On a boundary value problem to third order PDE with multiple characteristics

Yusufjon P. Apakov[a], **Stasys Rutkauskas**[b,c]

[a]Namangan Engineering Pedagogical Institute
Duslik str. 12, 160103, Namangan, Uzbekistan
apakov.1956@mail.ru

[b]Institute of Mathematics and Informatics of Vilnius University
Akademijos str. 4, LT-08663 Vilnius, Lithuania

[c]Vilnius Pedagogical University
Studentų str. 39, LT-08106, Vilnius, Lithuania
stasys.rutkauskas@mii.vu.lt

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Abstract. In the paper the second boundary value problem in a rectangular domain to equation $u_{xxx} - u_{yy} = f(x, y)$ with the multiple characteristics is considered. The considered equation is closely related with nonlinear equation $u_{xxx} + u_{yy} - \frac{\nu}{y}u_y = u_x u_{xx}$, which describes transonic flow of a gas around a revolution bodies. Using the fundamental solutions of corresponding homogeneous equation the Green function of analyzed problem is composed and thereby this problem is solved.

Keywords: PDE's of odd order, fundamental solutions, Green function and boundary value problems.

1 Introduction

First investigations of the third order differential equation

$$u_{xxx} - u_{yy} = f(x, y) \tag{1}$$

which possesses the multiple characteristics, are published in [1–3]. After a while the works [4,5], in which various boundary value problems are studded using potential theory, appear.

Let us observe, that equation (1) is conjugated to differential equation

$$u_{xxx} + u_{yy} = F(x, y)$$

that is related with the linear part of equation

$$u_{xxx} + u_{yy} - \frac{\nu}{y}u_y = u_x u_{xx}$$

describing transonic flow of a gas. Particularly, if $\nu = 0$, this equation describes the plane-parallel flow of a gas (see [6, 7]).

In the theory of boundary value problems to equation (1) the fundamental solutions of homogeneous equation

$$u_{xxx} - u_{yy} = 0$$

are significant. Such solutions $U(x,y;\xi,\eta)$ and $V(x,y;\xi,\eta)$ are composed in [8]. Here is shown that they can be expressed in the form

$$\begin{aligned} U(x,y;\xi,\eta) &= |y-\eta|^{\frac{1}{3}} f(t), \quad -\infty < t < \infty, \\ V(x,y;\xi,\eta) &= |y-\eta|^{\frac{1}{3}} \varphi(t), \quad t < 0, \end{aligned} \tag{2}$$

where

$$f(t) = \frac{2\sqrt[3]{2}}{\sqrt{3}\pi} t\Psi\left(\frac{1}{6},\frac{4}{3};\tau\right), \quad \varphi(t) = \frac{36\Gamma(\frac{1}{3})}{\sqrt{3}\pi} t\Phi\left(\frac{1}{6},\frac{4}{3};\tau\right), \quad \tau = \frac{4}{27}t^3, \quad t = \frac{x-\xi}{|y-\eta|^{\frac{2}{3}}},$$

and both $\Psi(a,b;x)$, $\Phi(a,b;x)$ are degenerate hypergeometric functions (see [9]), Γ is Gamma function. Taking in account the properties of these functions the following estimates for fundamental solution $U(x,y;\xi,\eta)$ are obtained:

$$\left|\frac{\partial^{h+k}U}{\partial x^h dy^k}\right| \leqslant C_{kh}|y-\eta|^{\frac{1-(-1)^k}{2}}|x-\xi|^{-\frac{1}{2}[2h+3k-1+\frac{3}{2}(1-(-1)^k)]}, \quad \text{if } \left|\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right| \to \infty,$$

where C_{kh} are constants, k, $h = 0,1,2,\ldots$. (There hold analogously estimates for $V(x,y;\xi,\eta)$ if $(x-\xi)|y-\eta|^{-\frac{2}{3}} \to -\infty$.)

In [10] there are considered some boundary value problems to equation (1) in the rectangular domain $D = \{(x,y)\colon 0 < x < p,\ 0 < y < l\}$, $p > 0$, $l > 0$. Here the solutions of the considered problems are composed by Fourier method under assumption that boundary value conditions on $y = 0$ and $y = l$ are homogeneous.

We shell solve in this paper the second boundary value problem to equation (1) in a rectangular domain using the Green function method.

2 Statement of the problem

Definition 1. We will say that solution $u(x,y)$ of equation (1) is regular in domain $D = \{(x,y)\colon 0 < x < p,\ 0 < y < l\}$, if it satisfies (1) in D and is from the class $C^{3,2}_{x,y}(D) \cap C^{1,1}_{x,y}(\overline{D})$.

Let us consider the following boundary value problem.

Problem F_2. To find the regular in domain solution $u(x,y)$ of equation (1) satisfying the boundary value conditions

$$u_y(x,0) = \varphi_1(x), \quad u_y(x,l) = \varphi_2(x), \tag{3}$$

$$u(0,y) = \psi_1(y), \quad u(p,y) = \psi_2(y), \quad u_x(p,y) = \psi_3(y), \tag{4}$$

where

$$\varphi_i(x) \in C[0,p],\ \ i=1,2, \quad \psi_j(y) \in C^3[0,l],\ \ j=1,2,$$
$$\psi_3(y) \in C^2[0,l], \quad f(x,y) \in C^{0,2}_{x,y}(\overline{D}),$$

and the following compatibility conditions are fulfilled:

$$\begin{aligned} &\varphi_1(0) = \psi_1'(0), \quad \varphi_1(p) = \psi_2'(0), \quad \varphi_1'(p) = \psi_3(0), \quad \varphi_2(0) = \psi_1'(l), \\ &\varphi_2(p) = \psi_2'(l), \quad \varphi_2'(p) = \psi_3(l), \quad f_y'(x,0) = f_y'(x,l) = 0. \end{aligned} \tag{5}$$

We shall note that, in work [11] analogical problem investigated in endless band.

3 Uniqueness of the solution

Theorem 1. *There exists no more than one solution of Problem F_2.*

Proof. Propose that Problem F_2 has two solutions $u_1(x,y)$ and $u_2(x,y)$. Then $u(x,y) = u_1(x,y)-u_2(x,y)$ satisfies the homogeneous equation $u_{xxx}-u_{yy} = 0$ and corresponding homogeneous boundary value conditions. We shell prove that $u(x,y) \equiv 0$ in D in such a case.

Let us consider the identity

$$\frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2\right) - \frac{\partial}{\partial y}(uu_y) + u_y^2 = 0.$$

Integrating it over domain D and taking in account the homogeneity of boundary value conditions we obtain that

$$\frac{1}{2}\int_0^l u_x^2(0,y)\,\mathrm{d}y + \iint_D u_y^2(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Hence, $u_y(x,y) = 0$, i.e. $u(x,y) = \phi(x)$, where $\phi(x)$ is arbitrary function. Since $u(x,0) = 0$, we get $\phi(x) = 0$ because of continuity of function $u(x,y)$. Therefore, $u(x,y) \equiv 0$ in D. □

4 The existence of the solution

Let us consider the adjoint differential operators

$$L \equiv \frac{\partial^3}{\partial \xi^3} - \frac{\partial^2}{\partial \eta^2}, \quad L^* \equiv -\frac{\partial^3}{\partial \xi^3} - \frac{\partial^2}{\partial \eta^2}.$$

Let φ, ψ be smooth enough functions. It is easy to check that the identity

$$\varphi L[\psi] - \psi L^*[\varphi] \equiv \frac{\partial}{\partial \xi}(\varphi\psi_{\xi\xi} - \varphi_\xi\psi_\xi + \varphi_{\xi\xi}\psi) - \frac{\partial}{\partial \eta}(\varphi\psi_\eta - \varphi_\eta\psi)$$

holds. Integrating it over domain D we get the equality

$$\begin{aligned}&\iint\limits_D \bigl[\varphi L[\psi] - \psi L^*[\varphi]\bigr]\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad= \iint\limits_D \frac{\partial}{\partial\xi}(\varphi\psi_{\xi\xi} - \varphi_\xi\psi_\xi + \varphi_{\xi\xi}\psi)\,\mathrm{d}\xi\,\mathrm{d}\eta - \iint\limits_D \frac{\partial}{\partial\eta}(\varphi\psi_\eta - \varphi_\eta\psi)\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (6)\end{aligned}$$

Let us choose the fundamental solution $U(x,y;\xi,\eta)$, which satisfies with respect to (ξ,η) the equation

$$L^*[U] \equiv -U_{\xi\xi\xi} - U_{\eta\eta} = 0 \quad \text{if } (x,y) \neq (\xi,\eta),$$

instead of function φ, and any regular solution $u(\xi,\eta)$ of equation (1) instead of function ψ. Since $U_\eta(x,y;\xi,\eta)$ has a singularity at the line $y=\eta$, we introduce the domains

$$\begin{aligned}D_1^\varepsilon &= \bigl\{(\xi,\eta)\colon 0<\xi<p,\ 0<\eta<y-\varepsilon\bigr\},\\ D_2^\varepsilon &= \bigl\{(\xi,\eta)\colon 0<\xi<p,\ y+\varepsilon<\eta<l\bigr\}\end{aligned}$$

such that $D = \lim_{\varepsilon\to0}(D_1^\varepsilon \cup D_2^\varepsilon)$. Then we obtain from equality (6) that

$$\begin{aligned}&\iint\limits_D U(x,y;\xi,\eta)f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &= \lim_{\varepsilon\to0+}\int_0^p\int_0^{y-\varepsilon} \frac{\partial}{\partial\xi}(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad+ \lim_{\varepsilon\to0+}\int_0^p\int_{y+\varepsilon}^{l} \frac{\partial}{\partial\xi}(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad- \lim_{\varepsilon\to0+}\int_0^p\int_0^{y-\varepsilon} \frac{\partial}{\partial\eta}(Uu_\eta - U_\eta u)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad- \lim_{\varepsilon\to0+}\int_0^p\int_{y+\varepsilon}^{l} \frac{\partial}{\partial\eta}(Uu_\eta - U_\eta u)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &= \lim_{\varepsilon\to0+}\int_0^{y-\varepsilon} (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}\,\mathrm{d}\eta\\ &\quad+ \lim_{\varepsilon\to0+}\int_{y+\varepsilon}^{l} (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}\,\mathrm{d}\eta\end{aligned}$$

$$
\begin{aligned}
&- \lim_{\varepsilon\to 0+} \int_0^p (Uu_\eta - U_\eta u)\big|_{\eta=0}^{\eta=y-\varepsilon} \,\mathrm{d}\xi - \lim_{\varepsilon\to 0+} \int_0^p (Uu_\eta - U_\eta u)\big|_{\eta=y+\varepsilon}^{\eta=l} \,\mathrm{d}\xi \\
&= \int_0^y (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p} \,\mathrm{d}\eta + \int_y^l (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p} \,\mathrm{d}\eta \\
&- \lim_{\varepsilon\to 0+} \int_0^p \big[U(x,y;\xi,y-\varepsilon)u_\eta(\xi,y-\varepsilon) - U(x,y;\xi,0)u_\eta(\xi,0)\big] \,\mathrm{d}\xi \\
&+ \lim_{\varepsilon\to 0+} \int_0^p \big[U_\eta(x,y;\xi,y-\varepsilon)u(\xi,y-\varepsilon) - U_\eta(x,y;\xi,0)u(\xi,0)\big] \,\mathrm{d}\xi \\
&- \lim_{\varepsilon\to 0+} \int_0^p \big[U(x,y;\xi,l)u_\eta(\xi,l) - U(x,y;\xi,y+\varepsilon)u_\eta(\xi,y+\varepsilon)\big] \,\mathrm{d}\xi \\
&+ \lim_{\varepsilon\to 0+} \int_0^p \big[U_\eta(x,y;\xi,l)u(\xi,l) - U_\eta(x,y;\xi,y+\varepsilon)u(\xi,y+\varepsilon)\big] \,\mathrm{d}\xi \\
&= \int_0^l (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p} \,\mathrm{d}\eta \\
&- \int_0^p \big[U(x,y;\xi,l)u_\eta(\xi,l) - U(x,y;\xi,0)u_\eta(\xi,0)\big] \,\mathrm{d}\xi \\
&+ \int_0^p \big[U_\eta(x,y;\xi,l)u(\xi,l) - U_\eta(x,y;\xi,0)u(\xi,0)\big] \,\mathrm{d}\xi \\
&+ \lim_{\varepsilon\to 0+} \int_0^p U_\eta(x,y;\xi,y-\varepsilon)u(\xi,y-\varepsilon) \,\mathrm{d}\xi \\
&- \lim_{\varepsilon\to 0+} \int_0^p U_\eta(x,y;\xi,y+\varepsilon)u(\xi,y+\varepsilon) \,\mathrm{d}\xi.
\end{aligned}
$$

So we get the relation

$$
\begin{aligned}
&\iint_D U(x,y;\xi,\eta)f(\xi,\eta) \,\mathrm{d}\xi \,\mathrm{d}\eta \\
&= \int_0^l \big[Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u\big]\big|_{\xi=0}^{\xi=p} \,\mathrm{d}\eta - \int_0^p U(x,y;\xi,\eta)u_\eta(\xi,\eta)\big|_{\eta=0}^{\eta=l} \,\mathrm{d}\xi
\end{aligned}
$$

$$+\int_0^p U_\eta(x,y;\xi,\eta)u(\xi,\eta)\big|_{\eta=0}^{\eta=l}\,\mathrm{d}\xi+\lim_{\varepsilon\to 0+}\int_0^p U_\eta(x,y;\xi,y-\varepsilon)u(\xi,y-\varepsilon)\,\mathrm{d}\xi$$
$$-\lim_{\varepsilon\to 0+}\int_0^p U_\eta(x,y;\xi,y+\varepsilon)u(\xi,y+\varepsilon)\,\mathrm{d}\xi \tag{7}$$

There holds following

Lemma. *Let φ be any function from $C[0,p]$. Then relation*

$$\lim_{\substack{x\to x_0\\ \eta\to y}}\int_0^p U_\eta(x,y;\xi,\eta)\varphi(\xi)\,\mathrm{d}\xi=-\varphi(x_0)\,\mathrm{sgn}(y-\eta)$$

holds with any $x_0\in(0,p)$.

Proof. Assume that $y>\eta$. Due to the continuity of $\varphi(x)$ at the point x_0, there exists $\delta=\delta(\varepsilon)$ such that $|\varphi(x)-\varphi(x_0)|<\varepsilon$, if only $|x-x_0|<\delta$. Using the relation (see [8])

$$U_\eta=-U^*\,\mathrm{sgn}(y-\eta),$$

where

$$U^*(x,y;\xi,\eta)=\frac{1}{|y-\eta|^{\frac{2}{3}}}f^*\left(\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right),\quad f^*(t)=\frac{t}{3\gamma}\Psi\left(\frac{7}{6},\frac{4}{3};\frac{4}{27}t^3\right),\quad \gamma=\frac{3\sqrt{3\pi}}{2^{\frac{1}{3}}}$$

one can rewrite the integral in the left-hand side of (7) as follows:

$$\begin{aligned}
&\int_0^p U_\eta(x,y;\xi,\eta)\varphi(\xi)\,\mathrm{d}\xi\\
&\quad=-\int_0^p U^*(x,y;\xi,\eta)\varphi(\xi)\,\mathrm{d}\xi=-\int_0^p\frac{1}{|y-\eta|^{\frac{2}{3}}}f^*\left(\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right)\varphi(\xi)\,\mathrm{d}\xi\\
&\quad=\left(-\int_0^{x_1}-\int_{x_1}^{x_2}-\int_{x_2}^{p}\right)\frac{1}{(y-\eta)^{\frac{2}{3}}}f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)\,\mathrm{d}\xi\\
&\quad=I_1+I_2+I_3.
\end{aligned}$$

(Here $x_1=x_0-\delta$,$x_2=x_0+\delta$.)

The main term I_2 of obtained sum can be rewritten as

$$\begin{aligned}
&-\varphi(x_0)\int_{x_1}^{x_2}\frac{1}{(y-\eta)^{\frac{2}{3}}}f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\mathrm{d}\xi\\
&-\int_{x_1}^{x_2}\frac{1}{(y-\eta)^{\frac{2}{3}}}f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\big[\varphi(\xi)-\varphi(x_0)\big]\,\mathrm{d}\xi\\
&\quad=I_{21}+I_{22}.
\end{aligned}$$

Let us calculate the limit of integral I_{21}, as $x \to x_0$, $\eta \to y-0$. Introduce to this the variable

$$t = \frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}.$$

Then $\xi = x - t(y-\eta)^{\frac{2}{3}}$ $\mathrm{d}\xi = -(y-\eta)^{\frac{2}{3}}\,\mathrm{d}t$, obviously, and we obtain that

$$I_{21} = -\varphi(x_0) \int\limits_{\frac{x-x_2}{(y-\eta)^{2/3}}}^{\frac{x-x_1}{(y-\eta)^{2/3}}} f^*(t)\,\mathrm{d}t.$$

If $|x-x_0| < \delta$, then upper limit of this integral is positive and lower limit is negative. Besides, the upper limit tends to $+\infty$ and the lower limit tends to $-\infty$, as $\eta \to y-0$. Therefore, taking in account the equality [8]

$$\int\limits_{-\infty}^{\infty} f^*(t)\,\mathrm{d}t = 1$$

we get that

$$\lim_{\substack{\eta\to y-0\\ x\to x_0}} I_{21} = -\varphi(x_0).$$

It remains to show that the rest integrals I_{21}, I_1, I_3 tend to zero, as $x \to x_0$, $\eta \to y-0$.

Let us consider integral I_{22}. Since

$$|I_{22}| \leqslant \int\limits_{x_1}^{x_2} \left| \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\right| \bigl|\varphi(\xi)-\varphi(x_0)\bigr|\,\mathrm{d}\xi$$

and $|\xi - x_0| < \delta$, we obtain that

$$|I_{22}| \leqslant \varepsilon \int\limits_{\frac{x-x_2}{(y-\eta)^{2/3}}}^{\frac{x-x_1}{(y-\eta)^{2/3}}} \bigl|f^*(t)\bigr|\,\mathrm{d}t.$$

Then the estimate $|f^*(t)| < C|t|^{-\frac{5}{2}}$ (see [8]) yields the equality

$$\lim_{\substack{\eta\to y-0\\ x\to x_0}} I_{22} = 0.$$

Let N be a constant such that $|\varphi(x)| \leqslant N$ $\forall x \in [0,l]$. Then

$$|I_1| < \left| \int\limits_0^{x_1} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)\,\mathrm{d}\xi \right| < N \int\limits_{\frac{x}{(y-\eta)^{2/3}}}^{\frac{x-x_1}{(y-\eta)^{2/3}}} \bigl|f^*(t)\bigr|\,\mathrm{d}t \to 0$$

as $x \to x_0$, $\eta \to y - 0$, because both upper and lower limits tend to $+\infty$, as $x \to x_0$, $\eta \to y - 0$.

Analogously

$$|I_3| < \left| \int\limits_{x_2}^{p} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)\,\mathrm{d}\xi \right| < N \int\limits_{\frac{x-x_2}{(y-\eta)^{2/3}}}^{\frac{x-p}{(y-\eta)^{2/3}}} \left|f^*(t)\right| \mathrm{d}t \to 0,$$

as $x \to x_0$, $\eta \to y - 0$.

Thus, lemma is proved in the case $y > \eta$. In the opposite case the proof of the relation

$$\lim_{\substack{x\to x_0 \\ \eta \to y}} \int\limits_a^b U_\eta(x,y;\xi,\eta)\varphi(\xi)\,\mathrm{d}\xi = \varphi(x_0).$$

is analogously. □

Further, using the Lemma we obtain from (7) that

$$\begin{aligned}
&\iint\limits_D U(x,y;\xi,\eta) f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad = \int\limits_0^l (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}\,\mathrm{d}\eta - \int\limits_0^p U(x,y;\xi,\eta)u_\eta(\xi,\eta)\big|_{\eta=0}^{\eta=l}\,\mathrm{d}\xi \\
&\qquad + \int\limits_0^p U_\eta(x,y;\xi,\eta)u(\xi,\eta)\big|_{\eta=0}^{\eta=l}\,\mathrm{d}\xi - 2u(x,y).
\end{aligned}$$

Thus,

$$\begin{aligned}
2u(x,y) = &\int\limits_0^l (Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}\,\mathrm{d}\eta - \int\limits_0^p (Uu_\eta - U_\eta u)\big|_{\eta=0}^{\eta=l}\,\mathrm{d}\xi \\
&- \iint\limits_D U(x,y;\xi,\eta)f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned} \tag{8}$$

Let $u(x,y)$ be any regular solution of equation (1) and $W(x,y;\xi,\eta)$ be any regular solution of the adjoint equation. Then putting $\varphi = W(x,y;\xi,\eta)$, $\psi = u(\xi,\eta)$ into (6) we obtain that

$$\begin{aligned}
0 = &\int\limits_0^l (Wu_{\xi\xi} - W_\xi u_\xi + W_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}\,\mathrm{d}\eta - \int\limits_0^p (Wu_\eta - W_\eta u)\big|_{\eta=0}^{\eta=l}\,\mathrm{d}\xi \\
&- \iint\limits_D W(x,y;\xi,\eta)f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned} \tag{9}$$

Both (8) and (9) yield the important relation

$$2u(x,y) = \int_0^l (Gu_{\xi\xi} - G_\xi u_\xi + G_{\xi\xi}u)|_{\xi=0}^{\xi=p}\,\mathrm{d}\eta - \int_0^p (Gu_\eta - G_\eta u)|_{\eta=0}^{\eta=l}\,\mathrm{d}\xi - \iint_D G(x,y;\xi,\eta)f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{10}$$

where

$$G(x,y;\xi,\eta) = U(x,y;\xi,\eta) - W(x,y;\xi,\eta).$$

Definition 2. We will say that $G(x,y;\xi,\eta)$ is the Green function of Problem F_2 in domain D if it satisfies the following conditions:

$$\begin{cases} L[G] = 0, \\ G_y(x,0;\xi,\eta) = G_y(x,l;\xi,\eta) = 0, \\ G(0,y;\xi,\eta) = G(p,y;\xi,\eta) = G_x(p,y;\xi,\eta) = 0 \end{cases} \tag{11}$$

with respect to variables (x,y);

$$\begin{cases} L^*[G] = 0, \\ G_\eta(x,y;\xi,0) = G_\eta(x,y;\xi,l) = 0, \\ G(x,y;0,\eta) = G(x,y;p,\eta) = G_\xi(x,y;0,\eta) = 0 \end{cases} \tag{12}$$

with respect to variables (ξ,η).

In order to compose the mentioned above Green function we solve the following subsidiary problem.

Problem F_0. To find a regular in domain D solution $u(x,y)$ of equation (1) satisfying conditions

$$u_y(x,0) = 0, \quad u_y(x,l) = 0, \quad 0 \leqslant x \leqslant p, \tag{13}$$

$$u(0,y) = u(p,y) = u_x(p,y) = 0, \quad 0 \leqslant y \leqslant l. \tag{14}$$

We seek for the solution of this problem of the shape

$$u(x,y) = \sum_{k=1}^{\infty} X_k(x)\cos\frac{k\pi}{l}y, \tag{15}$$

where $X_k(x)$ are unknown functions.

Let us express the function $f(x,y)$ into Fourier series

$$f(x,y) = \sum_{k=0}^{\infty} f_k(x)\cos\frac{k\pi}{l}y, \tag{16}$$

where

$$f_k(x) = \frac{2}{l}\int\limits_0^l f(x,y)\cos\frac{k\pi}{l}y\,\mathrm{d}y$$

Substituting both (15) and (16) into (1) we get that

$$\sum_{k=0}^{\infty}\bigl[X_k'''(x) + \lambda_k^3 X_k(x) - f_k(x)\bigr]\cos\frac{k\pi}{l}y = 0.$$

Therefore, we obtain the boundary value problem

$$\begin{cases} L[X_k] := X_k'''(x) + \lambda_k^3 X_k(x) = f_k(x), \\ X_k(0) = X_k(p) = X_k'(p) = 0 \end{cases} \tag{17}$$

with respect to unknown function $X_k(x)$; here $\lambda_k^3 = (k\pi l)^2$.

We shall solve problem (17) by Green function method.

Definition 3. We will say that $G_k(x,\xi)$ is the Green function of problem (17), if it satisfies the following conditions:

(i) both $G_k(x,\xi)$ and $\frac{\partial G_k(x,\xi)}{\partial x}$ are continuous on the square $0 \leqslant x \leqslant p$, $0 \leqslant \xi \leqslant p$;

(ii) $\frac{\partial^2 G_k(x,\xi)}{\partial x^2}$ is discontinuous at the line $x = \xi$ and

$$\frac{\partial^2 G_k(x,\xi)}{\partial x^2}\bigg|_{x=\xi+0} - \frac{\partial^2 G_k(x,\xi)}{\partial x^2}\bigg|_{x=\xi-0} = 1;$$

(iii) $G_k(x,\xi)$ satisfies with respect to x the equation

$$L[G_k] := \frac{\partial^3 G_k}{\partial\, x^3} + \lambda_k^3 G_k = 0$$

in both intervals $0 \leqslant x < \xi$ and $\xi < x \leqslant p$ for $\forall \xi \in (0,p)$;

(iv) It satisfies following boundary value conditions

$$G_k(0,\xi) = G_k(p,\xi) = G_{kx}(p,\xi) = 0$$

for $\forall \xi \in [0,p]$.

It is easy to verify that Green function $G_k(0,\xi)$ of problem (17) is of the shape:

$$\begin{aligned} G_k(x,\xi) &= \frac{1}{\bar\Delta}\biggl\{2e^{-\lambda_k(\frac{3}{2}p+x-\xi)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k p+\frac{\pi}{6}\right) \\ &\quad - 2e^{-\frac{\lambda_k}{2}(2x+\xi)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k\xi+\frac{\pi}{6}\right) \\ &\quad - 2e^{-\lambda_k(\frac{3}{2}p-\xi-\frac{x}{2})}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-x)+\frac{\pi}{6}\right] \\ &\quad + 2e^{-\frac{\lambda_k}{2}(\xi-x)}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(\xi-x)+\frac{\pi}{6}\right] \\ &\quad + 4e^{-\frac{\lambda_k}{2}(3p+\xi-x)}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-\xi)\right]\sin\frac{\sqrt{3}}{2}\lambda_k x\biggr\}, \quad 0\leqslant x\leqslant\xi, \end{aligned} \tag{18}$$

$$\begin{aligned} G_k(x,\xi) &= \frac{1}{\bar\Delta}\biggl\{-2e^{-\frac{\lambda_k}{2}(2x+\xi)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k\xi+\frac{\pi}{6}\right) \\ &\quad - 2e^{-\lambda_k(\frac{3}{2}p-\xi-\frac{x}{2})}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-x)+\frac{\pi}{6}\right]+e^{-\lambda_k(x-\xi)} \\ &\quad + 4e^{-\frac{\lambda_k}{2}(3p+\xi-x)}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-x)+\frac{\pi}{6}\right] \\ &\qquad \times\sin\left(\frac{\sqrt{3}}{2}\lambda_k\xi+\frac{\pi}{6}\right)\biggr\}, \quad \xi\leqslant x\leqslant p, \end{aligned}$$

where

$$\bar\Delta = 3\lambda_k^2\left(1-2e^{-\frac{3}{2}\lambda_k p}\sin\left(\frac{\sqrt{3}}{2}\lambda_k p+\frac{\pi}{6}\right)\right).$$

Hence, the solution of problem (17) is of the shape

$$X_k(x)=\int_0^p G_k(x,\xi)f_k(\xi)\,\mathrm{d}\xi. \tag{19}$$

Then taking in account (19) we get according to formula (15) the solution $u(x,y)$ of Problem F_0

$$u(x,y)=\sum_{k=1}^{\infty}\left(\int_0^p G_k(x,\xi)f_k(\xi)\,\mathrm{d}\xi\right)\cos\frac{\pi k}{l}y. \tag{20}$$

It stands to reason that we are in need of the proof of the uniform convergence in domain $D=\{(x,y)\colon 0<x<p,\ 0<y<l\}$ of the series in right-hand side of (20) together with its partial derivatives of needful order.

Let us note to this end that

$$\left|\sum_{k=1}^{\infty}\left(\int_0^p G_k(x,\xi)f_k(\xi)\,\mathrm{d}\xi\right)\cos\frac{\pi k}{l}y\right| \leqslant \sum_{k=1}^{\infty}\int_0^p \bigl|G_k(x,\xi)\bigr|\bigl|f_k(\xi)\bigr|\,\mathrm{d}\xi. \tag{21}$$

Note that well known estimate

$$\bigl|f_k(\xi)\bigr| \leqslant \frac{M_1}{k^2}, \quad M_1 = const > 0,$$

holds for $\forall\xi \in [0,p]$ because of the assumed smoothness of function $f(x,y)$. Further, it follows from (19) that

$$\bigl|G_k(x,\xi)\bigr| \leqslant \begin{cases} \dfrac{10}{3}\dfrac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2} + \dfrac{2}{3}\dfrac{e^{-\frac{1}{2}\lambda_k\delta_1}}{\lambda_k^2}, & 0 \leqslant x < \xi,\ 0 < \delta_1 < \xi - x,\\ \dfrac{8}{3}\dfrac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2} + \dfrac{1}{3}\dfrac{e^{-\frac{1}{2}\lambda_k\delta_2}}{\lambda_k^2}, & \xi < x \leqslant l,\ 0 < \delta_2 < x - \xi, \end{cases} \tag{22}$$

or

$$\bigl|G_k(x,\xi)\bigr| \leqslant \frac{10}{3}\frac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2} + \frac{2}{3}\frac{e^{-\frac{1}{2}\lambda_k\delta}}{\lambda_k^2} \leqslant M_2 k^{-\frac{4}{3}},$$

where M_2 is some constant independent of k. That jointly with (22) yields the estimate

$$\int_0^p \bigl|G_k(x,\xi)\bigr|\bigl|f_k(\xi)\bigr|\,\mathrm{d}\xi \leqslant pM_1M_2k^{-\frac{10}{3}}.$$

Thus, the series in right-hand side of (20) converges uniformly in D to $u(x,y)$ because of the last estimate, and equality (20) can be rewritten as follows:

$$u(x,y) = \int_0^p \left(\sum_{k=1}^{\infty} G_k(x,\xi)f_k(\xi)\cos\frac{\pi k}{l}y\right)\mathrm{d}\xi. \tag{23}$$

We shall prove that the expression (23) of function $u(x,y)$ can be thrice differentiable with respect to x, i.e.

$$\frac{\partial^3}{\partial x^3}u(x,y) = \int_0^p \left(\sum_{k=1}^{\infty}\frac{\partial^3}{\partial x^3}G_k(x,\xi)f_k(\xi)\cos\frac{\pi ky}{l}\right)\mathrm{d}\xi. \tag{24}$$

In this order it is enough to show that series

$$\sum_{k=1}^{\infty}\frac{\partial^3}{\partial x^3}G_k(x,\xi)f_k(\xi)\cos\frac{\pi ky}{l} \tag{25}$$

converges uniformly in domain D. According to equality $\frac{\partial^3 G_k}{\partial x^3} + \lambda_k^3 G_k = 0$ we obtain similarly as above that

$$\left|\frac{\partial^3}{\partial x^3} G_k(x,\xi) f_k(\xi) \cos\frac{\pi k y}{l}\right| \leqslant \left|\frac{\partial^3}{\partial x^3} G_k(x,\xi)\right| \left|f_k(\xi)\right| = \left|\lambda_k^3 G_k(x,\xi)\right| \left|f_k(\xi)\right| \leqslant \lambda_k^3 M_1 M_2 k^{-\frac{10}{3}} = \left(\frac{\pi}{l}\right)^2 M_1 M_2 k^{-\frac{4}{3}}.$$

That yields the uniform convergence of series (25), evidently. Therefore, derivative $\frac{\partial^3}{\partial x^3} u(x,y)$ is continuous in D and equality (24) holds.

The validity of the equality

$$\frac{\partial^2}{\partial y^2} u(x,y) = \int\limits_0^p \left(\sum_{k=1}^{\infty} \left(\frac{\pi k y}{l}\right)^2 G_k(x,\xi) f_k(\xi) \cos\frac{\pi k y}{l}\right) \mathrm{d}\xi$$

formally obtained from (23) follows because of estimate

$$\left(\frac{\pi k y}{l}\right)^2 \left|G_k(x,\xi)\right| \left|f_k(\xi)\right| \leqslant \left(\frac{\pi}{l}\right)^2 M_1 M_2 k^{-\frac{4}{3}}, \quad (x,\xi) \in D.$$

Hence, function $u(x,y)$ defined by (23) is the solution of subsidiary Problem F_0, really. Putting in (23)

$$f_n(\xi) = \frac{2}{l} \int\limits_0^l f(\xi,\eta) \cos\frac{\pi k}{l} \eta \, \mathrm{d}\eta$$

(see (16)) we get that

$$\begin{aligned} u(x,y) &= \int\limits_0^p \sum_{k=1}^{\infty} G_k(x,\xi) \cos\frac{\pi k y}{l} f_k(\xi) \, \mathrm{d}\xi \\ &= \int\limits_0^p \int\limits_0^l f(\xi,\eta) \frac{2}{l} \sum_{k=1}^{\infty} G_k(x,\xi) \cos\frac{\pi k}{l} \eta \cos\frac{\pi k}{l} y d\xi \, \mathrm{d}\eta \\ &= \int\limits_0^p \int\limits_0^l G(x,\xi,y,\eta) f(\xi,\eta) \, \mathrm{d}\xi \, \mathrm{d}\eta, \end{aligned}$$

where

$$G(x,\xi,y,\eta) = \frac{2}{l} \sum_{k=1}^{\infty} G_k(x,\xi) \cos\frac{\pi k}{l} \eta \cos\frac{\pi k}{l} y. \tag{26}$$

It is easily seen that function $G(x,\xi,y,\eta)$ satisfies conditions (11) and (12), i.e. it is Green function of boundary value Problem F_2 of equation (1). The convergence in D of series

(26) and its needful order derivatives follows from the estimates of function $G_k(x,\xi)$ given above.

According to Definition 2 of Green function $G(x,\xi,y,\eta)$ we obtain from (10) the solution $u(x,y)$ of considered Problem F_2 of the shape

$$\begin{aligned} 2u(x,y) = {} & \int_0^l G_{\xi\xi}(x,y,p,\eta)\psi_2(\eta)\,\mathrm{d}\eta - \int_0^l G_{\xi\xi}(x,y,0,\eta)\psi_1(\eta)\,\mathrm{d}\eta \\ & - \int_0^l G_{\xi}(x,y,p,\eta)\psi_3(\eta)\,\mathrm{d}\eta + \int_0^p G(x,y,\xi,0)\varphi_1(\xi)\,\mathrm{d}\xi \\ & - \int_0^p G(x,y,\xi,l)\varphi_2(\xi)\,\mathrm{d}\xi - \iint_D G(x,y,\xi,\eta)f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{27}$$

Hence, there holds

Theorem 2. *Let $\varphi_i(x) \in C[0,p]$, $i=1,2$, $\psi_j(y) \in C^3[0,l]$, $j=1,2$, $\psi_3(y) \in C^2[0,l]$ and $f(x,y) \in C^{0;2}_{x;y}(\overline{D})$, and let the following compatibility conditions* (5) *are fulfilled. Then there exists a unique solution $u(x,y)$ of Problem F_2 which can be represent by formula* (27).

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