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Quaternion rational Bézier curves

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Abstract. We extended the rational Bézier model for space curve, by allowing quaternion weights. These curves are Möbius invariant and have halved degree with respect to real Bézier curves. This simplify the analysis of curves. In general, these curves are in four dimensional space. We analyze when the quadratically parameterized quaternion curve is in usual three dimensional subspace.

Keywords: circle, inversion, quaternion, weight point.

Introduction

We analyze a rational Bézier curves with quaternion control points and weights. This class of curves offers two remarkable advantages.

- If the degree of the Bézier quaternion curve is n then the corresponding real curve have degree $2n$. So we need less control points and weights for the quaternion curve.
- A quaternion curve is invariant with respect to Möbius transformation.

The quaternion curves can be considered as generalization of Bézier construction for complex numbers similar to [3]. The quaternion curves of degree one are discussed in [5]. In this short paper, we restricted to the question: what conditions are necessary for the quadratic quaternion Bézier curve which guarantee that the curve is in tree-dimensional space.

An interesting generalization to the surface case is in a paper [2], where bi-linear quaternion Bézier surface are described. It turn out, that the corresponding surface is Darboux cyclide. These cyclides contains (in general case) six different families of circles. This could be useful in architectural freeform circular arc structures and discreet differential geometry.

1 Notations and definitions

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ the set of real numbers, complex numbers and quaternion numbers respectively.

In general, the quaternion set $\mathbb{H}$ can be represented as

$$\mathbb{H} = \left\{ q = [r, p\ ] \;\middle|\; r \in \mathbb{R}, p \in \mathbb{R}^3 \right\} = \mathbb{R}^4 \tag{1}$$

We denote real and imaginary parts of quaternion $q = [r, p]$ by $\mathrm{Re}(q) = r$, $\mathrm{Im}(q) = p$. The multiplication in the algebra $\mathbb{H}$ is defined as

$$[r_1, p_1][r_2, p_2] = [r_1 r_2 - p_1 \cdot p_2,\ r_1 p_2 + r_2 p_1 + p_1 \times p_2], \tag{2}$$

where $p_1 \cdot p_2, p_1 \times p_2$ are scalar and vector products in $\mathbb{R}^3$. We denote by $\bar{q} = [r, -p]$ a conjugate quaternion to $q = [r, p]$, $|q| = \sqrt{r^2 + p \cdot p} = \sqrt{q\bar{q}}$ is the length of the quaternion, $q^{-1} = \bar{q}/|q|^2 = [r/|q|^2, -p/|q|^2]$ denote the multiplicative inverse of q, i.e. $qq^{-1} = q^{-1}q = 1$. Denote the set of pure imaginary quaternions

$$\mathrm{Im}(\mathbb{H}) = \big\{[0, p\ ]\ \big|\ p \in \mathbb{R}^3\big\} = \mathbb{R}^3. \tag{3}$$

2 Quaternion rational Bezier curve

We analyze Bézier form curve with quaternion control points $a_k \in \mathbb{H}$ and quaternion weights $w_k \in \mathbb{H}$. Formally speaking, we are dealing with a quaternion function in homogeneous coordinates $(a_k w_k, w_k) \in \mathbb{H}^2$. The *quaternion rational Bézier curve* is defined in terms of Bernstein polynomials $B_k^n(t)$ as customary quotient:

$$c_h(t) = n(t)d(t)^{-1}, \quad \text{where } n(t) = \sum_{k=0}^{n} a_k w_k B_k^n(t),\ d(t) = \sum_{k=0}^{n} w_k B_k^n(t), \tag{4}$$

$$h = \big\{h_k = (a_k, w_k) \in \mathbb{H}^2,\ k = 0, \ldots, n\big\} \tag{5}$$

Here we consider $n(t), d(t)$ as quaternions, $d(t)^{-1}$ is an inverse quaternion and $n(t)d(t)^{-1}$ is the multiplication of two quaternions. So, in general, we have $c_h(t) \in \mathbb{H} = \mathbb{R}^4$. If we multiply the numerator and the denominator by the conjugated quaternion $\bar{d}(t)$:

$$c_h(t) = \frac{n(t)\bar{d}(t)}{d(t)\bar{d}(t)}, \tag{6}$$

we get a real curve in $\mathbb{R}^4$ with a real denominator of degree $2n$. Note, that the denominator is a real positive polynomial.

Remark 1. If we change the weights $w_0, w_1, \ldots, w_n$ to $1, w_1 w_0^{-1}, \ldots, w_n w_0^{-1}$ the parameterized curve $c_h(t)$ is the same. Moreover, if we change the parameter t to $s = \rho t/(1 - t + \rho t)$ $(\rho \in \mathbb{R})$ and weights $w_0, w_1, \ldots, w_n$ to $w_0, w_1/\rho, \ldots, w_n/\rho^n$ then the curve is the same too. In particular, if we take $\rho = -1$ and $s = -t/(1-2t) \in [0, 1]$ then $t = s/(2s - 1) \in [0, -\infty] \cap [\infty, 1]$. Therefor, the union of two curve parts

$$\big\{c_h(t),\ t \in [0, 1]\big\} \cup \big\{c_{\hat{h}}(s),\ s \in [0, 1]\big\}, \quad \text{where} \tag{7}$$

$$\hat{h} = \big\{\hat{h}_k = (a_k, (-1)^k w_k),\ k = 0, \ldots, n\big\} \tag{8}$$

contains all points of the same curve.

A rational Bézier curve $c(t)$ of degree one is a circular arc with two endpoints a_0, a_1. This case is well understood (see [5]).

In geometric modeling application we need geometric description of curves in $\mathbb{R}^3$ space. The only obvious geometric property is endpoints interpolation, i.e. $c(0) = a_0$,

$c(1) = a_n$. To find out an interpretation for weights and inner control points we restricted to quadratic quaternion rational Bézier curve (a real degree of the curve is 4). We note that these curves are invariant with respect to Möbius transformation. A general *Möbius transformation* is defined on $\mathbb{H}^2$:

$$M(q_1, q_2) = (aq_1 + bq_2, cq_1 + dq_2), \quad \text{where } a, b, c, d \in \mathbb{H}, \tag{9}$$

or on $\mathbb{H}$ as linear fractional function:

$$M(q) = (aq + b)(cq + d)^{-1}, \quad \text{where } a, b, c, d \in \mathbb{H}. \tag{10}$$

A Möbius transformation is conformal, i.e. it preserves angles between vectors (see [1]). Note that arbitrary rational curve is not Möbius invariant. For example, consider a real rational quartic which is intersection of two at some point A tangent cylinders. This quartic has singular point A. We can apply inversion with the center on the point A. Then one can show that the quartic will be transformed to the sextic. Therefor, this quartic curve can not be obtained as quadratic quaternionic Bézier curve.

3 Quadratic quaternion Bézier curves

With any quadratic quaternion Bézier curve $c_h(t)$, $h = \{h_k = (a_k, w_k),\ k = 0, 1, 2\}$ we associate a linear rational quaternion Bézier surface in $\mathbb{R}^4 = \mathbb{H}$:

$$s_h(x, y) = (a_0w_0u + a_1w_1x + a_2w_2y)(w_0u + w_1x + w_2y)^{-1}, \quad u = 1 - x - y. \tag{11}$$

We note that $s_h(2t(1-t), t^2) = c_h(t)$. We are looking for the condition on the points and weights which guarantee that the curve $c_h(t)$ is in $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$. Firstly, the endpoints a_0, a_2 should be in $\mathbb{R}^3$. If the surface $s_h(x, y)$ is in $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$ then the curve $c_h(t)$ is in $\mathbb{R}^3$ too (the inverse statement is not true).

Proposition 1. *Assume that the control points a_0, a_1, a_2 of Bézier curve $c_h(t) \in \mathbb{R}^3$ are in $\mathbb{R}^3$ then the corresponding parameterized surface $s_h(x, y)$ is a two dimensional sphere (or a plane) in $\mathbb{R}^3$. Therefore, the curve $c_h(t)$ is on a sphere (or a plane) too.*

Proof. We compute real control points of the curve:

$$\begin{aligned} c_h(t) &= nd^{-1} = \frac{n\bar{d}}{d\bar{d}} \\ &= \frac{a_0|w_0|^2(1-t)^2 + p_{01}2(1-t)^3t + p_{02}(1-t)^2t^2 + p_{12}2(1-t)t^3 + a_2|w_2|^2t^4}{|w_0|^2(1-t)^2 + w_{01}2(1-t)^3t + w_{02}(1-t)^2t^2 + w_{12}2(1-t)t^3 + |w_2|^2t^4} \end{aligned} \tag{12}$$

where

$$\begin{aligned} p_{01} &= a_0w_0\bar{w}_1 + a_1w_1\bar{w}_0, & w_{01} &= w_0\bar{w}_1 + w_1\bar{w}_0, \\ p_{02} &= a_0w_0\bar{w}_2 + a_2w_2\bar{w}_0 + 4a_1|w_1|^2, & w_{02} &= w_0\bar{w}_2 + w_2\bar{w}_0 + 4|w_1|^2, \\ p_{12} &= a_1w_1\bar{w}_2 + a_2w_2\bar{w}_1, & w_{12} &= w_1\bar{w}_2 + w_2\bar{w}_1. \end{aligned} \tag{13}$$

Since the curve $c_h(t)$ is in $\mathbb{R}^3$ we see that $p_{01}, p_{02}, p_{12}, w_{01}, w_{02}, w_{12} \in \mathbb{R}^3$. Similarly, we can compute real points for the corresponding surface:

$$s_h(x,y) = \frac{a_0|w_0|^2u^2 + P_{01}ux + P_{02}uy + P_{12}xy + a_1|w_1|^2x^2 + a_2|w_2|^2y^2}{|w_0|^2u^2 + W_{01}ux + W_{02}uy + W_{12}xy + |w_1|^2x^2 + |w_2|^2y^2}, \tag{14}$$

$$\text{where } u = 1 - x - y, \quad P_{01} = p_{01}, \quad P_{12} = p_{12}, \quad P_{02} = p_{02} - 4a_1|w_1|^2. \tag{15}$$

Since $a_1 \in \mathbb{R}^3$ we see that the surface $s_h(x,y)$ is in 3D space too. A curve $s_h(a + pt, b + rt)$, $a, p, b, r \in \mathbb{R}$ is a circle because it is a image of a line with a rational linear map (see for details in [5]). Hence the surface contains an infinite number of circles through any point on the surface so it must be a sphere (or a plane) (see [4]). □

In general case, the curve $c_h(t)$ may be in $\mathbb{R}^3$ but the midpoint $a_1 \notin \mathbb{R}^3$. The corresponding example we can get using bi-linear quaternion surface $b_H(s,t) = n(s,t)d(s,t)^{-1}$, where

$$\begin{aligned} n(s,t) &= b_0w_0(1-s)(1-t) + b_1w_1s(1-t) + b_2w_2(1-s)t + b_3w_3st, \\ d(s,t) &= w_0(1-s)(1-t) + w_1s(1-t) + w_2(1-s)t + w_3st, \\ H &= \{h_k = (b_k, w_k) \in \mathbb{H}^2,\ k = 0,1,2,3\}. \end{aligned} \tag{16}$$

One can find points h_k such that the surface $b_H(s,t)$ is in $\mathbb{R}^3$ (see [2]). The diagonal quadratic quaternion curve $b_H(t,t)$ will be in $\mathbb{R}^3$ too. It is known that in general the surface $b_H(s,t)$ is Darboux cyclide of degree 4. In general case, the diagonal curve $b_H(t,t)$ will be not on the sphere, so the corresponding midpoint of this curve is not in $\mathbb{R}^3$. The natural question that arises is whether the quaternion quadratic Bézier curve in $\mathbb{R}^3$ exist if $a_1 \notin \mathbb{R}^3$. The answer is affirmative:

Proposition 2. *Let* $a_0, a_2, v_0, v_2 \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$, $a_1 \in \mathbb{H} \setminus \mathrm{Im}(\mathbb{H})$. *We set*

$$\begin{aligned} &u_0 = 1, \qquad w_1 = (a_1 - a_0)^{-1}v_0u_0/2, \qquad w_2 = 2v_2^{-1}(a_2 - a_1)w_1 \quad \textit{and} \\ &h = \big\{(a_k, w_k),\ k = 0,1,2\big\}, \quad \textit{where } w_0 = \mathrm{Re}\big(-4a_1|w_1|^2\big)/\,\mathrm{Re}(a_0\bar{w}_2 + a_2w_2) \end{aligned} \tag{17}$$

Then $c_h(t) \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$ *with the following derivatives at the end points* $c_h'(0) = w_0v_0$, $c_h'(1) = v_2$.

Proof. First of all, we compute derivatives of the quadratic Bézier curve $c = c_h(t)$ at the end points. Since $c = nd^{-1}$ we take derivative of both sides for the equality $cd = n$ and obtain $c' = (n' - cd')d^{-1}$. Hence

$$c_h'(0) = 2(a_1 - a_0)w_1w_0^{-1}, \qquad c_h'(1) = 2(a_2 - a_1)w_1w_2^{-1}. \tag{18}$$

Now, we consider a quaternion curve $C(t)$ such that $C'(0) = v_0$ and $C'(1) = v_2$. Using the above equalities, for the curve $C(t)$ we can take weights as follows $u_0 = 1$, $w_1 = (a_1 - a_0)^{-1}v_0u_0/2$, $w_2 = 2v_2^{-1}(a_2 - a_1)w_1$. In order to obtain the curve in $\mathbb{R}^3$ we will replace the weight u_0 with some real constant. Since $c_h(t) \in \mathrm{Im}(\mathbb{H})$ we have $p_{01}, p_{02}, p_{12} \in \mathrm{Im}(\mathbb{H})$ (see formulas (13)). Note that $p_{01} - a_0$ is a tangent vector to the curve $c_h(t)$ at the point a_0 hence $p_{01} \in \mathrm{Im}(\mathbb{H})$ because $v_0 \in \mathrm{Im}(\mathbb{H})$. Similarly, we see that $p_{12} \in \mathrm{Im}(\mathbb{H})$. Now if we set $w_0 = \mathrm{Re}(-4a_1|w_1|^2)/\,\mathrm{Re}(a_0\bar{w}_2 + a_2w_2)$ we see that $\mathrm{Re}(p_{02}) = 0$, i.e. $p_{02} \in \mathrm{Im}(\mathbb{H})$ and $c_h(t) \in \mathrm{Im}(\mathbb{H})$ too. □

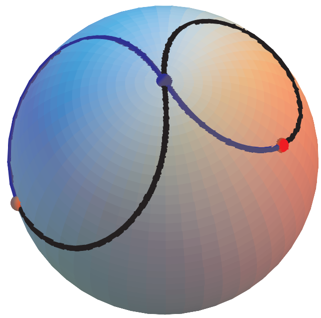

Fig. 1. Viviani's curve on a sphere presented as two Bézier curves: $c_h(t) \cup c_{\hat{h}}(s)$.

The next proposition show that any quadratic quaternion curve is a diagonal curve on a bi-linear surface, i.e. any Möbius invariant curve is on a Möbius invariant surface.

Proposition 3. *Any quadratic quaternion curve $c_h(t)$, $h = \{(a_k, w_k),\ k = 0,1,2\}$ can be represented (not uniquely) as a diagonal curve $b_H(t,t)$, where $H = \{(b_k, u_k),\ k = 0,1,2,3\}$. Moreover, if $c_h(t) \in \mathrm{Im}(\mathbb{H})$ then $b_H(t,s) \in \mathrm{Im}(\mathbb{H})$.*

Sketch of proof. Since $b_H(t,t) = c_h(t)$ we have $b_0 = a_0, b_3 = a_2$. Let us take $u_0 = w_0$, $u_3 = w_2$ and choose any $b_1, u_1 \in \mathbb{H}$ then we compute u_2, b_2 by the formulas

$$u_2 = 2w_1 - u_1, \qquad b_2 = (2a_1w_1 - b_1u_1)u_2^{-1}. \tag{19}$$

We can verify that $b_H(t,t) = c_h(t)$. Now we suppose $c_h(t) \in \mathrm{Im}(\mathbb{H})$. We would like $b_H(t,s) \in \mathrm{Im}(\mathbb{H})$ too. Let $u_1 = \lambda u_0, \lambda \in \mathbb{R}$ and take $b_1 \in \mathrm{Im}(\mathbb{H})$ such that $(b_1 - b_3)u_1u_3^{-1} \in \mathrm{Im}(\mathbb{H})$. Then we compute u_2, b_2 by the formulas (19). It is easy to see that $\mathrm{Re}(b_2)$ is linear function in λ. Let us take λ such that $\mathrm{Re}(b_2)(\lambda) = 0$ then one can check that $b_H(t,s) \in \mathrm{Im}(\mathbb{H})$. $\square$

3.1 Viviani's type of the space curve

Viviani's curve is a space curve named after the Italian mathematician Vincenzo Viviani, the intersection of a sphere with a cylinder that is tangent both to the sphere and its center. We can generalized this definition and say that intersection of a sphere with any tangent quadric is Viviani's type of a space quartic on the sphere. This curve is rational because after stereographic projection from singular point to a plane we get a conic. Moreover, any rational quartic on the sphere must be singular hence it is of Viviani's type. This quartic we can parameterized using quadratic quaternion Bézier curve. Firstly, we parameterize a sphere. We take arbitrary $a_0, a_1, a_1, v_0 \in \mathbb{R}^3$ then set $w_0 = 1$, $w_1 = (a_1 - a_0)^{-1}v_0$, $w_2 = (a_2 - a_1)^{-1}(a_1 - a_0)$ and $h = \{h_k = (a_k, w_k),\ k = 0,1,2\}$. One can check that $s_h(x,y) \in \mathbb{R}^3$ is the unique sphere which contains three points a_0, a_1, a_2 and v_0 is a tangent vector to the sphere at the point a_0. Also the Bézier curve $c_h(t)$ is on this sphere. Moreover, $c_h(1/2) = a_1$ is a singular point of the curve $c_h(t)$. Indeed,

$$\begin{aligned} c_h(1/2) - a_1 &= (a_0 + 2a_1w_1 + a_2w_2)(1 + 2w_1 + w_2)^{-1} - a_1 \\ &= (a_0 - a_1 + (a_2 - a_1)w_2)(1 + 2w_1 + w_2)^{-1} = 0. \end{aligned} \tag{20}$$

The full curve can be presented as two Bézier curves $c_h(t) \cup c_{\hat{h}}(s)$ as explained in the Remark 1.

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REZIUMĖ

Kvaternioninės racionalios Bézier kreivės

Severinas Zubė

Darbe yra apibrėžiamos kvaternioninės racionalios Bézier kreivės. Kadangi kontroliniai taškai yra bet kokie kvaternionai, tai kreivės patenka į keturmatę erdvę. Taikymuose dažniausiai yra reikalingos kreivės trimatėje erdvėje. Darbe pagrindinis dėmesys yra sutelktas į kvartikas. Nagrinėjamas klausimas, kad jos guli trimatėje erdvėje.

Raktiniai žodžiai: Bézier kreivės, kvaternionai.