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Limit Theorems for Twists of L-Functions of Elliptic Curves. II

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Abstract. In the paper, a limit theorem for the argument of twisted with Dirichlet character L-functions of elliptic curves with an increasing modulus of the character is proved.

Keywords: Dirichlet character, elliptic curve, L-function of elliptic curve, probability measure, weak convergence.

AMS Subject Classification: 11M41; 44A15.

1 Introduction

In [3], we began to study limit theorems for twisted with Dirichlet character L-functions of elliptic curves with an increasing modulus of the character, and obtained a limit theorem of such a type for the modulus of these twists. Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z},$$

with non-zero discriminant $\Delta = -16(4a^3 + 27b^2)$. For each prime p, denote by E_p the reduction of the curve E modulo p which is a curve over the finite field $\mathbb{F}_p$, and define $\lambda(p)$ by

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ is the number of points of E_p. The L-function $L_E(s)$, $s = \sigma + it$, of the elliptic curve E is defined by the Euler product

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

Since, by the classical Hasse result,

$$|\lambda(p)| \leqslant 2\sqrt{p} \tag{1.1}$$

for all primes, the product defining $L_E(s)$ converges uniformly on compact subset of the half-plane $\{s \in \mathbb{C}\colon \sigma > \frac{3}{2}\}$ and define there an analytic function without zeros. Moreover, in [1], the Taniyama–Shimura conjecture has been proved, therefore, the function $L_E(s)$ is analytically continued to an entire function, and satisfies the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{s} \Gamma(s) L_E(s) = w \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L_E(2-s),$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function, N is the conductor of the curve E, and $w = \pm 1$.

The twist $L_E(s,\chi)$ with Dirichlet character χ for the function $L_E(s)$ is defined similarly. For $\sigma > \frac{3}{2}$, we have that

$$L_E(s,\chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}, \tag{1.2}$$

and function $L_E(s,\chi)$ is also analytically continued to an entire function.

Suppose that the modulus q of the character χ is a prime number, and is not fixed. Denoting by χ_0 the principal character modulo q, for $Q \geqslant 2$, define

$$M_Q = \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} 1,$$

and put

$$\mu_Q(\dots) = M_Q^{-1} \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0 \\ \dots}} 1,$$

where in place of dots we will write a condition satisfied by a pair $(q, \chi (\mathrm{mod}\, q))$. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Then in [3], the weak convergence of the frequency,

$$\hat{P}_Q(A) = \mu_Q(|L_E(s,\chi)| \in A), \quad A \in \mathcal{B}(\mathbb{R}),$$

as $Q \to \infty$, has been obtained. To state a limit theorem, we need some additional notation and definitions. For $p \nmid \Delta$, let $\alpha(p)$ and $\beta(p)$ be conjugate complex numbers such that $\alpha(p)\beta(p) = p$ and $\alpha(p) + \beta(p) = \lambda(p)$. Then (1.2), for $\sigma > \frac{3}{2}$, can be rewritten in the form

$$L_E(s,\chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-1}. \tag{1.3}$$

As in [3], we use the notation $\eta = \eta(\tau) = i\tau/2$, $\tau \in \mathbb{R}$, and, for primes p and $k \in \mathbb{N}$,

$$d_\tau(p^k) = \frac{\eta(\eta+1)\cdots(\eta+k-1)}{k!}.$$

For $p \nmid \Delta$ and $k \in \mathbb{N}$, we set

$$a_\tau(p^k) = \sum_{l=0}^{k} d_\tau(p^l)\alpha^l(p)d_\tau(p^{k-l})\beta^{k-l}(p), \tag{1.4}$$

$$b_\tau(p^k) = \sum_{l=0}^{k} d_\tau(p^l)\overline{\alpha}^l(p)d_\tau(p^{k-l})\overline{\beta}^{k-l}(p), \tag{1.5}$$

where $\overline{\alpha}(p)$ and $\overline{\beta}(p)$ denote the conjugates of $\alpha(p)$ and $\beta(p)$, respectively. For $p \mid \Delta$ and $k \in \mathbb{N}$, we define

$$a_\tau(p^k) = b_\tau(p^k) = d_\tau(p^k)\lambda^k(p). \tag{1.6}$$

Let $a_\tau(m)$ and $b_\tau(m)$, $m \in \mathbb{N}$, be multiplicative functions defined by (1.4)–(1.6), i.e.,

$$a_\tau(m) = \prod_{p^l \| m} a_\tau(p^l), \qquad b_\tau(m) = \prod_{p^l \| m} b_\tau(p^l),$$

where $p^l \parallel m$ means that $p^l \mid m$ but $p^{l+1} \nmid m$.

On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ define the probability measure $\hat{P}$ by the characteristic transforms [5],

$$w_k(\tau) = \int\limits_{\mathbb{R}\setminus\{0\}} |x|^{i\tau}\mathrm{sgn}^k\, \mathrm{d}\hat{P} = \sum_{m=1}^{\infty} \frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R},\ k = 0, 1.$$

Theorem 1 [see [3]]**.** *Suppose that $\sigma > \frac{3}{2}$. Then $\hat{P}_Q$ converges weakly to $\hat{P}$ as $Q \to \infty$.*

The other results for L-functions with increasing modulus of the character are shortly discussed in [3].

The aim of this paper is to prove a limit theorem for the argument of the function $L_E(s,\chi)$. The estimate (1.1) and (1.3) show that $L_E(s,\chi) \neq 0$ for $\sigma > \frac{3}{2}$. Thus, for $\sigma > \frac{3}{2}$, $\arg L_E(s,\chi)$ is well defined. For $k \in \mathbb{Z}$, let $\theta = \theta(k) = \frac{k}{2}$, for primes p and $l \in \mathbb{N}$,

$$d_k(p^l) = \frac{\theta(\theta+1)\cdots(\theta+l-1)}{l!},$$

and $d_k(1) = 1$. Now similarly to (1.4) and (1.5), for $p \nmid \Delta$ and $l \in \mathbb{N}$, we define

$$a_k(p^l) = \sum_{j=0}^{l} d_k(p^j)\alpha^j(p)d_k(p^{l-j})\beta^{l-j}(p),$$

$$b_k(p^l) = \sum_{j=0}^{l} d_{-k}(p^j)\overline{\alpha}^j(p)d_{-k}(p^{l-j})\overline{\beta}^{l-j}(p).$$

If $p \mid \Delta$, then, for $l \in \mathbb{N}$, we set

$$a_k(p^l) = d_k(p^l)\lambda^l(p), \quad b_k(p^l) = d_{-k}(p^l)\lambda^l(p).$$

Moreover, for $m \in \mathbb{N}$, we set

$$a_k(m) = \prod_{p^l \| m} a_k(p^l), \quad b_k(m) = \prod_{p^l \| m} b_k(p^l).$$

Thus, $a_k(m)$ and $b_k(m)$ are multiplicative functions. Denote by γ the unit circle on the complex plane. Furthermore, let P be a probability measure on $(\gamma, \mathcal{B}(\gamma))$ defined by the Fourier transform

$$g(k) \stackrel{\text{def}}{=} \int_\gamma x^k \, \mathrm{d}P = \sum_{m=1}^\infty \frac{a_k(m) b_k(m)}{m^{2\sigma}}, \quad k \in \mathbb{Z}, \ \sigma > \frac{3}{2}.$$

The main result of this paper is the following statement.

Theorem 2. *Suppose that $\sigma > \frac{3}{2}$. Then*

$$P_Q(A) \stackrel{\text{def}}{=} \mu_Q\bigl(\exp\{i \arg L_E(s,\chi)\} \in A\bigr), \quad A \in \mathcal{B}(\gamma),$$

converges weakly to P as $Q \to \infty$.

We recall that a distribution function $F(x)$ is said to be a distribution function mod 1 if

$$F(x) = \begin{cases} 1, & \text{if } x \geqslant 1, \\ 0, & \text{if } x < 0. \end{cases}$$

Let $F_n(x)$, $n \in \mathbb{N}$, and $F(x)$ be distribution functions mod 1. We say that $F_n(x)$, as $n \to \infty$, converges weakly mod 1 to $F(x)$, if at all continuity points x_1, x_2, $0 \leqslant x_1 \leqslant x_2 < 1$, of $F(x)$

$$\lim_{n \to \infty} (F_n(x_2) - F_n(x_1)) = F(x_2) - F(x_1).$$

Denote by $L(s,\chi)$ the Dirichlet L-functions. Elliott in [2], for $\sigma > \frac{1}{2}$, obtained the weak convergence mod 1, as $Q \to \infty$, for

$$\mu_Q\left(\frac{1}{2\pi} \arg L(s,\chi) \leqslant x (\operatorname{mod} 1)\right).$$

From Theorem 2, the following corollary follows.

Corollary 1. Suppose that $\sigma > \frac{3}{2}$. Then

$$\mu_Q\left(\frac{1}{2\pi} \arg L_E(s,\chi) \leqslant x (\operatorname{mod} 1)\right)$$

converges weakly mod 1 to the distribution function mod 1 defined by the Fourier transform $g(k)$ as $Q \to \infty$.

Differently from Dirichlet L-functions, we do not have any information on the convergence of the series defining the function $L_E(s,\chi)$, $\chi \neq \chi_0$, in the region $\sigma > 1$. Therefore, we can prove Theorem 2 only in the half-plane of absolute convergence of the mentioned series. Of course, we have a conjecture that the statement of Theorem 2 remains also true for $\sigma > 1$, however, at the moment we can not prove this.

2 Fourier Transform

Let $g_Q(k)$, $k \in \mathbb{Z}$, denote the Fourier transform of P_Q i.e., $g_Q(k) = \int_\gamma x^k \,\mathrm{d}P$. Then the definition of P_Q implies the equality

$$g_Q(k) = \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi=\chi(\bmod q) \\ \chi \neq \chi_0}} \mathrm{e}^{ik \arg L_E(s,\chi)}. \tag{2.1}$$

For the proof of Theorem 2, we need the asymptotics of $g_Q(k)$ as $Q \to \infty$. In this section, we give an expression for $g_Q(k)$ convenient for the investigation of its asymptotics.

For any fixed $\delta > 0$, denote by R the region $\{s \in \mathbb{C}\colon \sigma \geqslant \frac{3}{2} + \delta\}$. For $s \in R$, we have that

$$\begin{aligned}
(L_E(s,\chi))^{\frac{1}{2}} \left(\overline{L_E(s,\chi)}\right)^{-\frac{1}{2}} &= \left(L_E(s,\chi)\right)^{\frac{1}{2}} \left(L_E(\overline{s},\overline{\chi})\right)^{-\frac{1}{2}} \\
&= \left|L_E(s,\chi)\right|^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2} i \arg L(s,\chi)} \left|L_E(s,\chi)\right|^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2} i \arg L(s,\chi)} \\
&= \mathrm{e}^{i \arg L(s,\chi)}.
\end{aligned}$$

Therefore, for $s \in R$ and $k \in \mathbb{Z} \setminus \{0\}$, formula (1.3) yields

$$\begin{aligned}
\mathrm{e}^{ik \arg L_E(s,\chi)} &= \exp\Bigg\{-\frac{k}{2} \sum_{p|\Delta} \left(\log\left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right) - \log\left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right) \\
&\quad - \frac{k}{2} \sum_{p\nmid\Delta} \left(\log\left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right) + \log\left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)\right) \\
&\quad + \frac{k}{2} \sum_{p\nmid\Delta} \left(\log\left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right) + \log\left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right)\Bigg\} \\
&= \prod_{p|\Delta} \exp\left\{ -\theta \log\left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)\right\} \\
&\quad \times \prod_{p\nmid\Delta} \exp\left\{ -\theta\left(\log\left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right) + \log\left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)\right)\right\} \\
&\quad \times \prod_{p|\Delta} \exp\left\{\theta \log\left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right\} \\
&\quad \times \prod_{p\nmid\Delta} \exp\left\{\theta\left(\log\left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right) + \log\left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right)\right\} \\
&= \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-\theta} \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{\theta} \\
&\quad \times \prod_{p\nmid\Delta} \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-\theta} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-\theta}
\end{aligned}$$

$$\times \prod_{p \nmid \Delta} \left(1 - \frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{\theta} \left(1 - \frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{\theta}. \tag{2.2}$$

Here the multi-valued functions $\log(1-z)$ and $(1-z)^{\pm\theta}$ in the region $|z|<1$ are defined by continuous variation along any path lying in this region from the values $\log(1-z)|_{z=0} = 0$ and $(1-z)^{\pm\theta}|_{z=0} = 1$, respectively.

In the disc $|z|<1$, by the definition of $d_k(p^l)$ we have that

$$(1-z)^{\pm\theta} = \sum_{l=0}^{\infty} d_{\mp k}(p^l) z^l.$$

Therefore, (2.2) implies that, for $s \in R$,

$$\begin{aligned} \mathrm{e}^{ik \arg L_E(s,\chi)} &= \prod_{p|\Delta} \sum_{j=0}^{\infty} \frac{d_k(p^j)\lambda^j(p)\chi^j(p)}{p^{js}} \prod_{p\nmid\Delta} \sum_{l=0}^{\infty} \frac{d_k(p^l)\alpha^l(p)\chi^l(p)}{p^{ls}} \\ &\times \sum_{v=0}^{\infty} \frac{d_k(p^v)\beta^v(p)\chi^v(p)}{p^{vs}} \prod_{p|\Delta} \sum_{j=0}^{\infty} \frac{d_{-k}(p^j)\lambda^j(p)\overline{\chi}^j(p)}{p^{j\overline{s}}} \\ &\times \prod_{p\nmid\Delta} \sum_{l=0}^{\infty} \frac{d_{-k}(p^l)\overline{\alpha}^l(p)\overline{\chi}^l(p)}{p^{l\overline{s}}} \sum_{v=0}^{\infty} \frac{d_{-k}(p^v)\overline{\beta}^v(p)\overline{\chi}^v(p)}{p^{v\overline{s}}}. \end{aligned} \tag{2.3}$$

Let $\hat{a}_k(m)$ and $\hat{b}_k(m)$ be multiplicative functions with respect to m defined, for primes $p \nmid \Delta$ and $l \in \mathbb{N}$, by

$$\hat{a}_k(p^l) = \sum_{j=0}^{l} d_k(p^j)\alpha^j(p)\chi(p^j) d_k(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j}), \tag{2.4}$$

$$\hat{b}_k(p^l) = \sum_{j=0}^{k} d_{-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j) d_{-k}(p^{l-j})\overline{\beta}^{l-j}(p)\overline{\chi}(p^{l-j}), \tag{2.5}$$

and, for primes $p \mid \Delta$ and $l \in \mathbb{N}$, by

$$\hat{a}_k(p^l) = d_k(p^l)\lambda^l(p)\chi(p^l), \quad \hat{b}_k(p^l) = d_{-k}(p^l)\lambda^l(p)\overline{\chi}(p^l). \tag{2.6}$$

For $l \in \mathbb{N}$, we have that

$$\begin{aligned} |d_{\pm k}(p^l)| &\leqslant \frac{|\theta|(|\theta|+1)\cdots(|\theta|+l-1)}{l!} = \theta \prod_{j=2}^{l} \left(1 + \frac{|\theta|-1}{j}\right) \\ &\leqslant |\theta| \prod_{j=1}^{l} \left(1 + \frac{|\theta|}{j}\right) \leqslant |\theta| \exp\left\{|\theta| \sum_{j=1}^{l} \frac{1}{j}\right\} \leqslant (l+1)^c, \end{aligned} \tag{2.7}$$

where the constant c depends on k, only. By the definition of $\alpha(p)$ and $\beta(p)$, we have that $|\alpha(p)| = |\beta(p)| = \sqrt{p}$. Therefore, for $p \nmid \Delta$ and $l \in \mathbb{N}$, (2.4) and (2.5) imply the bounds

$$|\hat{a}_k(p^l)| \leqslant p^{\frac{l}{2}} \sum_{j=0}^{l} (j+1)^c (l-j+1)^c \leqslant p^{\frac{l}{2}} (l+1)^{2c+1} \tag{2.8}$$

and

$$|\hat{b}_k(p^l)| \leqslant p^{\frac{l}{2}}(l+1)^{2c+1}. \tag{2.9}$$

It is known [4] that, for $p \mid \Delta$, the numbers $\lambda(p)$ are equal to 1 or 0. Thus, by (2.6)–(2.7) we have that, for $p \mid \Delta$,

$$|\hat{a}_k(p^l)| \leqslant (l+1)^c, \quad |\hat{b}_k(p^l)| \leqslant (l+1)^c. \tag{2.10}$$

Now the multiplicativity of $\hat{a}_k(m)$ and $\hat{b}_k(m)$, and the estimates (2.8)–(2.10) show that

$$\hat{a}_k(m) = \prod_{p^l \| m} |\hat{a}_k(p^l)| \leqslant m^{\frac{1}{2}} \prod_{p^l \| m} (l+1)^{2c+1} = m^{\frac{1}{2}} d^{2c+1}(m), \tag{2.11}$$

$$|\hat{b}_k(m)| \leqslant m^{\frac{1}{2}} d^{2c+1}(m), \tag{2.12}$$

where $d(m)$ is the divisor function. Since

$$d(m) = \mathrm{O}_\varepsilon(m^\varepsilon) \tag{2.13}$$

with every $\varepsilon > 0$, the latter estimates imply, for every fixed $k \in \mathbb{Z} \setminus \{0\}$ and $s \in R$, the absolute convergence of the series

$$\sum_{m=1}^{\infty} \frac{\hat{a}_k(m)}{m^s} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{\hat{b}_k(m)}{m^s}.$$

Therefore, in view of (2.3), we conclude that, for every fixed $k \in \mathbb{Z} \setminus \{0\}$ and and $s \in R$,

$$\begin{aligned} \mathrm{e}^{ik \arg L_E(s,\chi)} &= \prod_{p|\Delta} \sum_{j=0}^{\infty} \frac{d_k(p^j)\lambda^j(p)\chi^j(p)}{p^{js}} \prod_{p\nmid\Delta} \sum_{l=0}^{\infty} \frac{\hat{a}_k(p^l)}{p^{ls}} \\ &\times \prod_{p|\Delta} \sum_{j=0}^{\infty} \frac{d_{-k}(p^j)\lambda^j(p)\overline{\chi}^j(p)}{p^{j\overline{s}}} \prod_{p\nmid\Delta} \sum_{l=0}^{\infty} \frac{\hat{b}_k(p^l)}{p^{l\overline{s}}} \\ &= \prod_{p} \sum_{l=0}^{\infty} \frac{\hat{a}_k(p^j)}{p^{js}} \prod_{p} \sum_{l=0}^{\infty} \frac{\hat{b}_k(p^l)}{p^{l\overline{s}}} = \sum_{m=1}^{\infty} \frac{\hat{a}_k(m)}{m^s} \sum_{n=1}^{\infty} \frac{\hat{b}_k(n)}{n^{\overline{s}}}. \end{aligned}$$

This and (2.1) give an expresion for the Fourier transform

$$g_Q(k) = \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} \sum_{m=1}^{\infty} \frac{\hat{a}_k(m)}{m^s} \sum_{n=1}^{\infty} \frac{\hat{b}_k(n)}{n^{\overline{s}}}. \tag{2.14}$$

3 Proof of Theorem 2

Having (2.14), we are in position to obtain the asymptotics for $g_Q(k)$ as $Q \to \infty$. First we modify the right-hand side of (2.14). Let $c_1 = 2c+1$. Then, using (2.11)–(2.13), we find that, for $s \in R$, any fixed $k \in \mathbb{Z} \setminus \{0\}$ and $N \in \mathbb{N}$,

$$\sum_{m>N} \frac{\hat{a}_k(m)}{m^s} = \mathrm{O}\Bigg(\sum_{m>N} \frac{d^{c_1}(m)}{m^{1+\delta}} \Bigg) = \mathrm{O}_\varepsilon\Bigg(\sum_{m>N} \frac{1}{m^{1+\delta-\varepsilon}} \Bigg) = \mathrm{O}_\varepsilon(N^{-\delta+\varepsilon}),$$

and

$$\sum_{m>N} \frac{\hat{b}_k(m)}{m^{\bar{s}}} = \mathrm{O}_\varepsilon(N^{-\delta+\varepsilon}).$$

Therefore, for any fixed $k \in \mathbb{Z} \setminus \{0\}$ and $s \in R$, (2.14) can be rewritten as

$$\begin{aligned} g_Q(k) &= \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \Bigg(\Bigg(\sum_{m \leqslant N} \frac{\hat{a}_k(m)}{m^s} + \mathrm{O}_\varepsilon(N^{-\delta+\varepsilon}) \Bigg) \\ &\quad \times \Bigg(\sum_{n \leqslant N} \frac{\hat{b}_k(n)}{n^{\bar{s}}} + \mathrm{O}_\varepsilon(N^{-\delta+\varepsilon}) \Bigg) \Bigg) \\ &= \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \Bigg(\sum_{m \leqslant N} \frac{\hat{a}_k(m)}{m^s} \sum_{n \leqslant N} \frac{\hat{b}_k(n)}{n^{\bar{s}}} \Bigg) \\ &\quad + \mathrm{O}_\varepsilon\Bigg(N^{-\delta+\varepsilon} \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \Bigg(\Bigg| \sum_{m \leqslant N} \frac{\hat{a}_k(m)}{m^s} \Bigg| + \Bigg| \sum_{n \leqslant N} \frac{\hat{b}_k(n)}{n^{\bar{s}}} \Bigg| \Bigg) \Bigg) \\ &\quad + \mathrm{O}_\varepsilon(N^{-\delta+\varepsilon}). \end{aligned} \tag{3.1}$$

Since, in view of (2.11)–(2.13), for any fixed $k \in \mathbb{Z} \setminus \{0\}$ and $s \in R$,

$$\sum_{m \leqslant N} \frac{\hat{a}_k(m)}{m^s} = \mathrm{O}\Bigg(\sum_{m=1}^{\infty} \frac{d^{c_1}(m)}{m^{1+\delta}} \Bigg) = \mathrm{O}(1), \qquad \sum_{n \leqslant N} \frac{\hat{b}_k(n)}{n^{\bar{s}}} = \mathrm{O}(1),$$

we find that

$$\frac{1}{M_Q} \sum_{q \leqslant Q} \Bigg(\Bigg| \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \frac{\hat{a}_k(m)}{m^s} \Bigg| + \Bigg| \sum_{n \leqslant N} \frac{\hat{b}_k(n)}{n^{\bar{s}}} \Bigg| \Bigg) = \mathrm{O}(1).$$

Substituting this in (3.1), we obtain that, for any fixed $k \in \mathbb{Z} \setminus \{0\}$ and $s \in R$,

$$g_Q(k) = \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \Bigg(\sum_{m \leqslant N} \frac{\hat{a}_k(m)}{m^s} \sum_{n \leqslant N} \frac{\hat{b}_k(n)}{n^{\bar{s}}} \Bigg) + \mathrm{O}\left(N^{-\delta+\varepsilon}\right). \tag{3.2}$$

From the multiplicativity of the functions $\hat{a}_k(m)$ and $\hat{b}_k(m)$, and the complete multiplicativity of Dirichlet characters we deduce that

$$\begin{aligned} \hat{a}_k(m) &= \prod_{p^l \| m} \hat{a}_k(p^l) = \prod_{\substack{p^l \| m \\ p \nmid \Delta}} \Bigg(\sum_{j=0}^{l} d_k(p^j)\alpha^j(p)\chi(p^j) d_k(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j}) \Bigg) \\ &\quad \times \prod_{\substack{p^l \| m \\ p \mid \Delta}} d_k(p^l)\lambda^l(p)\chi(p^l) \\ &= \Bigg(\prod_{p^l \| m} \chi(p^l) \Bigg) \prod_{\substack{p^l \| m \\ p \nmid \Delta}} \Bigg(\sum_{j=0}^{l} d_k(p^j)\alpha^j(p) d_k(p^{l-j})\beta^{l-j}(p) \Bigg) \end{aligned}$$

$$\times \prod_{\substack{p^l \| m \\ p | \Delta}} d_k(p^l)\lambda^l(p) = a_k(m)\chi(m),$$

and similarly $\hat{b}_k(m) = b_k(m)\overline{\chi}(m)$, where the multiplicative functions $a_k(m)$ and $b_k(m)$ are defined in Section 1. Thus, (3.2) becomes

$$g_Q(k) = \sum_{m \leqslant N} \frac{a_k(m)}{m^s} \sum_{n \leqslant N} \frac{b_k(n)}{n^s} \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \chi(m)\overline{\chi}(n). \tag{3.3}$$

If $m = n$, then we have that

$$\sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \chi(m)\overline{\chi}(n) = \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} |\chi(m)|^2 = M_Q - \sum_{\substack{q | m \\ q \leqslant N}} (q-2)$$
$$= M_Q + \mathrm{O}\biggl(\sum_{q \leqslant N} q\biggr) = M_Q + \mathrm{O}\left(N^2\right).$$

Therefore, taking $N = \log Q$, and using the estimate [3]

$$M_Q = \frac{Q^2}{2 \log Q} + \mathrm{O}\left(\frac{Q^2}{\log^2 Q}\right)$$

as well as (2.11) and (2.12) type estimates for $a_k(m)$ and $b_k(m)$, we find that, for any fixed $k \in \mathbb{Z} \setminus \{0\}$ and $s \in R$,

$$\sum_{\substack{m \leqslant N \\ m=n}} \sum_{n \leqslant N} \frac{a_k(m)}{m^s} \frac{b_k(n)}{n^{\overline{s}}} \frac{1}{M_Q} \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \chi(m)\overline{\chi}(n)$$
$$= \sum_{m \leqslant N} \frac{a_k(m) b_k(m)}{m^{2\sigma}} (1 + \mathrm{o}(1)) = \sum_{m=1}^{\infty} \frac{a_k(m) b_k(m)}{m^{2\sigma}} + \mathrm{o}(1) \tag{3.4}$$

as $Q \to \infty$. It remains to consider the case $m \neq n$. For this, we will apply the relation

$$\sum_{\chi = \chi (\mathrm{mod}\, q)} \chi(m)\overline{\chi}(n) = \begin{cases} q-1 & \text{if } m \equiv n (\mathrm{mod}\, q), \\ 0 & \text{if } m \not\equiv n (\mathrm{mod}\, q), \end{cases}$$

provided that $(m, q) = 1$. So, for $m \neq n$ and $m, n \leqslant N$, we have that

$$\sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \chi(m)\overline{\chi}(n) = \sum_{q \leqslant Q} \sum_{\chi = \chi (\mathrm{mod}\, q)} \chi(m)\overline{\chi}(n) - \sum_{q \leqslant Q} \sum_{\chi = \chi_0 (\mathrm{mod}\, q)} \chi(m)\overline{\chi}(n)$$
$$= \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ q | (m-n)}} \chi(m)\overline{\chi}(n) + \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ q \nmid (m-n)}} \chi(m)\overline{\chi}(n) + \mathrm{O}\biggl(\sum_{q \leqslant Q} 1\biggr)$$
$$= \mathrm{O}\biggl(\sum_{q \leqslant N} q\biggr) + \mathrm{O}\left(\frac{Q}{\log Q}\right) = \mathrm{O}\left(\frac{Q}{\log Q}\right).$$

Therefore, we obtain that, for any fixed $k \in \mathbb{Z} \setminus \{0\}$ and $s \in R$,

$$\begin{aligned}
&\sum_{\substack{m\leqslant N\\ m\neq n}}\sum_{n\leqslant N}\frac{a_k(m)b_k(n)}{m^s n^{\overline{s}}}\frac{1}{M_Q}\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}}\chi(m)\overline{\chi}(n)\\
&\quad = \mathrm{O}\biggl(\frac{1}{Q}\sum_{m\leqslant N}\frac{|a_k(m)|}{m^{\frac{3}{2}+\delta}}\sum_{m\leqslant N}\frac{|b_k(m)|}{m^{\frac{3}{2}+\delta}}\biggr)\\
&\quad = \mathrm{O}\biggl(\frac{1}{Q}\biggl(\sum_{m\leqslant N}\frac{d^{c_1}(m)}{m^{1+\delta}}\biggr)^2\biggr) = \mathrm{o}(1)
\end{aligned}$$

as $Q \to \infty$. Now this, (3.4) and (3.3) show that, for any fixed $k \in \mathbb{Z}$, uniformly in $s \in R$,

$$g_Q(k) = \sum_{m=1}^{\infty}\frac{a_k(m)b_k(m)}{m^{2\sigma}} + \mathrm{o}(1)$$

as $Q \to \infty$. The last relation implies the weak convergence of P_Q to the probability measure defined the Fourier transform

$$\sum_{m=1}^{\infty}\frac{a_k(m)b_k(m)}{m^{2\sigma}}$$

as $Q \to \infty$. The same arguments also prove Corollary 1.

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