Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition*

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Abstract. Two-dimensional parabolic equation with nonlocal condition is solved by alternating direction method in the rectangular domain. Values of the solution on the boundary points are bind with the integral of the solution in whole two-dimensional domain. Because of this nonlocal condition, the classical alternating direction method is complemented by the solution of low dimension system of algebraic equations. The peculiarities of the method are considered.

Keywords: parabolic equation, nonlocal integral condition, alternating-direction method, finite-difference method.

1 Introduction and statement of the problem

The paper deals with the initial problem for parabolic equation with an integral boundary condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad x, y \in \Omega = \{0 < x, y < 1\},\ \ 0 < t \leqslant T, \tag{1}$$

$$u(0,y,t) = \mu_1(y,t), \quad u(1,y,t) = \mu_2(y,t), \tag{2}$$

$$u(x,1,t) = \mu_3(x,t), \tag{3}$$

$$u(x,0,t) = \gamma(x) \iint\limits_{\Omega} u(\xi,\eta,t)\,\mathrm{d}\xi d\eta + \mu_4(x,t), \quad x \in \Gamma_1, \tag{4}$$

$$u(x,y,0) = \varphi(x,y). \tag{5}$$

Motivation and possible applications of the problem are indicated in [1]. Parabolic equations with nonlocal conditions of different types at present are an intensively considered field both in the theory of differential equations and in numerical analysis. In papers [2–10], linear and nonlinear two-dimensional parabolic equations with nonlocal conditions are solved by the finite difference method.

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The particularity of our problem (1)–(5) under consideration is that the value of the solution in nonlocal condition (4) at the boundary points is linked with a two-dimensional integral of the solution. This is the difference of our research from the analogous researches in the above mentioned articles [2–10]. Solution of two-dimensional parabolic equation with a nonlocal condition of type (4) by the finite difference method has considered rather in little. In [11], equation (1) with $f = 0$ and the integral condition.

$$u(x,y,t) = \iint\limits_{\Omega} K(x,y,\xi,\eta)u(\xi,\eta,t)\,\mathrm{d}\xi d\eta, \quad x,y \in \partial\Omega, \tag{6}$$

is solved in rectangle domain by the finite difference method under the assumption

$$\iint\limits_{\Omega} \bigl|K(x,y,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\bigr| < \rho < 1. \tag{7}$$

Condition (4) is a partial case of condition (6).

The main result of our paper is the fact that we show that two-dimensional parabolic equations with a nonlocal condition of type (4) can be successfully solved by an efficient alternating direction method, and that condition (7) is not always necessary for this purpose.

2 Statement of a difference problem

Let us introduce the notation:

$$\begin{aligned}
&\Lambda_1 u_{ij}^n = \frac{u_{i-1,j}^n - 2u_{ij}^n + u_{i+1,j}^n}{h^2},\\
&\Lambda_2 u_{ij}^n = \frac{u_{i,j-1}^n - 2u_{ij}^n + u_{i,j+1}^n}{h^2},\\
&u_{ij}^n = u(x_i,y_j,t_n), \quad i,j = 0,1,\dots,N, \ \ n = 0,1,2,\dots,M,\\
&h = \frac{1}{N}, \quad \tau = \frac{T}{M},
\end{aligned}$$

$$\rho_i = \begin{cases} 1, & i \neq 0, N,\\ 1/2, & i = 0, N. \end{cases}$$

Let us write the Peaceman–Rachford alternating direction method [12] for a differential problem (1)–(5).

$$\frac{u_{ij}^{n+1/2} - u_{ij}^n}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+1/2} + \Lambda_2 u_{ij}^n + f_{ij}^{n+1/2}, \quad i,j = 1,\dots,N-1, \tag{8}$$

$$u_{0j}^{n+1/2} = \mu_{1_j}^{n+1/2}, \quad u_{Nj}^{n+1/2} = \mu_{2_j}^{n+1/2}, \tag{9}$$

and

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+1/2} + \Lambda_2 u_{ij}^{n+1} + f_{ij}^{n+1}, \quad i,j = 1,\dots,N-1, \tag{10}$$

$$u_{iN}^{n+1} = \mu_{3_i}^{n+1}, \tag{11}$$

$$u_{i0}^{n+1} = h^2\gamma_i \sum_{k=1}^{N-1}\sum_{j=0}^{N-1} \rho_i u_{kj}^{n+1} + g_{i0}^{n+1} + \mu_{4_i}^{n+1}, \quad i = 1,\dots N-1. \tag{12}$$

Formula (12) is a trapezoidal rule for a two-dimensional integral. The quantity g_{i0}^{n+1} in this formula is defined as follows

$$g_{i0}^{n+1} = h^2\gamma_i \left(\sum_{i=0}^{N-1} \rho_i \mu_{3i}^{n+1} + \frac{1}{2}\sum_{j=1}^{N-1} \left(\mu_{1j}^{n+1} + \mu_{2j}^{n+1}\right) \right). \tag{13}$$

Note that problem (10)–(12) differs from the usual one-dimensional difference problems of the alternating direction method. Namely, we cannot solve system (10) for a single fixed value $i = 1,2,\dots,N-1$ separately – in nonlocal condition (12) interconnected.

3 Algorithm for solving difference equations

We present an algorithm how to find u_{ij}^{n+1} when the values of u_{ij}^{n} are known. The first part of algorithm is problem (8)–(9) realized by the classical Thomas algorithm, namely, we have to solve a system of difference equations with a three-diagonal matrix for each value of the index j separately, $j = 1,2,\dots,N-1$.

In the second part of algorithm we have to solve system (10)–(12) witch, as mentioned above, due to condition (12), cannot be solved separately with a single fixed value of the index i. Therefore, we solve this system (10)–(12) by a modified Thomas algorithm, described in [6], in which the algorithm was applied to the system of difference equations with a nonlocal condition, simpler than condition (12).

First of all we write system of equation (10) in the form

$$Bu_{i-1,j}^{n+1} - Cu_{ij}^{n+1} + Bu_{i+1,j}^{n+1} = F_{ij}^{n+1}, \tag{14}$$

where

$$B = \frac{\tau}{2h^2}, \quad C = \frac{\tau}{h^2} + 1, \quad F_{ij}^{n+1} = -\frac{\tau}{2}\left(\Lambda_1 u_{ij}^{n+1/2} + f_{ij}^{n+1}\right) - u_{ij}^{n+1/2}.$$

According to the modified Thomas algorithm, first of all we write the solution of system (10)–(12) to each fixed value $i = 1,2,\dots,N-1$ in the form

$$u_{ij}^{n+1} = \tilde{\alpha}_j u_{ij-1}^{n+1} + \tilde{\beta}_{ij}^{n+1}, \quad j = 1,\dots,N. \tag{15}$$

Note that in the general case, the coefficient $\tilde{\alpha}_j$ should also depend on indices i and n, however, regarding the specificity of equation (10) (coefficients B, C in equation (14) do not depend on i, j and n), the coefficient $\tilde{\alpha}_j$ in formula (15) will not depend on other indices. Using equation (14) and condition (11), we calculate by the classical Thomas algorithm:

$$\tilde{\alpha}_j = \frac{B}{C - B\tilde{\alpha}_{j+1}}, \quad j = 1, 2, \ldots, N-1; \quad \tilde{\alpha}_N = 0, \tag{16}$$

$$\tilde{\beta}_{ij}^{n+1} = \frac{B\tilde{\beta}_{i,j+1}^{n+1} - F_{ij}^{n+1}}{C - B\tilde{\alpha}_{j+1}}, \quad i, j = 1, 2, \ldots, N-1; \quad \tilde{\beta}_{iN}^{n+1} = \mu_{3i}^{n+1}. \tag{17}$$

Further we use theoretically the Thomas algorithm once more for each $i = 1, 2, \ldots, N-1$, but we look for u_{ij}^{n+1} in the following form

$$u_{ij}^{n+1} = \alpha_j u_{i0}^{n+1} + \beta_{ij}^{n+1}, \quad j = 0, 1, \ldots, N, \tag{18}$$

where $\alpha_0 = 1$, $\beta_{i0}^{n+1} = 0$. From expressions (15) and (18) we derive

$$\alpha_j = \tilde{\alpha}_j \alpha_{j-1}, \quad j = 1, 2, \ldots, N, \tag{19}$$

$$\beta_{ij}^{n+1} = \tilde{\alpha}_j \beta_{ij-1}^{n+1} + \tilde{\beta}_{ij}^{n+1}, \quad i = 1, 2, \ldots, N-1, \ \ j = 1, 2, \ldots, N. \tag{20}$$

We require now that solution (18) would satisfy nonlocal condition (12). By substituting expressions (18) into condition (12), we obtain

$$u_{i0}^{n+1} = h^2 \gamma_i \sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \rho_i \left(\alpha_j u_{k0}^{n+1} + \beta_{kj}^{n+1}\right) + g_{i0}^{n+1} + \mu_{4i}^{n+1}. \tag{21}$$

For each value of the index $i = 1, 2, \ldots, N-1$. In expressions (21) the quantities u_{i0}^{n+1}, $i = 1, 2, \ldots, N-1$ are unknown. These values are found by solving the system of equations (21) which we rewrite in the following shape:

$$Au_0^{n+1} = F, \tag{22}$$

where

$$A = \begin{pmatrix} 1 - h\gamma_1\alpha & -h\gamma_1\alpha & \cdots & -h\gamma_1\alpha \\ -h\gamma_2\alpha & 1 - h\gamma_2\alpha & \cdots & -h\gamma_2\alpha \\ \cdots & \cdots & \cdots & \cdots \\ -h\gamma_{N-1}\alpha & -h\gamma_{N-1}\alpha & \cdots & 1 - h\gamma_{N-1}\alpha \end{pmatrix},$$

u_0^{n+1} is the $(N-1)$-order vector $u_0^{n+1} = \{u_{i0}^{n+1}\}$,

$$\alpha = h \sum_{j=0}^{N-1} \rho_j \alpha_j = h\left(\frac{\alpha_0}{2} + \sum_{j=1}^{N-1} \alpha_j\right). \tag{23}$$

Thus, in order to realize the second part of the algorithm, i.e. to solve the system of equations (10)–(12) with a nonlocal condition, first, we need to find the coefficients

$\tilde{\alpha}_j$, $\tilde{\beta}_{ij}^{n+1}$ by the Thomas algorithm, and then to calculate the coefficients α_j, β_{ij}^{n+1} by formulas (19), (20). To achieve this aim, the number of arithmetic operations is proportional to N^2, i.e. proportional to the number of unknowns in one layer. Afterwards we have to solve the system of $(N-1)$-order linear algebraic equations (22). To this end, the number of arithmetic operations is proportional to N^3 (Gausian elimination) or N^2 (iterative methods). After finding u_{i0}^{n+1}, we have only to make use of formula (18).

Let us consider some main properties of the system of equations (22).

Lemma 1. *For each $\tau > 0$ and $h > 0$ there exists a strict estimate*

$$0 < \alpha < \frac{1}{2}. \tag{24}$$

Proof. Taking into consideration the values of coefficients B and C, we get from formula (16)

$$0 < \tilde{\alpha}_{N-j} < \frac{j}{j+1}, \quad j = 1, 2, \ldots, N-1.$$

Regarding the condition $\alpha_0 = 1$, it follows from formula (19) that

$$0 < \alpha_j < \frac{N-j}{N}, \quad j = 1, 2, \ldots, N-1.$$

These estimates and formula (23) directly yield the proposition of the lemma. □

Lemma 2. *If $-\infty < \gamma(x) < 2$, then determinant $\det A$ of the system* (22) *is a positive number.*

Proof. We estimate directly that for each $N = 2, 3, \ldots$.

$$\det A = 1 - h\alpha \sum_{i=1}^{N-1} \gamma_i$$

is true. According to the assumption of the lemma on the function $\gamma(x)$, we get

$$h\alpha \sum_{i=1}^{N-1} \gamma_i < 1.$$

Hence follows that $\det A > 0$. □

Lemma 3. *If $|\gamma(x)| \leqslant 2$, then matrix A is diagonally dominant.*

Proof. The condition of diagonal domination of matrix A is as follows

$$|1 - h\gamma_i\alpha| > (N-2)h|\gamma_i|\alpha. \tag{25}$$

Hence, if $0 \leqslant \gamma_i \leqslant 2$, condition (25) becomes as follows

$$1 > (1-h)\gamma_i\alpha.$$

The latter condition will always be true, if $0 < \alpha < 1/2$, $0 \leqslant \gamma_i \leqslant 2$.

If $-2 \leqslant \gamma_i < 0$, then condition (25) is written in the form

$$1 > (1 - 3h)|\gamma_i|\alpha.$$

The inequalities $|\gamma_i| \leqslant 2$ and $0 < \alpha < 1/2$ are sufficient for this condition. □

Remark. If $|\gamma(x)| \leqslant 2$, the system of equations (22) can be solved by a stable algorithm.

4 Numerical results

Problem (1)–(5) with the function $\gamma(x) = ce^x$, choosing different values of c, has been solved by the method described in this paper. The problem with $\gamma(x) = c$ was solved as well. In this case, in fact there is no need to solve the system of equation (22), because it follows from the condition $\gamma(x) = c$ that u_{i0}^{n+1} with all the values of i is an unknown constant, so the system of equations (22) degenerates to a single equation. Expressions of the functions $f(x)$, $\varphi(x)$ and $\mu_i(x)$, $i = 1, \ldots, 4$ were selected so that the function

$$u^*(x, y, t) = \sin(\pi x)\sin(\pi y)e^{2t}$$

were an exact solution of problem (1)–(5).

Numerical results are written in Tables 1–2.

Table 1. $h = 1/40, \tau = 1/1600, T = 1$.

$\gamma(x)$	0	1	-1	e^{3x}	$-e^{3x}$	-100	$0.5e^x$	e^x
r	0.0058	0.0055	0.0059	$1.8 \cdot 10^7$	0.0072	0.0133	0.0056	0.0053
$\gamma(x)$	$1.1e^x$	$1.3e^x$	$1.5e^x$	$1.8e^x$	$2e^x$	$2.5e^x$	$3e^x$	
r	0.0053	0.0053	0.0073	0.0124	0.0192	0.1813	72.227	

Table 2. $\gamma(x) = e^x, T = 1$.

h	$1/10$	$1/20$	$1/40$	$1/80$
τ	$1/10$	$1/40$	$1/160$	$1/640$
r	0.5166	0.1280	0.0329	0.0084

The errors

$$r = \max_{i,j} \left|u^*\left(x_i, y_j, t^n\right) - u_{ij}^n\right|$$

as $t_n = T$ are presented in these tables.

One of the main goals of the numerical experiment was to obtain information on the stability of a difference scheme. As far as it is known, when solving a parabolic equation with any type of nonlocal conditions by the finite difference method, one of the most important things is that the stability of a difference scheme depends on the parameters or functions presented in nonlocal conditions.

The authors of papers [3,9,13] have proved that while considering nonlocal conditions of different types in one or two-dimensional case, the sufficient stability condition of a difference scheme in a certain energetic norm can be

$$\left|\lambda(S)\right| < 1, \tag{26}$$

where S is a transition matrix of the difference scheme expressed in the shape

$$u^{n+1} = Su^n + \varphi^n,$$

$\lambda(S)$ – eigenvalue of matrix S.

The structure of the spectrum of the matrix S for difference scheme (8)–(12) considered by us has not yet been analyzed, therefore we could observe the fact of stability or instability only from the numerical result. Note that the spectrum of difference operator with nonlocal conditions can be very complicated [14, 15].

In addition we can compare our numerical results with the theoretical result on the stability of a difference scheme for a one-dimensional parabolic equation that formally corresponds to problem (1)–(5):

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t),\\
&u(0,t) = \gamma \int_0^1 u(x,t)\,\mathrm{d}x + \mu_4(t),\\
&u(1,t) = \mu_3(t), \quad u(x,0) = \varphi(x).
\end{aligned}$$

The stability condition of finite difference scheme for this problem is [13]:

$$-\infty < \gamma < 2,$$

which has a connection (at random or naturally) with the matrix property $\det A > 0$ of system (22) (see Lemma 2.

Table 1 indicates, that the numerical results, in the example solved by us, allow us to think that there is a certain analogy between problems (1)–(5) and (22) on the subject of stability.

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