigma+\varepsilon},
 \end{aligned}$$

$$\int_1^T |S_1(\sigma + it)|^2 |S_2(\sigma + it)| t^{-\frac{1}{4}} dt \ll_{\lambda} T^{\frac{5}{4} - \frac{\sigma}{2} + \varepsilon},$$

$$\int_1^T |S_1(\sigma + it) S_2(\sigma + it)| t^{-\frac{1}{2}} dt \ll_{\lambda} T^{\frac{7}{8} - \frac{\sigma}{2} + \varepsilon}.$$

These estimates together with Lemmas 2 and 3 prove the theorem.

## References

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## REZIUMĖ

### Periodinės dzeta funkcijos su racionaliuoju parametru

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Straipsnyje gautas liekamojo nario įvertis periodinės dzeta funkcijos su racionaliuoju parametru ketvirtojo momento asimptotinėje formulėje.

*Raktiniai žodžiai:* artutinė funkcinė lygtis, periodinė dzeta funkcija, Rymano dzeta funkcija.