

# Zeros of the Lerch Transcendent Function\*

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Received August 29, 2011; revised January 3, 2012; published online April 1, 2012

**Abstract.** We investigate the distribution of zeros of the Lerch transcendent function  $\Phi(q, s, \alpha) = \sum_{n=0}^{\infty} q^n (n + \alpha)^{-s}$ . We find an upper and lower estimates of zeros of the function  $\Phi(q, s, \alpha)$  in any rectangle  $\{s : \sigma_1 < \operatorname{Re} s < \sigma_2 \leq 1.73 \dots, 0 < \operatorname{Im} s \leq T\}$ . Further we are interested in a computer calculations concerning the zeros of  $\Phi(q, s, \alpha)$  in  $\{s : \operatorname{Re} s > 1, 0 < \operatorname{Im} s \leq 1000\}$ .

**Keywords:** polylogarithm, Lerch transcendent, zero distribution.

**AMS Subject Classification:** 11M41.

## 1 Introduction

Let  $s = \sigma + it$  denotes a complex variable. The Lerch transcendent function is the analytic continuation of the series

$$\Phi(q, s, \alpha) = \sum_{n=0}^{\infty} q^n (n + \alpha)^{-s},$$

which converges for any real number  $\alpha > 0$  if  $q$  and  $s$  are complex numbers with either  $|q| < 1$ , or  $|q| = 1$  and  $\sigma > 1$ . Here we consider  $\Phi(q, s, \alpha)$  as a function of  $s$  with the parameters  $q \in \mathbb{C}$ ,  $0 < |q| \leq 1$ , and  $0 < \alpha \leq 1$ . Special cases include the Riemann zeta-function  $\zeta(s) = \Phi(1, s, 1)$ , the Hurwitz zeta-function  $\zeta(s, \alpha) = \Phi(1, s, \alpha)$ , the polylogarithm function  $Li_s(q) = q\Phi(q, s, 1)$ , and the Lerch zeta-function  $L(\lambda, \alpha, s) = \Phi(\exp(2\pi i\lambda), s, \alpha)$ .

The Riemann zeta-function has no zeros in the right-half-plane  $\sigma \geq 1$ . In the left-half-plane  $\sigma \leq 0$  it has only trivial zeros at even negative integers. The famous Riemann hypothesis (RH) asserts that the remaining, nontrivial, zeros lie on the critical line  $\sigma = 1/2$ .

The Hurwitz zeta-function  $\zeta(s, \alpha)$  has infinitely many zeros in  $1 < \sigma < 1 + \alpha$  if  $\alpha$  is transcendental or rational  $\neq 1/2, 1$  (Davenport and Heilbronn [2]). This

\* Supported by grant No. MIP-94 from the Research Council of Lithuania.

result was extended by Cassels [1] for  $\alpha$  algebraic irrational. Let  $1/2 < \sigma_1 < \sigma_2 < 1$ . Then Voronin [16] (for rational  $\alpha \neq 1/2, 1$ ) and Gonek [10] (for transcendental  $\alpha$ ) proved that the number of zeros of  $\zeta(s, \alpha)$  in the rectangle  $\sigma_1 < \sigma < \sigma_2$ ,  $0 < t < T$  is approximately equal to  $T$  for sufficiently large  $T$ . Gonek [11] also showed that for certain values of  $\alpha$  the proportion of zeros of  $\zeta(s, \alpha)$  on  $\sigma = 1/2$  is definitely less than 1. In the complex  $s$ -plane trajectories of zeros  $\rho = \rho(\alpha)$  of the Hurwitz zeta function were considered in [8] and [9]. Based on these trajectories, the classification of nontrivial zeros of the Riemann zeta function were introduced. For the zero distribution of the Lerch zeta-function see [4, 5, 6, 7, 12].

Fornberg and Kölbig [3] investigated trajectories of zeros  $\rho = \rho(q)$  of the polylogarithm function  $Li_s(q)$  for real  $q$  with  $|q| < 1$ . They found that some trajectories tend towards the zeros of  $\zeta(s)$  as  $q \rightarrow -1$ , and approach these zeros closely as  $q \rightarrow 1 - \delta$  for small but finite  $\delta > 0$ . However, the later trajectories appear to descend to the point  $s = 1$  as  $\delta \rightarrow 0$ . Both, for  $q \rightarrow -1$  and  $q \rightarrow 1$ , there are trajectories which do not tend towards zeros of  $\zeta(s)$ .

Next we consider the zeros of  $\Phi(q, s, \alpha)$  for  $0 < \alpha < 1$  and  $q \in \mathbb{C}$ ,  $0 < |q| < 1$ . Let  $N_\Phi(\sigma_1, \sigma_2, T) = N_\Phi(\sigma_1, \sigma_2, T, q, \alpha)$  denote the number of zeros of  $\Phi(q, s, \alpha)$  in the region  $\{s: \sigma_1 < \operatorname{Re} s < \sigma_2, 0 < \operatorname{Im} s \leq T\}$ . Let  $\sigma_0 = \sigma_0(q, \alpha)$  be a real number defined by the equality

$$\sum_{n=1}^{\infty} \frac{|q|^n}{(n/\alpha + 1)^{\sigma_0}} = 1.$$

It is easy to see that  $\sigma_0 \leq c = 1.73\dots$ , where  $\zeta(c) = \sum_{n=1}^{\infty} n^{-c} = 2$ , and that  $\sigma_0$  can take any value between  $-\infty$  and  $c$ .

**Theorem 1.** *Let  $q \in \mathbb{C}$ ,  $0 < |q| < 1$ . Let  $0 < \alpha < 1$  be a transcendental number. Then we have that, for any fixed strip  $\sigma_1 < \sigma < \sigma_2 \leq \sigma_0$ ,*

$$T \ll N_\Phi(\sigma_1, \sigma_2, T) \ll T$$

*and  $\Phi(q, s, \alpha)$  has no zeros for  $\sigma > \sigma_0$ .*

The theorem is proved in Section 3.

Wiener and Wintner [17, Section 4] pointed to a possible relationship between the behaviour of the zeros in the right-half-plane  $\sigma > 1$  of the polylogarithm function and the Riemann Hypothesis. They proved that the Riemann Hypothesis is true if there exists a number  $0 < \varepsilon < 1$  such that  $\sum_{n=1}^{\infty} q^n n^{-s} \neq 0$  for  $\sigma > 1$  and  $1 - \varepsilon < q < 1$ . However, Montgomery [13] pointed that the polylogarithm function  $Li_s(e^{-1/N})$  has zeros in the region  $\sigma > 1$  for all sufficiently large integers  $N$ , making Wiener and Winter theorem vacuous. Theorem 1 shows that the Lerch transcendent function  $\Phi(q, s, \alpha)$  also has zeros in the region  $\sigma > 1$  for  $0.92 < q < 1$  and transcendental  $\alpha$ ,  $1/2 < \alpha < 1$ . In the next section, we try to find explicit zeros in  $\sigma > 1$ . We see that it is relatively easy to find zeros if  $\alpha \neq 1$ . In the case  $\alpha = 1$  the zeros in the right half-plane,  $\sigma > 1$  currently are out of reach.

## 2 Calculations

The calculations of this section were done with programme MATHEMATICA. To calculate the number  $N$  of zeros of  $\Phi(q, s, \alpha)$  inside the contour  $\Gamma$  we have used the well known formula

$$N = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\Phi(q, s, \alpha))'_s}{\Phi(q, s, \alpha)} ds.$$

If the interior of the contour  $\Gamma$  contains one zero  $\rho$ , then we find this zero using the following expression

$$\rho = \frac{1}{2\pi i} \int_{\Gamma} s \frac{(\Phi(q, s, \alpha))'_s}{\Phi(q, s, \alpha)} ds.$$

The zero  $\rho$  can be adjusted by MATHEMATICA command *FindRoot*.

Let  $R = \{s: \text{Re } s > 1, 0 < \text{Im } s \leq 1000\}$ . In Table 1, we present the number of zeros of function  $\Phi(q, s, \alpha)$  for chosen  $q$  and  $\alpha$  in the region  $R$ . For example, we see that  $\Phi(0.99, s, 0.9)$  has 34 zeros in  $R$ . In Table 1, the last column describes zeros of the Hurwitz zeta-function, and the last row describes zeros of the polylogarithm function. In view of Montgomery's result [13] we expect that  $\Phi(q, s, 1)$  has zeros in  $\sigma > 1$  for  $q \geq 0.9$ . If so, then Table 1 possibly indicates the different behaviour of zeros of  $\Phi(q, s, \alpha)$  in  $\sigma > 1$  dependently on  $\alpha = 1$  or  $\alpha \neq 1$ .

**Table 1.** Number of zeros of the function  $\Phi(q, s, \alpha)$  in the region  $R$ .

$\alpha \setminus q$	<b>0.9</b>	<b>0.95</b>	<b>0.99</b>	<b>1</b>
<b>0.9</b>	2	8	34	40
<b>0.95</b>	4	10	37	46
<b>0.99</b>	14	27	41	45
<b>1</b>	0	0	0	0

In Table 2, we present zeros of functions  $\Phi(0.9, s, 0.9)$ ,  $\Phi(0.9, s, 0.95)$ ,  $\Phi(0.9, s, 0.99)$ . In this table numbers were rounded up to two decimal places.

## 3 Proof of Theorem 1

First we formulate theorems of Kronecker and Rouché (see Tichmarsh [15, Section 8.3] and Tichmarsh [14, Section 3.42]).

**Lemma 1 [Kronecker's theorem].** *Let  $a_1, a_2, \dots, a_N$  be linearly independent real numbers, i.e. numbers such that relation  $\lambda_1 a_1 + \dots + \lambda_N a_N = 0$  is possible only if  $\lambda_1 = \dots = \lambda_N = 0$ . Let  $b_1, \dots, b_N$  be any real numbers, and  $\varepsilon$  a given positive number. Then we can find a number  $t$  and integers  $x_1, \dots, x_N$  such that  $|ta_n - b_n - x_n| < \varepsilon, n = 1, \dots, N$ .*

**Table 2.** Coordinates of zeros of the function  $\Phi(q, s, \alpha)$  in the region  $R$ .

	$\Phi(0.9, s, 0.9)$	$\Phi(0.9, s, 0.95)$	$\Phi(0.9, s, 0.99)$
1	1.02 + 550.55 <i>i</i>	1.07 + 108.39 <i>i</i>	1.05 + 480.29 <i>i</i>
2	1.02 + 609.75 <i>i</i>	1.01 + 135.21 <i>i</i>	1.11 + 525.79 <i>i</i>
3	–	1.09 + 169.68 <i>i</i>	1.08 + 588.57 <i>i</i>
4	–	1.07 + 196.67 <i>i</i>	1.06 + 616.03 <i>i</i>
5	–	–	1.11 + 651.27 <i>i</i>
6	–	–	1.11 + 696.71 <i>i</i>
7	–	–	1.13 + 724.38 <i>i</i>
8	–	–	1.05 + 759.64 <i>i</i>
9	–	–	1.15 + 787.05 <i>i</i>
10	–	–	1.02 + 805.00 <i>i</i>
11	–	–	1.12 + 849.96 <i>i</i>
12	–	–	1.17 + 895.31 <i>i</i>
13	–	–	1.09 + 958.10 <i>i</i>
14	–	–	1.00 + 985.50 <i>i</i>

**Lemma 2 [Rouché’s theorem].** *Suppose that  $f(s)$  and  $g(s)$  are analytic functions inside and on a regular closed curve  $\gamma$ , and that  $|f(s)| > |g(s)|$  for all  $s \in \gamma$ . Then  $f(s) + g(s)$  and  $f(s)$  have the same number of zeros inside  $\gamma$ .*

The next lemma will be useful in the proof of Theorem 1.

**Lemma 3.** *Let  $q \in \mathbb{C}$ ,  $0 < |q| < 1$ , and  $0 < \alpha < 1$  be a transcendental number. Let  $\sigma'$  be a real number. Let  $a(n)$  be a sequence of complex numbers such that  $|a(n)| = 1$ . Let  $\Phi_a(q, s, \alpha) = \sum_{n=0}^{\infty} a(n)q^n(n + \alpha)^{-s}$ . Then for any  $\varepsilon > 0$  there exist  $\tau \in \mathbb{R}$  such that*

$$|\Phi(q, s + i\tau, \alpha) - \Phi_a(q, s, \alpha)| < \varepsilon$$

for  $\text{Re } s \geq \sigma'$ .

*Proof.* The Dirichlet series of the Lerch transcendent function converges absolutely for any  $s$  if  $|q| < 1$ . Therefore, for given  $\sigma'$  there is a positive integer  $N$  such that, for any real number  $u$  and  $\sigma \geq \sigma'$ ,

$$\left| \sum_{n=N+1}^{\infty} \frac{q^n}{(n + \alpha)^{s+iu}} - \sum_{n=N+1}^{\infty} \frac{q^n a(n)}{(n + \alpha)^s} \right| \leq 2 \sum_{n=N+1}^{\infty} \frac{|q|^n}{(n + \alpha)^\sigma} < \frac{\varepsilon}{2}. \tag{3.1}$$

Let  $A = \sum_{n=0}^N |q|^n / (n + \alpha)^\sigma$ . There is a sequence of real numbers  $b(n)$  such that  $e^{-2\pi i b(n)} = a(n)$ . The numbers  $\log(n + \alpha)$  are linearly independent over  $\mathbb{Q}$  since  $\alpha$  is the transcendental number. By Kronecker’s theorem (Lemma 1), there exist a real number  $\tau$  and integers  $x_n$  such that

$$\left| \frac{\tau \log(n + \alpha)}{2\pi} - b(n) - x_n \right| < \frac{\varepsilon}{8\pi A}.$$

In view of the inequality  $|e^z - 1| \leq 2|z|$ , where  $|z| < 1$ , we obtain

$$|(n + \alpha)^{-i\tau} - a(n)| = |e^{-2\pi i(\tau \log(n+\alpha)/2\pi - b(n) - x_n)} - 1| < \frac{\varepsilon}{2A}.$$

By above we see that there is  $\tau$  such that, for  $\text{Re } s \geq \sigma'$ ,

$$\left| \sum_{n=0}^N \frac{q^n}{(n + \alpha)^{s+i\tau}} - \sum_{n=0}^N \frac{q^n a(n)}{(n + \alpha)^s} \right| \leq \sum_{n=0}^N \frac{|q|^n}{(n + \alpha)^{\sigma'}} |(n + \alpha)^{-i\tau} - a(n)| < \frac{\varepsilon}{2}.$$

This and inequality (3.1) in view of triangle inequality, prove Lemma 3.  $\square$

*Proof of Theorem 1.* For fixed  $q$  and  $\alpha$  the function  $\Phi(q, s, \alpha)$  is bounded in any right half-plane, of complex numbers. This together with Theorem 9.62 of Titchmarsh [14] give the bound

$$N_{\Phi}(\sigma_1, \sigma_2, T) \ll T.$$

Further, if the strip  $\sigma_1 < \sigma < \sigma_2$  contains a zero of  $\Phi(q, s, \alpha)$  then, arguing as in Lemma 1 of [4], we get the bound

$$N_{\Phi}(\sigma_1, \sigma_2, T) \gg T.$$

Next we will show that the function  $\Phi(q, s, \alpha)$  has a zero in the strip  $\sigma_1 < \sigma < \sigma_2$ . We consider an auxiliary function  $\Phi_a(q, \sigma, \alpha) = \sum_{n=0}^{\infty} a(n)q^n(n+\alpha)^{-\sigma}$ . For fixed  $\sigma, q$  and  $\alpha$ , let  $V$  be a set of values taken by  $\Phi_a(q, \sigma, \alpha)$  for independent  $a(0), a(1), \dots$ , where  $a(n) \in \mathbb{C}$  and  $|a(n)| = 1$ . If  $\sigma < \sigma_0$ , then by Tichmarsh [15, Section 11.5, p. 297] we see that

$$V = \left\{ z : |z| \leq \sum_{n=0}^{\infty} |q|^n(n + \alpha)^{-\sigma} \right\}.$$

Thus for  $\sigma_1 < \sigma' < \sigma_2, q$ , and  $\alpha$  there is a sequence  $a(1), a(2), \dots$ , such that  $\Phi_a(q, \sigma', \alpha) = 0$ .

Let  $0 < \varepsilon' < \min(\sigma' - \sigma_1, \sigma_2 - \sigma')$  be such that  $\Phi_a(q, s, \alpha) \neq 0$  for  $|s - \sigma'| = \varepsilon'$ . Let

$$\varepsilon = \min_{|s - \sigma'| = \varepsilon'} |\Phi_a(q, s, \alpha)|.$$

By Lemma 3 there is a real shift  $\tau$  such that

$$|\Phi(q, s + i\tau, \alpha) - \Phi_a(q, s, \alpha)| < \varepsilon$$

for  $\text{Re } s \geq \sigma_1$ . Hence Rouché’s theorem gives that  $\Phi(q, s, \alpha)$  has a zero in the disk  $|s - \sigma' - i\tau| < \varepsilon'$ , which is contained in the strip  $\sigma_1 < \sigma < \sigma_2$ . By this Theorem 1 is proved.  $\square$

### 4 Conclusions

Let  $0 < q < 1$  and  $1/2 < \alpha \leq 1$ . We expect that the Lerch transcendent function  $\Phi(q, s, \alpha)$  has zeros in  $\text{Re } s > 1$ , if  $q$  is sufficiently near to 1. For  $\alpha = 1$  this is due to Montgomery [13]. Here we prove the case when  $\alpha$  is a transcendental number. However, computer calculations indicate the different behaviour of zeros of  $\Phi(q, s, \alpha)$  in  $\text{Re } s > 1$  dependently on  $\alpha = 1$  or  $\alpha \neq 1$ .

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