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О совместном распределении значений дзета-функций Гурвица<sup>1</sup>

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## Аннотация

Хорошо известно, что некоторые дзета и  $L$ -функции универсальны в смысле Воронины, т.е., ими приближается широкий класс аналитических функций. Некоторые из этих функций также совместно универсальны. В этом случае, набор аналитических функций одновременно приближается набором дзета-функций. В статье рассматривается проблема, связанная со совместной универсальностью дзета-функций Гурвица. Известно, что дзета-функции Гурвица  $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$  совместно универсальны, если параметры  $\alpha_1, \dots, \alpha_r$  алгебраически независимы над полем рациональных чисел  $\mathbb{Q}$ , или в более общем случае, если множество  $\{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\}$  линейно независимо над  $\mathbb{Q}$ . Мы рассматриваем случай произвольных параметров  $\alpha_1, \dots, \alpha_r$  и получаем, что существует непустое замкнутое множество функций  $F_{\alpha_1, \dots, \alpha_r}$  пространства  $H^r(D)$  аналитических в полосе  $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$  такое, что для любых компактных множеств  $K_1, \dots, K_r \subset D$ , функций  $(f_1, \dots, f_r) \in F_{\alpha_1, \dots, \alpha_r}$  и всякого  $\varepsilon > 0$  множество  $\{\tau \in \mathbb{R} : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\}$  имеет положительную нижнюю плотность. Также рассматривается случай положительной плотности этого множества.

*Ключевые слова:* вероятностная мера, дзета-функция Гурвица, пространство аналитических функций, слабая сходимости, универсальность.

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On joint value distribution of Hurwitz zeta-functions<sup>2</sup>

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## Abstract

It is well known that some zeta and  $L$ -functions are universal in the Voronin sense, i.e., they approximate a wide class of analytic functions. Also, some of them are jointly universal. In this case, a collection of analytic functions is simultaneously approximated by a collection of zeta-functions. In the paper, a problem related to joint universality of Hurwitz zeta-functions is discussed. It is known that the Hurwitz zeta-functions  $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$  are jointly universal if the parameters  $\alpha_1, \dots, \alpha_r$  are algebraically independent over the field of rational numbers  $\mathbb{Q}$ , or, more generally, if the set  $\{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\}$  is linearly independent over  $\mathbb{Q}$ . We consider the case of arbitrary parameters  $\alpha_1, \dots, \alpha_r$  and obtain that there exists a non-empty closed set  $F_{\alpha_1, \dots, \alpha_r}$  of the space  $H^r(D)$  of analytic functions on the strip  $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$  such that, for every compact sets  $K_1, \dots, K_r \subset D$ ,  $f_1, \dots, f_r \in F_{\alpha_1, \dots, \alpha_r}$  and  $\varepsilon > 0$ , the set  $\{\tau \in \mathbb{R} : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\}$  has a positive lower density. Also, the case of positive density of the latter set is discussed.

*Keywords:* Hurwitz zeta-function, probability measure, space of analytic functions, universality, weak convergence.

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## 1. Introduction

The Hurwitz zeta-function  $\zeta(s, \alpha)$ ,  $s = \sigma + it$ , with parameter  $\alpha$ ,  $0 < \alpha \leq 1$ , is defined, for  $\sigma > 1$ , by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

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and has the analytic continuation to the whole complex plane, except for a simple pole at the point  $s = 1$  with residue 1. For  $\alpha = 1$ , the Hurwitz zeta-function reduces to the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).$$

Thus,  $\zeta(s, \alpha)$  is a generalization of the Riemann zeta-function. The function  $\zeta(s)$  has the Euler product over primes

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

while the function  $\zeta(s, \alpha)$ , except for the values  $\alpha = 1$  and  $\alpha = \frac{1}{2}$ , has no such a product. This fact reflects in value distribution differences of the functions  $\zeta(s)$  and  $\zeta(s, \alpha)$ . For example, it is well known that  $\zeta(s) \neq 0$ , while the function  $\zeta(s, \alpha)$  has infinitely many zeros for all  $\alpha \neq 1, \frac{1}{2}$  in the half plane  $\sigma > 1$ . On the other hand, the functions  $\zeta(s)$  and  $\zeta(s, \alpha)$  for some classes of the parameter  $\alpha$  have a common property of the approximation of a wide class of analytic functions. This interesting property is called universality, and for the function  $\zeta(s)$  was obtained by S. M. Voronin [12]. For modern statements of universality theorems it is convenient to use the following notation. Let  $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ . Denote by  $\mathcal{K}$  the class of compact subsets of the strip  $D$  with connected complements, and by  $H_0(K)$  with  $K \in \mathcal{K}$  the class of continuous non-vanishing functions on  $K$  that are analytic in the interior of  $K$ . Then the modern Voronin universality theorem, see, for example, [7], says that for every  $K \in \mathcal{K}$ ,  $f \in H_0(K)$  and  $\varepsilon > 0$ ,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

The later inequality shows that there are infinitely many shifts  $\zeta(s + i\tau, \alpha)$  approximating with accuracy  $\varepsilon$  a given function  $f(s) \in H_0(K)$ . Yuri Vladimirovich Linnik knew the Voronin theorem and highly valued it. Moreover, Il'dar Abdulovich Ibragimov informed the second author that Yu. V. Linnik had a conjecture that all Dirichlet series satisfying some natural growth conditions are universal in the Voronin sense. Now this conjecture is called the Linnik-Ibragimov conjecture (or problem), see, for example, [11].

The universality of the Hurwitz zeta-function differs slightly from that of the function  $\zeta(s)$ . Denote by  $H(K)$  with  $K \in \mathcal{K}$  the class of continuous functions on  $K$  that are analytic in the interior of  $K$ . Thus,  $H_0(K) \subset H(K)$  for all  $K \in \mathcal{K}$ . Then the following universality theorem for the function  $\zeta(s, \alpha)$  is known.

**THEOREM 1.** *Suppose that the parameter  $\alpha$  is transcendental or rational  $\neq 1, \frac{1}{2}$ . Let  $K \in \mathcal{K}$  and  $f(s) \in H(K)$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0. \quad (1)$$

The theorem in the case of rational  $\alpha$  was already known to Voronin [14]. In a slightly different form, the theorem was obtained independently by S. M. Gonek and B. Bagchi in their theses [5], [1].

Unfortunately, the universality of  $\zeta(s, \alpha)$  with algebraic irrational parameter  $\alpha$  is an open problem. This problem is closely connected to linear independence over the field of rational numbers  $\mathbb{Q}$  of the set  $L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ . Denote by  $H(D)$  the space of analytic functions on  $D$  endowed with the topology of uniform convergence on compacta. Then, in [2], the following result towards to universality problem of  $\zeta(s, \alpha)$  with algebraic irrational  $\alpha$  was obtained.

**THEOREM 2.** *Suppose that the parameter  $\alpha$  is algebraic irrational. Then there exists a closed non-empty set  $F_\alpha \subset H(D)$  such that, for every compact set  $K \subset D$ ,  $f(s) \in F_\alpha$  and  $\varepsilon > 0$ , the inequality (1) is true.*

Some of zeta-functions are also jointly universal. In this case, a collection of analytic functions are simultaneously approximated by a collection of zeta-functions. The first joint universality results belong to S.M. Voronin. In [13], he considered the joint functional independence of Dirichlet  $L$ -functions, and, for this, he applied their joint universality. It is clear, that in the case of joint universality, the approximating zeta-functions must be in some sense independent. For Hurwitz zeta-functions this independence in [10] was described by the algebraic independence over  $\mathbb{Q}$  of the parameters  $\alpha_1, \dots, \alpha_r$ . In [8], the algebraic independence was replaced by the linear independence over  $\mathbb{Q}$  for the set

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\}.$$

Thus, the following theorem is known [8].

**THEOREM 3.** *Suppose that the set  $L(\alpha_1, \dots, \alpha_r)$  is linearly independent over  $\mathbb{Q}$ . For  $j = 1, \dots, r$ , let  $K_j \in \mathcal{K}$  and  $f_j(s) \in H(K_j)$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

The aim of this paper is to prove a joint generalization of Theorem 2, i.e., to prove a certain theorem on joint approximation by the functions  $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$  without using any independence condition.

**THEOREM 4.** *Suppose that the numbers  $\alpha_j$ ,  $0 < \alpha_j < 1$ ,  $\alpha_j \neq \frac{1}{2}$ ,  $j = 1, \dots, r$ , are arbitrary. Then there exists a closed non-empty set  $F_{\alpha_1, \dots, \alpha_r} \subset H^r(D)$  such that, for every compact sets  $K_1, \dots, K_r \subset D$ ,  $(f_1, \dots, f_r) \in F_{\alpha_1, \dots, \alpha_r}$  and  $\varepsilon > 0$ ,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Theorem 4 has the following modification.

**THEOREM 5.** *Suppose that the numbers  $\alpha_j$ ,  $0 < \alpha_j < 1$ ,  $\alpha_j \neq \frac{1}{2}$ ,  $j = 1, \dots, r$ , are arbitrary. Then there exists a closed non-empty set  $F_{\alpha_1, \dots, \alpha_r} \subset H^r(D)$  such that, for every compact sets  $K_1, \dots, K_r \subset D$  and  $(f_1, \dots, f_r) \in F_{\alpha_1, \dots, \alpha_r}$ , the limit*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0$$

*exists for all but at most countably many  $\varepsilon > 0$ .*

For the proof of above theorems we will apply the probabilistic approach. This is influenced in a certain sense by Yu. V. Linnik who was an expert not only in number theory but also in probability theory and mathematical statistics.

## 2. Auxiliary results

In this section, we will prove a joint limit theorem for the functions  $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$  in the space of analytic functions. Denote by  $\mathcal{B}(\mathbb{X})$  the Borel  $\sigma$ -field of the space  $\mathbb{X}$ , and, for  $A \subset \mathcal{B}(H^r(D))$ , define

$$P_{T,\underline{\alpha}}(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \underline{\zeta}(s + i\tau, \underline{\alpha}) \in A \},$$

where  $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$  and

$$\underline{\zeta}(s, \underline{\alpha}) = (\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)).$$

**THEOREM 6.** *Suppose that the numbers  $\alpha_j, 0 < \alpha_j < 1, \alpha_j \neq \frac{1}{2}, j = 1, \dots, r$ , are arbitrary. Then, on  $(H^r(D), \mathcal{B}(H^r(D)))$ , there exists a probability measure  $P_{\underline{\alpha}}$  such that  $P_{T,\underline{\alpha}}$  converges weakly to  $P_{\underline{\alpha}}$  as  $T \rightarrow \infty$ .*

We divide the proof of Theorem 6 into lemmas.

Denote by  $\gamma$  the unit circle on the complex plane, and define the set

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where  $\gamma_m = \gamma$  for all  $m \in \mathbb{N}_0$ . By the classical Tikhonov theorem, the infinite-dimensional torus  $\Omega$  with the product topology and pointwise multiplication is a compact topological Abelian group. Define one more set

$$\Omega^r = \prod_{j=1}^r \Omega_j,$$

where  $\Omega_j = \Omega$  for all  $j = 1, \dots, r$ . Then again by the Tikhonov theorem,  $\Omega^r$  is a compact topological Abelian group. Denote by  $\underline{\omega} = (\omega_1, \dots, \omega_r), \omega_1 \in \Omega_1, \dots, \omega_r \in \Omega_r$ , the elements of  $\Omega^r$ . Moreover, let  $\omega_j(m)$  be the  $m$ -th component of the element  $\omega_j \in \Omega, j = 1, \dots, r, m \in \mathbb{N}_0$ .

For  $A \in \mathcal{B}(\Omega^r)$ , define

$$Q_{T,\underline{\alpha}}(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0) \in A \}.$$

**LEMMA 1.** *On  $(\Omega^r, \mathcal{B}(\Omega^r))$ , there exists a probability measure  $Q_{\underline{\alpha}}$  such that  $Q_{T,\underline{\alpha}}$  converges weakly to  $Q_{\underline{\alpha}}$  as  $T \rightarrow \infty$ .*

**PROOF.** We apply the Fourier transform method. The dual group of  $\Omega^r$  is isomorphic to

$$\mathcal{G} = \bigoplus_{j=1}^r \bigoplus_{m \in \mathbb{N}_0} \mathbb{Z}_{mj},$$

where  $\mathbb{Z}_{mj} = \mathbb{Z}$  for all  $j = 1, \dots, r, m \in \mathbb{N}_0$ . The element  $\underline{k} = (k_{mj} : k_{mj} \in \mathbb{Z}, j = 1, \dots, r, m \in \mathbb{N}_0)$  in  $\mathcal{G}$ , where only a finite number of integers  $k_{mj}$  are distinct from zero, acts on  $\Omega^r$  by

$$\omega \rightarrow \omega^{\underline{k}} = \prod_{j=1}^r \prod_{m \in \mathbb{N}_0} \omega_j^{k_{mj}}(m).$$

Therefore, the Fourier transform  $g_T(\underline{k})$  of  $Q_{T,\underline{\alpha}}$  is of the form

$$g_T(\underline{k}) = \int_{\Omega^r} \left( \prod_{j=1}^r \prod_{m \in \mathbb{N}_0}' \omega_j^{k_{jm}}(m) \right) dQ_{T,\underline{\alpha}},$$

where the sign “'” shows that only a finite number of integers  $k_{mj}$  are distinct from zero. Thus, by the definition of  $Q_{T,\underline{\alpha}}$ ,

$$g_T(\underline{k}) = \frac{1}{T} \int_0^T \prod_{j=1}^r \prod'_{m \in \mathbb{N}_0} (m + \alpha_j)^{-i\tau k_{mj}} d\tau = \frac{1}{T} \int_0^T \exp \left\{ -i\tau \sum_{j=1}^r \sum'_{m \in \mathbb{N}_0} k_{mj} \log(m + \alpha_j) \right\} d\tau. \quad (2)$$

Define two collections of integers

$$\{\underline{k}'\} = \left\{ k_{mj} : \sum_{j=1}^r \sum'_{m \in \mathbb{N}_0} k_{mj} \log(m + \alpha_j) = 0 \right\}$$

and

$$\{\underline{k}''\} = \left\{ k_{mj} : \sum_{j=1}^r \sum'_{m \in \mathbb{N}_0} k_{mj} \log(m + \alpha_j) \neq 0 \right\}.$$

Obviously, in view of (2),

$$g_T(\underline{k}) = 1 \quad (3)$$

for  $\underline{k} \in \{\underline{k}'\}$ . If  $\underline{k} \in \{\underline{k}''\}$ , then integrating in (2), we find that

$$g_T(\underline{k}) = \frac{1 - \exp \left\{ -iT \sum_{j=1}^r \sum'_{m \in \mathbb{N}_0} k_{mj} \log(m + \alpha_j) \right\}}{iT \sum_{j=1}^r \sum'_{m \in \mathbb{N}_0} k_{mj} \log(m + \alpha_j)}.$$

This and (3) show that

$$\lim_{T \rightarrow \infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in \{\underline{k}'\}, \\ 0 & \text{if } \underline{k} \in \{\underline{k}''\}. \end{cases}$$

The right-hand side of the later equality is continuous in the discrete topology. Therefore, by a continuity theorem for probability measures on compact groups, we obtain that  $Q_{T,\underline{\alpha}}$ , as  $T \rightarrow \infty$ , converges weakly to a probability measure  $Q_{\underline{\alpha}}$  on  $(\Omega^r, \mathcal{B}(\Omega^r))$  defined by the Fourier transform

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in \{\underline{k}'\}, \\ 0 & \text{if } \underline{k} \in \{\underline{k}''\}. \end{cases}$$

The lemma is proved.  $\square$

Unfortunately, the limit measure  $Q_{\underline{\alpha}}$  in Lemma 1 is given by its Fourier transform, we do not know the explicit form of  $Q_{\underline{\alpha}}$ , and this reflects in Theorems 4 and 5 with non-effective set  $F_{\alpha_1, \dots, \alpha_r}$ . For example, if the set  $L(\alpha_1, \dots, \alpha_r)$  is linearly independent over  $\mathbb{Q}$ , then

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and we have that the limit measure  $Q_{\underline{\alpha}}$  coincides with the Haar measure on  $(\Omega^r, \mathcal{B}(\Omega^r))$ .

The next lemma is a joint limit theorem in the space  $H^r(D)$  for absolutely convergent Dirichlet series.

Let  $\sigma_0$  be a fixed number. For  $m \in \mathbb{N}_0$  and  $n \in \mathbb{N}$ , set

$$v_n(m, \alpha_j) = \exp \left\{ - \left( \frac{m + \alpha_j}{n + \alpha_j} \right)^{\sigma_0} \right\}, \quad j = 1, \dots, r,$$

and define the functions

$$\zeta_n(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

It is known [9] that the series for  $\zeta_n(s, \alpha_j)$  are absolutely convergent for  $\sigma > \frac{1}{2}$ . For brevity, let

$$\underline{\zeta}_n(s, \underline{\alpha}) = (\zeta_n(s, \alpha_1), \dots, \zeta_n(s, \alpha_r)),$$

and

$$P_{T,n,\underline{\alpha}}(A) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \underline{\zeta}_n(s + i\tau, \underline{\alpha}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

LEMMA 2. *On  $(H^r(D), \mathcal{B}(H^r(D)))$ , there exists a probability measure  $P_{n,\underline{\alpha}}$  such that  $P_{T,n,\underline{\alpha}}$  converges weakly to  $P_{n,\underline{\alpha}}$  as  $T \rightarrow \infty$ .*

PROOF. For  $\omega_j \in \Omega_j$ , define the functions

$$\zeta_n(s, \omega_j, \alpha_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m)v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

Since  $|\omega_j(m)| = 1$ , the series for  $\zeta_n(s, \omega_j, \alpha_j)$  is also absolutely convergent for  $\sigma > \frac{1}{2}$ . Let

$$\underline{\zeta}_n(s, \omega, \underline{\alpha}) = (\zeta_n(s, \omega_1, \alpha_1), \dots, \zeta_n(s, \omega_r, \alpha_r)).$$

Consider the function  $u_{n,\underline{\alpha}} : \Omega^r \rightarrow H^r(D)$  given by the formula

$$u_{n,\underline{\alpha}}(\omega) = \underline{\zeta}_n(s, \omega, \underline{\alpha}).$$

In virtue of the absolute convergence of the series for  $\zeta_n(s, \omega_j, \alpha_j)$ ,  $j = 1, \dots, r$ , the function  $u_{n,\underline{\alpha}}$  is continuous. Moreover,

$$u_{n,\underline{\alpha}} \left( ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0) \right) = \underline{\zeta}_n(s + i\tau, \underline{\alpha}).$$

Therefore, for every  $A \in \mathcal{B}(H^r(D))$ ,

$$\begin{aligned} P_{T,n,\underline{\alpha}}(A) &= \\ &= \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \left\{ ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0) \right\} \in u_{n,\underline{\alpha}}^{-1}A \right\} = \\ &= Q_{T,\underline{\alpha}}(u_{n,\underline{\alpha}}^{-1}A). \end{aligned}$$

Hence,  $P_{T,n,\underline{\alpha}} = Q_{T,\underline{\alpha}}u_{n,\underline{\alpha}}^{-1}$ . Therefore, Theorem 5.1 of [3], Lemma 1 and the continuity of the function  $u_{n,\underline{\alpha}}$  imply that  $P_{T,n,\underline{\alpha}}$  converges weakly to the measure  $P_{n,\underline{\alpha}} = Q_{\underline{\alpha}}u_{n,\underline{\alpha}}^{-1}$  as  $T \rightarrow \infty$ , where  $Q_{\underline{\alpha}}$  is the limit measure in Lemma 1.  $\square$

The next step of the proof of Theorem 6 consists of the approximation of  $\zeta(s, \underline{\alpha})$  by  $\underline{\zeta}_n(s, \underline{\alpha})$ . For this, we recall the metric in the space  $H^r(D)$ . It is known, see, for example, [4], that there exists a sequence of compact sets  $\{K_l : l \in \mathbb{N}\} \subset D$  such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$  for all  $l \in \mathbb{N}$ , and, for every compact set  $K \subset D$ , there exists  $K_l$  such that  $K \subset K_l$ . Let, for  $g_1, g_2 \in H(D)$ ,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then  $\rho$  is a metric in the space  $H(D)$  inducing the topology of uniform convergence on compacta. Now, setting, for  $\underline{g}_1 = (g_{11}, \dots, g_{1r}), \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$ ,

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j})$$

gives a metric in the space  $H^r(D)$  inducing its product topology.

LEMMA 3. *The equality*

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho \left( \zeta(s + i\tau, \underline{\alpha}), \zeta_n(s + i\tau, \underline{\alpha}) \right) d\tau = 0$$

holds.

PROOF. The proof of the lemma does not depend on the arithmetic of the numbers  $\alpha_1, \dots, \alpha_r$ , and can be found in [8], Lemma 7.  $\square$

Now, we consider the sequence  $\{P_{n, \underline{\alpha}} : n \in \mathbb{N}\}$ , where  $P_{n, \underline{\alpha}}$  is the limit measure in Lemma 2.

LEMMA 4. *The sequence  $P_{n, \underline{\alpha}}$  is tight, i.e., for every  $\varepsilon > 0$ , there exists a compact set  $K = K_\varepsilon \subset H^r(D)$  such that*

$$P_{n, \underline{\alpha}}(K) > 1 - \varepsilon$$

for all  $n \in \mathbb{N}$ .

PROOF. For an arbitrary  $\alpha$ ,  $0 < \alpha < 1$ , define

$$P_{T, n, \alpha}(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \zeta_n(s + i\tau, \alpha) \in A \}, \quad A \in \mathcal{B}(H(D)),$$

and denote by  $P_{n, \alpha}$  the limit measure of  $P_{T, n, \alpha}$  as  $T \rightarrow \infty$ . Then, in [2], it was obtained that the sequences  $\{P_{n, \alpha} : n \in \mathbb{N}\}$  is tight. Hence, the sequences

$$\{P_{n, \alpha_j} : n \in \mathbb{N}\}, \quad j = 1, \dots, r,$$

are tight. Clearly,  $P_{n, \alpha_j}$  are the marginal measures of the measure  $P_{n, \underline{\alpha}}$ , i.e.,

$$P_{n, \alpha_j}(A) = P_{n, \underline{\alpha}} \left( \underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D) \right), \quad A \in \mathcal{B}(H(D)), \quad (4)$$

$j = 1, \dots, r$ . Since the sequence  $\{P_{n, \alpha_j}\}$  is tight, for every  $\varepsilon > 0$ , there exists a compact set  $K_j = K_j(\varepsilon) \subset H(D)$  such that

$$P_{n, \alpha_j}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r, \quad (5)$$

for all  $n \in \mathbb{N}$ . We put  $K = K_1 \times \dots \times K_r$ . Then the set  $K$  is compact in the space  $H^r(D)$ . Moreover, in view of (4) and (5),

$$\begin{aligned} P_{n, \underline{\alpha}}(H^r(D) \setminus K) &= P_{n, \underline{\alpha}} \left( \bigcup_{j=1}^r \left( \underbrace{H(D) \times \dots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \dots \times H(D) \right) \right) \\ &\leq \sum_{j=1}^r P_{n, \underline{\alpha}} \left( \underbrace{H(D) \times \dots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \dots \times H(D) \right) \\ &= \sum_{j=1}^r P_{n, \alpha_j}(H(D) \setminus K_j) \leq \sum_{j=1}^r \frac{\varepsilon}{r} = \varepsilon \end{aligned}$$

for all  $n \in \mathbb{N}$ . Therefore,

$$P_{n, \underline{\alpha}}(K) \geq 1 - \varepsilon$$



for all  $n \in \mathbb{N}$ . The lemma is proved.  $\square$

PROOF. [Proof of Theorem 6] We will use the language of convergence in distribution ( $\xrightarrow{\mathcal{D}}$ ). Let the random variable  $\theta$  be defined on a certain probability space with measure  $\mu$ , and be uniformly distributed on  $[0, 1]$ . Define the  $H^r(D)$ -valued random element by the formula

$$X_{T,n,\underline{\alpha}} = X_{T,n,\underline{\alpha}}(s) = \zeta_n(s + i\theta T, \underline{\alpha}).$$

Moreover, let  $X_{n,\underline{\alpha}} = X_{n,\underline{\alpha}}(s)$  be the  $H^r(D)$ -valued random element having the distribution  $P_{n,\underline{\alpha}}$ . Then the assertion of Lemma 2 can be written in the form

$$X_{T,n,\underline{\alpha}} \xrightarrow[T \rightarrow \infty]{\mathcal{D}} X_{n,\underline{\alpha}}. \tag{6}$$

Since the sequence  $\{P_{n,\underline{\alpha}} : n \in \mathbb{N}\}$  is tight, by the Prokhorov theorem ([3, Theorem 6.1]), it is relatively compact. Therefore, there is a subsequence  $\{P_{n_k,\underline{\alpha}}\} \subset \{P_{n,\underline{\alpha}}\}$  such that  $P_{n_k,\underline{\alpha}}$  converges weakly to a certain probability measure  $P_{\underline{\alpha}}$  on  $(H^r(D), \mathcal{B}(H^r(D)))$  as  $k \rightarrow \infty$ . In other words, we have the relation

$$X_{n_k,\underline{\alpha}} \xrightarrow[k \rightarrow \infty]{\mathcal{D}} P_{\underline{\alpha}}. \tag{7}$$

Define one more  $H^r(D)$ -valued random element  $X_{T,\underline{\alpha}}$  by the formula

$$X_{T,\underline{\alpha}} = X_{T,\underline{\alpha}}(s) = \zeta(s + i\theta T, \underline{\alpha}).$$

Then, the application of Lemma 3 shows that, for every  $\varepsilon > 0$ ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \mu \{ \underline{\rho}(X_{T,\underline{\alpha}}, X_{T,n,\underline{\alpha}}) \geq \varepsilon \} \\ &= \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \underline{\rho} \left( \zeta(s + i\tau, \underline{\alpha}), \zeta_n(s + i\tau, \underline{\alpha}) \right) \geq \varepsilon \right\} \\ &\leq \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{\varepsilon T} \int_0^T \underline{\rho} \left( \zeta(s + i\tau, \underline{\alpha}), \zeta_n(s + i\tau, \underline{\alpha}) \right) d\tau = 0. \end{aligned}$$

The latter equality together with relations (6) and (7) shows that all hypotheses of Theorem 4.2 of [3] are satisfied. Therefore, we obtain the relation

$$X_{T,\underline{\alpha}} \xrightarrow[T \rightarrow \infty]{\mathcal{D}} P_{\underline{\alpha}},$$

which is equivalent to the weak convergence of  $P_{T,\underline{\alpha}}$  to  $P_{\underline{\alpha}}$  as  $T \rightarrow \infty$ . The theorem is proved.  $\square$

### 3. Proof of Theorems 4 and 5

Theorems 4 and 5 follow easily from Theorem 6. For this, the notion of the support of a probability measure is applied. Denote by  $F_{\alpha_1, \dots, \alpha_r}$  the support of the limit measure  $P_{\underline{\alpha}}$  in Theorem 6. We remind that  $F_{\alpha_1, \dots, \alpha_r} \subset H^r(D)$  is a minimal closed set such that  $P_{\underline{\alpha}}(F_{\alpha_1, \dots, \alpha_r}) = 1$ . The set  $F_{\alpha_1, \dots, \alpha_r}$  consists of all elements  $\underline{g} \in H^r(D)$  such that, for every open neighborhood  $G$  of  $\underline{g}$ , the inequality  $P_{\underline{\alpha}}(G) > 0$  is satisfied.

Also, we will use two equivalents of the weak convergence of probability measures. We recall that a set  $A$  is a continuity set of the probability measure  $P$  if  $P(\partial A) = 0$ , where  $\partial A$  is the boundary of the set  $A$ .

LEMMA 5. *Let  $P_n$ ,  $n \in \mathbb{N}$ , and  $P$  be the probability measures on  $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ . Then the following statements are equivalent:*

1°  $P_n$  converges weakly to  $P$  as  $n \rightarrow \infty$ ;

2° For every open set  $G \subset \mathbb{X}$ ,

$$\liminf_{n \rightarrow \infty} P_n(G) \geq P(G);$$

3° For every continuity set  $A$  of the measure  $P$ ,

$$\lim_{n \rightarrow \infty} P_n(A) = P(A).$$

The lemma is Theorem 2.1 of [3].

PROOF. [Proof of Theorem 4] Suppose that  $F_{\alpha_1, \dots, \alpha_r}$  is the support of the measure  $P_{\underline{\alpha}}$ . Then  $F_{\alpha_1, \dots, \alpha_r}$  is non-empty closed set of the space  $H^r(D)$ .

Let  $(f_1, \dots, f_r) \in F_{\alpha_1, \dots, \alpha_r}$ ,  $K_1, \dots, K_r$  are compact sets of the strip  $D$  and  $\varepsilon > 0$ . Define

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then the set  $G_\varepsilon$  is an open neighborhood of the element  $(f_1, \dots, f_r)$  which belongs to the support of the measure  $P_{\underline{\alpha}}$ . Therefore,

$$P_{\underline{\alpha}}(G_\varepsilon) > 0. \quad (8)$$

Moreover, in view of Theorem 6, and 1° and 2° of Lemma 5, we have that

$$\liminf_{T \rightarrow \infty} P_{T, \underline{\alpha}}(G_\varepsilon) \geq P_{\underline{\alpha}}(G_\varepsilon).$$

This, the definitions of  $P_{T, \underline{\alpha}}$  and  $G_\varepsilon$ , and (7) show that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

□

PROOF. [Proof of Theorem 5] We use the same notation as in the proof of Theorem 4. We observe that the boundaries  $\partial G_{\varepsilon_1}$  and  $\partial G_{\varepsilon_2}$  do not intersect for different positive  $\varepsilon_1$  and  $\varepsilon_2$ . Therefore,  $P_{\underline{\alpha}}(G_\varepsilon) > 0$  for at most countably many  $\varepsilon > 0$ . This shows that that the set  $G_\varepsilon$  is a continuity set of the measure  $P_{\underline{\alpha}}$  for all but at most countably many  $\varepsilon > 0$ . Therefore, using Theorem 6, 1° and 3° of Lemma 5, and inequality (7), we obtain that the limit

$$\lim_{T \rightarrow \infty} P_{T, \underline{\alpha}}(G_\varepsilon) = P_{\underline{\alpha}}(G_\varepsilon) > 0$$

exists for all but at most countably many  $\varepsilon > 0$ . Thus, the definitions of  $P_{T, \underline{\alpha}}$  and  $G_\varepsilon$  prove the theorem. □

## 4. Conclusions

The Hurwitz zeta-function  $\zeta(s, \alpha)$  depends on the parameter  $\alpha$  whose arithmetic properties influence the analytic behavior of  $\zeta(s, \alpha)$ , including the universality. The universality problem is related to the linear independence over  $\mathbb{Q}$  of the set

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}.$$

If the parameter  $\alpha$  is algebraic irrational, then we have not much information on the set  $L(\alpha)$ , it is only known by the Cassels theorem that at least 51 percent of elements  $L(\alpha)$  in the sense of density

are linearly independent over  $\mathbb{Q}$ . However, there is not any idea how to use the Cassels theorem for the proof of universality.

A similar situation arises in the investigation of the joint universality for Hurwitz zeta-functions. The linear independence of the set

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\}$$

leads to joint universality for the functions  $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ . In the paper, we search a way how to avoid involving of the set  $L(\alpha_1, \dots, \alpha_r)$ . Without using any information about the set  $L(\alpha_1, \dots, \alpha_r)$ , we prove that there exists a closed non-empty set of analytic functions such that the collections of those functions can be approximated by shifts  $(\zeta(s + i\tau, \alpha_1), \dots, \zeta(s + i\tau, \alpha_r))$ . It remains a very difficult problem to describe the mentioned set of analytic functions.

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