

## Notion of a virtual derivative\*

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**Abstract.** Formal graphical procedures to calculate function's derivative are proposed. This can be applied to calculate expressions of geometric objects, construct approximate schemes in numerical analysis of solution of differential equation.

**Keywords:** derivative, graphs, virtual graphs, ordering of virtual graphs, symmetry coefficient.

### 1 Introduction

Finding derivatives of functions becomes a fatal obstacle in more complex analysis. Partial derivatives of a function over Euclidean space were used in Riemann's geometry from the 19th century [1]. There was an attempt to escape such cumbersome expressions with the help of notions from algebraic geometry [2]. We can call such method as an algebraic specification of geometric objects. It has its own difficulties. V. Arnold freely applied such abstract notions in the analysis of infinite dimensional spaces and described the flow of an ideal fluid governed by Euler equations as geodesic in the group of diffeomorphisms [3]. To calculate concrete derivatives for a detailed study we need an easy notation of derivatives. Everyone who tried to define concrete topologies in some infinite dimensional spaces will agree with this. Specialists of algebraic geometry like S. Lang knew the shortages of algebraic specifications and suggested the derivative's notion for a function over a Banach space [4]. These new derivatives needed a new more precise marking. That is obvious for derivatives of higher order. I started to mark derivatives by graphs, and applied this to calculate derivatives for the Runge–Kutta algorithm [5]. Later I applied the graphs of derivatives to describe asymptotic expansions in the Banach algebra of probabilities on a lattice [6]. In this work we present an overview of methods to mark derivatives, define more explicitly the natural ordering of graphs. Graph constructions for other new geometric or asymptotic problems remain to be found.

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\*This work is dedicated to the 70th anniversary of Prof. Mifodijus Sapagovas.

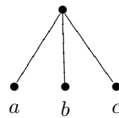
## 2 Virtual derivative

Every mapping can be pictured as an arrow:



The target of the arrows will be upper directed, therefore we shall use the arrows without heads.

A mapping of several arguments will be pictured by means of the corresponding number of arrows having the common vertex:



We choose an order of arguments pictured in the graph, i.e. we know which argument is the first, second, and so on. The composition of mappings will be pictured as a graph with wedges corresponding to the mappings under consideration. We get a tree with the top point marking the target space, and the roots marking the arguments in the source spaces. Every root will be called *an entrance of the graph*. The top vertices would be *an outlet of the graph*. The *youngest mappings* start with entrances, and the *eldest mapping* ends with the top vertex of the tree. We pretend a composition of mappings to call as a *joint mapping*.

If each mapping with  $n$  arguments is written as  $\langle x_1, x_2, \dots, x_n \rangle \circ f$ , then the tree of joint mapping can be written as

$$\langle \langle x_1, \dots, x_{n_1} \rangle \circ f_1, \langle x_{(n_1+1)}, \dots, x_{(n_1+n_2)} \rangle \circ f_2, \dots, \langle x_{n(k-1)+1}, \dots, x_{n(k-1)+n_k} \rangle \circ f_k \rangle \circ f.$$

The arguments of a derivative will be understood as *increments*  $\Delta x$  for argument  $x$  of taken function.

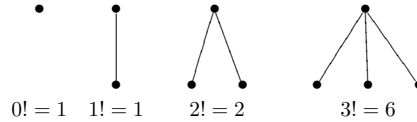
The function's  $n$ -order derivative is defined as symmetric  $n$ -linear form. Therefore it coincides with a symmetric polynome

$$\langle \Delta_1, \Delta_2, \dots, \Delta_n \rangle \circ f^{(n)} = \frac{1}{n!} \sum_{\sigma \in n!} \langle \Delta_{\sigma(1)}, \Delta_{\sigma(2)}, \dots, \Delta_{\sigma(n)} \rangle \circ f^{(n)}.$$

Each term of this sum will be called a *concrete derivative graph*. The *virtual graph* is understood as a class of similar concrete derivative graphs. The *symmetry number*  $S$  of a virtual graph helps to calculate the number of similar members in such class. This number we shall call a *weight of virtual graph*.

We identify the function's  $n$ th order derivative with the collection of all *virtual graphs* with  $n$  entrances. The virtual graph denotes the whole class of similar concrete derivative graphs.

At the beginning we picture virtual graphs of the simple mapping for the first three derivatives



Usually we don't picture the virtual graphs, it is enough that we can pick a needed virtual graph and calculate its weight number. The drawing of a virtual graph and the calculation of its weight number can be easily done automatically using a computer.

For the derivative of the simple mapping all members are similar, and we get the virtual graph identic with concrete derivative graph, only without ordering of argument increments

$$1 = \frac{n!}{S}$$

We shall choose a concrete derivative graph with increasing order

$$\langle \Delta_1, \Delta_2, \dots, \Delta_n \rangle$$

as *representing* for the virtual graph.

The derivative of some joint mapping is found by changing the wedges of the mapping tree by the virtual graphs of derivatives for each composed simple function. Such changing must be correct: each increment of any vertex must produce increment of elder vertexes, and is produced by increment of some argument. At this moment such formulation will be sufficient for the drawing of virtual graphs.

The number of similar members for each virtual graph is calculated as a fraction over graph's symmetry number

$$\frac{n!}{S}$$

Each vertex  $\ell$  has a symmetry number  $S_\ell$  coinciding with the factorial of the degree of the taken vertex

$$S_\ell = |\ell|!$$

The symmetry number of the whole graph is calculated as a product of its vertices symmetry numbers

$$S = \prod S_\ell.$$

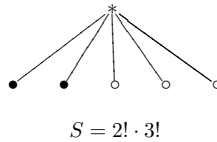
We shall say that a virtual graph is *totally asymmetric* if the only identical replacement  $Id \in n!$  doesn't change the representing concrete derivative graph. Each replacement  $\sigma \in n!$  will provide a different similar member, therefore the weight number for totally asymmetric virtual graph will be  $n!$ . It is hard imagine such possibility as interesting, but it will be usual for more complex cases of derivative calculation.

We shall say that a virtual graph is *totally symmetric*, if every replacement  $\sigma \in n!$  provides the same representing concrete derivative graph. In such case all similar members coincide, and we shall have the unitary weight 1 for totally symmetric virtual graph.

We remark that for the linear function of two arguments  $F(y, z) = y + z$ , the derivatives for composition  $F(f(x), g(x))$  will be calculated with usual binomial coefficients

$$D^n = \sum_{0 \leq k \leq n} \frac{n!}{k!(n-k)!} f^{(k)} g^{(n-k)}.$$

The virtual graph for such function must be drawn with coloured entrances and its weight coincides with binomial coefficient. We can draw the virtual graph for the derivative of order  $n = 5$  with  $k = 2$  black entrances corresponding the increment of argument  $f$  and  $n - k = 3$  white entrances corresponding the increment of argument  $g$ :



**Proposition 1.** *The  $n$ th order derivative of a joint function can be found using the collection of  $n$ -degree virtual graphs. Their weights are equal to the cardinality of the whole symmetry group  $n!$  divided by the symmetry number  $S$  of each virtual graph.*

*Proof.* We must check that our procedure of virtual graph drawing provides all needed concrete derivative graphs in the derivative calculation, and only such graphs. It is enough to check that we can get all concrete derivative graphs, and then apply calculation of similar concrete derivative graphs.

We apply the induction on the order  $n$  of a joint function derivative. For the zero order derivative the proposition is trivial. We shall check the case  $n = 1$ . Calculating the first derivative, all simple functions will be replaced by derivatives of first order. One must be sure that we have got all concrete derivative graph, and every concrete derivative graph is provided by such procedure.

The induction step from  $n$  to  $n + 1$ : Taking the derivative of any member presented by some concrete derivative graph, we differentiate some vertex, and get additional entrance of the concrete derivative graph of next degree  $(n + 1)$ . Also we must check that all concrete derivative graph of degree  $(n + 1)$  can be obtained in such manner from some concrete derivative graph of degree  $n$ . It is done by distraction anyone entrance from the taken concrete derivative graph. □

### 3 Ordering of virtual graphs

For easy virtual graphs recognition we need to choose simple ordering for all  $n$ -degree virtual graphs. If this ordering will be useful for a wide class of users, it may become standard. We prefer to order all virtual graphs, and secondly we induce this order to the set of virtual graphs of given degree  $n$ . It will be called a *natural order*. It is hard to imagine how somebody could choose the best ordering only in the set of  $n$ -degree graphs.

We order the virtual derivative graphs lexicographically. At first we order the virtual derivative of the eldest function. If we have some of the eldest functions, then we choose

the order between them. In such case we shall say that the top vertex is *ordered by the colour*. Each colour corresponds to some sort of the eldest functions.

Then we order the vertices of one colour by the degree of this vertex. We begin from the vertex of 0-degree, and then go to the higher degree. If the degree is higher than 1, then we order at first the vertex from the left argument, and after we go to the next argument to the right.

The ordering of new vertices is the same: the colour, degree, the younger vertex first from the left, and so on. Therefore virtual graph will be represented by concrete graphs having at left the younger graphs with first colour and smaller degree.

#### 4 The problems for the future

In geometrical calculations local coordinate change provides coordinate change for various geometric objects. Such new change is calculated as derivatives of joint function. The possible equality of joint functions compels us to construct some virtual graphs. In such a way we obtain new weights. They are obtained from earlier weights with some concrete addition operator. The earlier weights can be called as binomial and they present a free object for the derivative calculation task. The question remains open, for what derivative calculations such free object exists.

We shall give two examples of another free derivative calculation. The first one provides the derivatives of inverse function  $g = f^{-1}$ . Let these functions operate over the points  $x \in X$  and  $y \in Y$

$$y = f(x), \quad x = g(y).$$

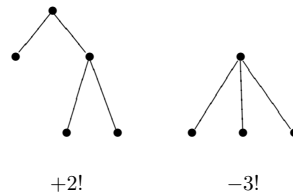
Then the derivatives of inverse function is calculated

$$\begin{aligned} Dg(y) &= (Df(g(y)))^{-1}, \\ D^2g(y) &= -\langle Dg(y), Dg(y) \rangle \bullet D^2f(g(y)) \bullet Dg(y), \\ D^3g(y) &= +3\langle Dg(y), \langle Dg(y), Dg(y) \rangle \bullet D^2f(g(y)) \bullet Dg(y) \rangle \bullet D^2f(g(y)) \bullet Dg(y) \\ &\quad - \langle Dg(y), Dg(y), Dg(y) \rangle \bullet D^3f(g(y)) \bullet Dg(y). \end{aligned}$$

The virtual derivative graphs are produced from the initial graph having only one vertex and one wedge for the identic composition  $1 = f \circ g$ .



The virtual derivative graphs are drawn without first order derivatives, but these derivatives must be written in final expression. The sign and weight of virtual derivative graph are immediately appointed in the same manner as in the previous case, cf. Valiukevičius [5, Table V]. We shall draw only virtual derivative graphs having degree  $n = 3$ . Under the graphs we shall write the graph's signed symmetry number. The first virtual graph will have a weight number +3 and second virtual graph will have a negative unitary number -1. A reader can verify that such weights are good for the corresponding members in the derivative formula.



The second example provides the derivatives of a solution of a differential equation

$$y' = f(y).$$

The following derivatives are obtained

$$\begin{aligned} y^{(2)} &= f(y) \bullet Df(y), \\ y^{(3)} &= f(y) \bullet Df(y) \bullet Df(y) + \langle f(y), f(y) \rangle \bullet D^2 f(y), \\ y^{(4)} &= f(y) \bullet Df(y) \bullet Df(y) \bullet Df(y) + \langle f(y), f(y) \rangle \bullet D^2 f(y) \bullet Df(y) \\ &\quad + 3 \langle f(y) \bullet Df(y), f(y) \rangle \bullet D^2 f(y) + \langle f(y), f(y), f(y) \rangle \bullet D^3 f(y). \end{aligned}$$

The virtual derivative graphs are provided from the initial graph having only one vertex representing the field  $f$ .



The virtual derivative graph is composed of field derivatives  $f^{(k)}$ , and each instance of derivative is provided by concrete derivative graph with ordered vertices.

In this case the degree  $n$  of virtual graph is defined by cardinality of graph vertex set, and the “binomial” weight is calculated with the earlier graph’s symmetry number and additionally with the new *complexity number*. We define the graph’s cardinality as the number of its vertices. If some vertex  $k$  has the younger graphs with cardinality  $c_1, c_2, \dots, c_{|k|}$  then the complexity number of taken vertex is defined by the product of these cardinalities

$$\tau_k = c_1 \cdots c_{|k|}.$$

For the whole graph of  $n$ -degree the complexity number is equal to the product of its vertices complexity numbers

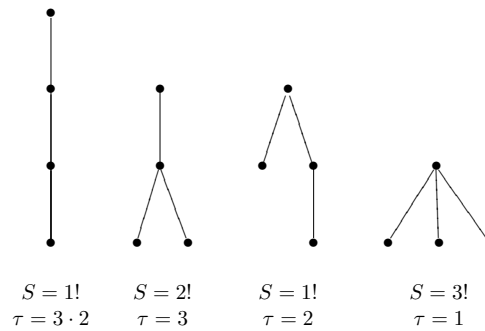
$$\tau = \tau_1 \cdots \tau_n.$$

For the  $n$ -degree virtual derivative graph the weight is calculated as  $(n - 1)!$  divided by the symmetry number and complexity number

$$\frac{(n - 1)!}{S\tau}.$$

The graphs are drawn in Valiukevičius [5, Table VII]. Now we shall draw only the virtual derivative graphs of degree  $n = 4$ . Under the graphs we shall write the graph’s symmetry

number and complexity number. All virtual graphs have the unitary weight numbers, only the third graph has the weight number 3. A reader can verify that such weights are good for the corresponding members in the derivative formula.



## References

1. D. Gromoll, W. Klingenberg, W. Meyer, *Riemannische Geometrie im Großen*, Springer, Berlin, Heidelberg, N.Y., 1968; Mir, Moscow, 1971.
2. A.M. Vinogradov, I.S. Krasil'shchik, V.V. Lychagin, *Vvedenie v Geometriyu Nelineinykh Differentsial'nykh Uravnenii*, Nauka, Moscow, 1986.
3. V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier*, **16**(1), pp.319–361, 1966.
4. S. Lang, *Introduction to Differentiable Manifolds*, Interscience, N.Y., London, 1962; Mir, Moscow, 1967.
5. G. Valiukevičius, Kombinatoriniai keliai išvestinėms skaičiuoti, preprint, Vilnius Univ. Library, Vilnius, 1979.
6. G. Valiukevičius, *Conspects on Analysis*, pp.4–21, Vilnius, 1989.